

Volume I and Volume II



Bruce H. Edwards

Expanded Study
and Solutions Guide

CALCULUS

Seventh Edition

Larson / Hostetler / Edwards

Expanded Study and Solutions Guide

CALCULUS

SEVENTH EDITION

Larson / Hostetler / Edwards

All Chapters

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STUDY AND SOLUTIONS GUIDE, VOLUME I
CALCULUS, SEVENTH EDITION

by Bruce Edwards, Larson, Hostetler and Edwards
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STUDY AND SOLUTIONS GUIDE, VOLUME II
CALCULUS, SEVENTH EDITION

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CHAPTER P

Preparation for Calculus

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CHAPTER P

Preparation for Calculus

Section P.1 Graphs and Models

Solutions to Odd-Numbered Exercises

1. $y = -\frac{1}{2}x + 2$

x-intercept: (4, 0)

y-intercept: (0, 2)

Matches graph (b)

3. $y = 4 - x^2$

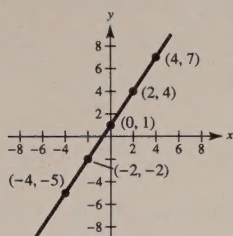
x-intercepts: (2, 0), (-2, 0)

y-intercept: (0, 4)

Matches graph (a)

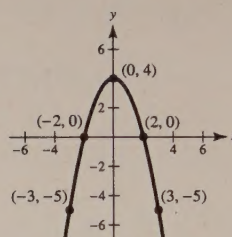
5. $y = \frac{3}{2}x + 1$

x	-4	-2	0	2	4
y	-5	-2	1	4	7



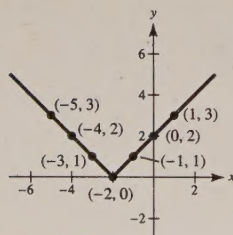
7. $y = 4 - x^2$

x	-3	-2	0	2	3
y	-5	0	4	0	-5



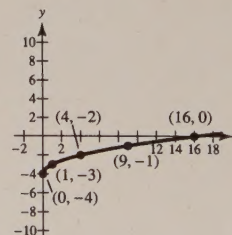
9. $y = |x + 2|$

x	-5	-4	-3	-2	-1	0	1
y	3	2	1	0	1	2	3



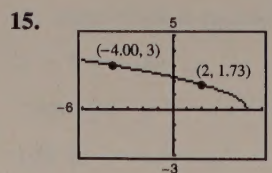
11. $y = \sqrt{x} - 4$

x	0	1	4	9	16
y	-4	-3	-2	-1	0



13.
$$\begin{array}{l} X_{\min} = -3 \\ X_{\max} = 5 \\ X_{\text{scl}} = 1 \\ Y_{\min} = -3 \\ Y_{\max} = 5 \\ Y_{\text{scl}} = 1 \end{array}$$

Note that $y = 4$ when $x = 0$.



(a) $(2, y) = (2, 1.73)$ ($y = \sqrt{5-2} = \sqrt{3} \approx 1.73$)

(b) $(x, 3) = (-4, 3)$ ($3 = \sqrt{5-(-4)}$)

17. $y = x^2 + x - 2$

y-intercept: $y = 0^2 + 0 - 2$
 $y = -2; (0, -2)$

x-intercepts: $0 = x^2 + x - 2$
 $0 = (x+2)(x-1)$
 $x = -2, 1; (-2, 0), (1, 0)$

21. $y = \frac{3(2 - \sqrt{x})}{x}$

y-intercept: None. x cannot equal 0.

x-intercepts: $0 = \frac{3(2 - \sqrt{x})}{x}$
 $0 = 2 - \sqrt{x}$
 $x = 4; (4, 0)$

25. Symmetric with respect to the y-axis since

$$y = (-x)^2 - 2 = x^2 - 2.$$

29. Symmetric with respect to the origin since

$$(-x)(-y) = xy = 4.$$

33. Symmetric with respect to the origin since

$$-y = \frac{-x}{(-x)^2 + 1}$$

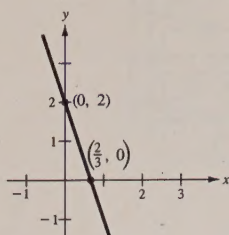
$$y = \frac{x}{x^2 + 1}.$$

37. $y = -3x + 2$

Intercepts:

$$\left(\frac{2}{3}, 0\right), (0, 2)$$

Symmetry: none



19. $y = x^2\sqrt{25-x^2}$

y-intercept: $y = 0^2\sqrt{25-0^2}$
 $y = 0; (0, 0)$

x-intercepts: $0 = x^2\sqrt{25-x^2}$
 $0 = x^2\sqrt{(5-x)(5+x)}$
 $x = 0, \pm 5; (0, 0); (\pm 5, 0)$

23. $x^2y - x^2 + 4y = 0$

y-intercept:

$$0^2(y) - 0^2 + 4y = 0$$

$$y = 0; (0, 0)$$

x-intercept:

$$x^2(0) - x^2 + 4(0) = 0$$

$$x = 0; (0, 0)$$

27. Symmetric with respect to the x-axis since

$$(-y)^2 = y^2 = x^3 - 4x.$$

31. $y = 4 - \sqrt{x+3}$

No symmetry with respect to either axis or the origin.

35. $y = |x^3 + x|$ is symmetric with respect to the y-axis

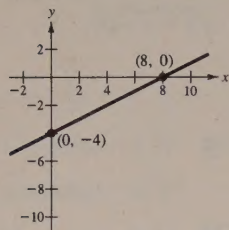
since $y = |(-x)^3 + (-x)| = |-(x^3 + x)| = |x^3 + x|.$

39. $y = \frac{x}{2} - 4$

Intercepts:

$(8, 0), (0, -4)$

Symmetry: none

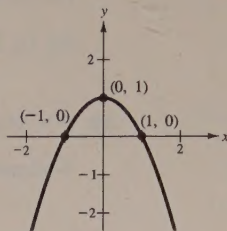


41. $y = 1 - x^2$

Intercepts:

$(1, 0), (-1, 0), (0, 1)$

Symmetry: y-axis

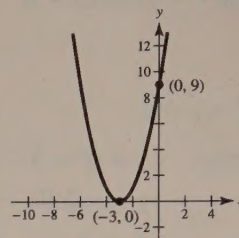


43. $y = (x + 3)^2$

Intercepts:

$(-3, 0), (0, 9)$

Symmetry: none

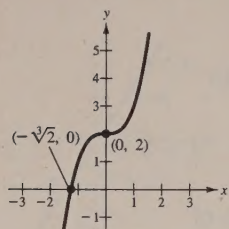


45. $y = x^3 + 2$

Intercepts:

$(-\sqrt[3]{2}, 0), (0, 2)$

Symmetry: none

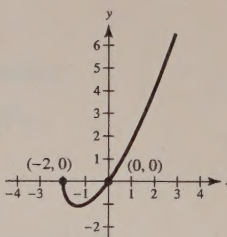


47. $y = x\sqrt{x+2}$

Intercepts:

$(0, 0), (-2, 0)$

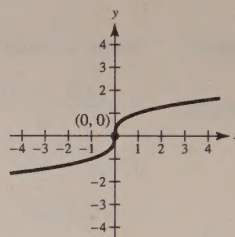
Symmetry: none

Domain: $x \geq -2$ 

49. $x = y^3$

Intercepts: $(0, 0)$

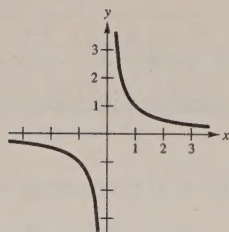
Symmetry: origin



51. $y = \frac{1}{x}$

Intercepts: none

Symmetry: origin

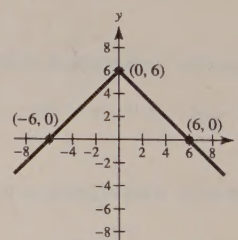


53. $y = 6 - |x|$

Intercepts:

$(0, 6), (-6, 0), (6, 0)$

Symmetry: y-axis



55. $y^2 - x = 9$

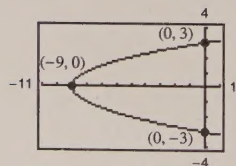
$y^2 = x + 9$

$y = \pm\sqrt{x+9}$

Intercepts:

$(0, 3), (0, -3), (-9, 0)$

Symmetry: x-axis



57. $x + 3y^2 = 6$

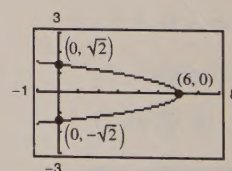
$3y^2 = 6 - x$

$y = \pm\sqrt{2 - \frac{x}{3}}$

Intercepts:

$(6, 0), (0, \sqrt{2}), (0, -\sqrt{2})$

Symmetry: x-axis



59. $y = (x + 2)(x - 4)(x - 6)$ (other answers possible)

63. $x + y = 2 \Rightarrow y = 2 - x$

$$2x - y = 1 \Rightarrow y = 2x - 1$$

$$2 - x = 2x - 1$$

$$3 = 3x$$

$$1 = x$$

The corresponding y-value is $y = 1$.

Point of intersection: $(1, 1)$

67. $x^2 + y = 6 \Rightarrow y = 6 - x^2$

$$x + y = 4 \Rightarrow y = 4 - x$$

$$6 - x^2 = 4 - x$$

$$0 = x^2 - x - 2$$

$$0 = (x - 2)(x + 1)$$

$$x = 2, -1$$

The corresponding y-values are $y = 2$ (for $x = 2$) and $y = 5$ (for $x = -1$).

Points of intersection: $(2, 2), (-1, 5)$

71. $y = x^3$

$$y = x$$

$$x^3 = x$$

$$x^3 - x = 0$$

$$x(x + 1)(x - 1) = 0$$

$$x = 0, x = -1, \text{ or } x = 1$$

The corresponding y-values are $y = 0, y = -1$, and $y = 1$.

Points of intersection: $(0, 0), (-1, -1), (1, 1)$

61. Some possible equations:

$$y = x$$

$$y = x^3$$

$$y = 3x^3 - x$$

$$y = \sqrt[3]{x}$$

65. $x + y = 7 \Rightarrow y = 7 - x$

$$3x - 2y = 11 \Rightarrow y = \frac{3x - 11}{2}$$

$$7 - x = \frac{3x - 11}{2}$$

$$14 - 2x = 3x - 11$$

$$-5x = -25$$

$$x = 5$$

The corresponding y-value is $y = 2$.

Point of intersection: $(5, 2)$

69. $x^2 + y^2 = 5 \Rightarrow y^2 = 5 - x^2$

$$x - y = 1 \Rightarrow y = x - 1$$

$$5 - x^2 = (x - 1)^2$$

$$5 - x^2 = x^2 - 2x + 1$$

$$0 = 2x^2 - 2x - 4 = 2(x + 1)(x - 2)$$

$$x = -1 \text{ or } x = 2$$

The corresponding y-values are $y = -2$ and $y = 1$.

Points of intersection: $(-1, -2), (2, 1)$

73. $y = x^3 - 2x^2 + x - 1$

$$y = -x^2 + 3x - 1$$

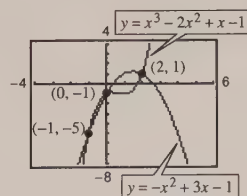
$$x^3 - 2x^2 + x - 1 = -x^2 + 3x - 1$$

$$x^3 - x^2 - 2x = 0$$

$$x(x - 2)(x + 1) = 0$$

$$x = -1, 0, 2$$

$(-1, -5), (0, -1), (2, 1)$



75. $C = R$

$$5.5\sqrt{x} + 10,000 = 3.29x$$

$$(5.5\sqrt{x})^2 = (3.29x - 10,000)^2$$

$$30.25x = 10.8241x^2 - 65,800x + 100,000,000$$

$$0 = 10.8241x^2 - 65,830.25x + 100,000,000 \quad \text{Use the Quadratic Formula.}$$

$$x \approx 3133 \text{ units}$$

The other root, $x \approx 2949$, does not satisfy the equation $R = C$.

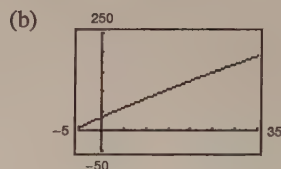
This problem can also be solved by using a graphing utility and finding the intersection of the graphs of C and R .

77. (a) Using a graphing utility, you obtain

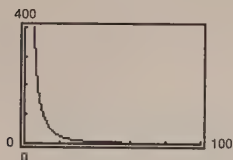
$$y = -0.0153t^2 + 4.9971t + 34.9405.$$

(c) For the year 2004, $t = 34$ and

$$y \approx 187.2 \text{ CPI.}$$



79. $y = \frac{10,770}{x^2} - 0.37$



If the diameter is doubled, the resistance is changed by approximately a factor of $\frac{1}{4}$. For instance, $y(20) \approx 26.555$ and $y(40) \approx 6.36125$.

81. False; x -axis symmetry means that if $(1, -2)$ is on the graph, then $(1, 2)$ is also on the graph.

83. True; the x -intercepts are

$$\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, 0 \right).$$

85. Distance from the origin = $K \times$ Distance from $(2, 0)$

$$\sqrt{x^2 + y^2} = K\sqrt{(x - 2)^2 + y^2}, K \neq 1$$

$$x^2 + y^2 = K^2(x^2 - 4x + 4 + y^2)$$

$$(1 - K^2)x^2 + (1 - K^2)y^2 + 4K^2x - 4K^2 = 0$$

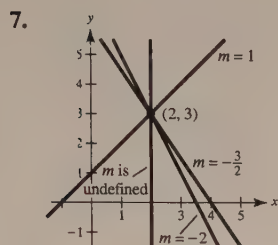
Note: This is the equation of a circle!

Section P.2 Linear Models and Rates of Change

1. $m = 1$

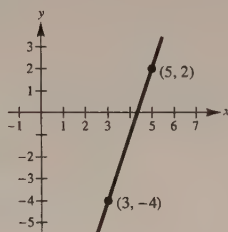
3. $m = 0$

5. $m = -12$



$$9. m = \frac{2 - (-4)}{5 - 3}$$

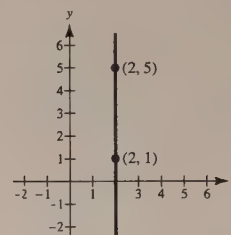
$$= \frac{6}{2} = 3$$



$$11. m = \frac{5 - 1}{2 - 2}$$

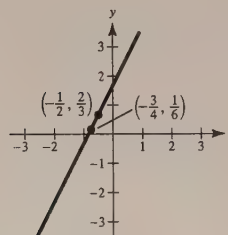
$$= \frac{4}{0}$$

undefined



$$13. m = \frac{2/3 - 1/6}{-1/2 - (-3/4)}$$

$$= \frac{1/2}{1/4} = 2$$



15. Since the slope is 0, the line is horizontal and its equation is $y = 1$. Therefore, three additional points are $(0, 1)$, $(1, 1)$, and $(3, 1)$.

17. The equation of this line is

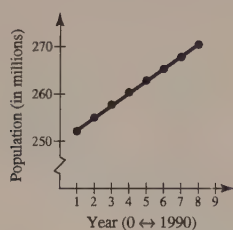
$$y - 7 = -3(x - 1)$$

$$y = -3x + 10.$$

Therefore, three additional points are $(0, 10)$, $(2, 4)$, and $(3, 1)$.

19. Given a line L , you can use any two distinct points to calculate its slope. Since a line is straight, the ratio of the change in y -values to the change in x -values will always be the same. See Section P.2 Exercise 93 for a proof.

21. (a)



(b) The slopes of the line segments are

$$\frac{255.0 - 252.1}{2 - 1} = 2.9$$

$$\frac{257.7 - 255.0}{3 - 2} = 2.7$$

$$\frac{260.3 - 257.7}{4 - 3} = 2.6$$

$$\frac{262.8 - 260.3}{5 - 4} = 2.5$$

$$\frac{265.2 - 262.8}{6 - 5} = 2.4$$

$$\frac{267.7 - 265.2}{7 - 6} = 2.5$$

$$\frac{270.3 - 267.7}{8 - 7} = 2.6$$

The population increased most rapidly from 1991 to 1992.

 $(m = 2.9)$

23. $x + 5y = 20$

$$y = -\frac{1}{5}x + 4$$

Therefore, the slope is $m = -\frac{1}{5}$ and the y-intercept is $(0, 4)$.

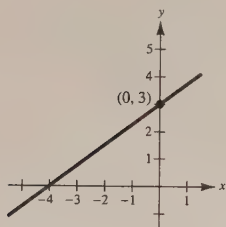
25. $x = 4$

The line is vertical. Therefore, the slope is undefined and there is no y-intercept.

27. $y = \frac{3}{4}x + 3$

$$4y = 3x + 12$$

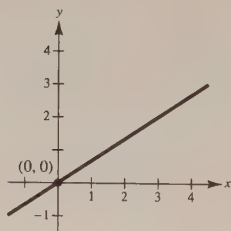
$$0 = 3x - 4y + 12$$



29. $y = \frac{2}{3}x$

$$3y = 2x$$

$$2x - 3y = 0$$

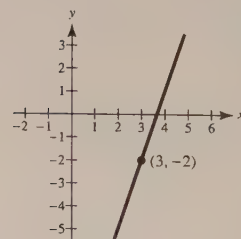


31. $y + 2 = 3(x - 3)$

$$y + 2 = 3x - 9$$

$$y = 3x - 11$$

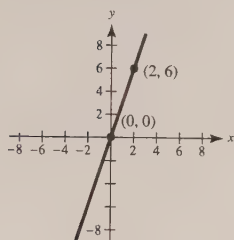
$$y - 3x + 11 = 0$$



33. $m = \frac{6 - 0}{2 - 0} = 3$

$$y - 0 = 3(x - 0)$$

$$y = 3x$$

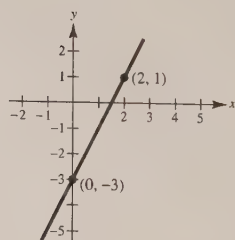


35. $m = \frac{1 - (-3)}{2 - 0} = 2$

$$y - 1 = 2(x - 2)$$

$$y - 1 = 2x - 4$$

$$0 = 2x - y - 3$$

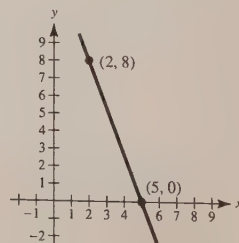


37. $m = \frac{8 - 0}{2 - 5} = -\frac{8}{3}$

$$y - 0 = -\frac{8}{3}(x - 5)$$

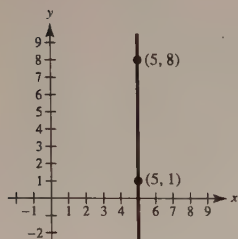
$$y = -\frac{8}{3}x + \frac{40}{3}$$

$$3y + 8x - 40 = 0$$



39. $m = \frac{8-1}{5-5}$ Undefined.

Vertical line $x = 5$

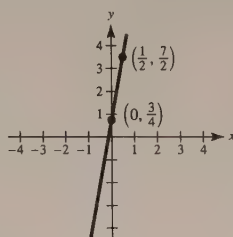


41. $m = \frac{7/2 - 3/4}{1/2 - 0} = \frac{11/4}{1/2} = \frac{11}{2}$

$$y - \frac{3}{4} = \frac{11}{2}(x - 0)$$

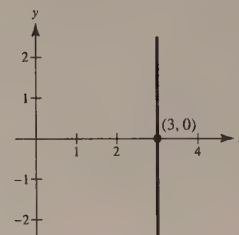
$$y = \frac{11}{2}x + \frac{3}{4}$$

$$22x - 4y + 3 = 0$$



43. $x = 3$

$$x - 3 = 0$$



45. $\frac{x}{2} + \frac{y}{3} = 1$

$$3x + 2y - 6 = 0$$

47. $\frac{x}{a} + \frac{y}{a} = 1$

$$\frac{1}{a} + \frac{2}{a} = 1$$

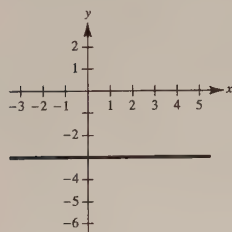
$$\frac{3}{a} = 1$$

$$a = 3 \Rightarrow x + y = 3$$

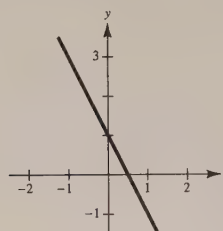
$$x + y - 3 = 0$$

49. $y = -3$

$$y + 3 = 0$$



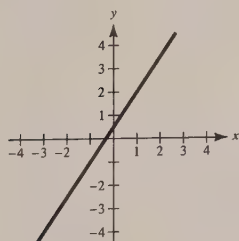
51. $y = -2x + 1$



53. $y - 2 = \frac{3}{2}(x - 1)$

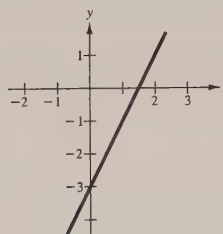
$$y = \frac{3}{2}x + \frac{1}{2}$$

$$2y - 3x - 1 = 0$$

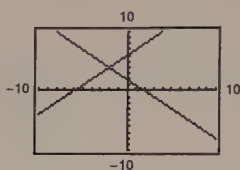


55. $2x - y - 3 = 0$

$$y = 2x - 3$$



57. (a)

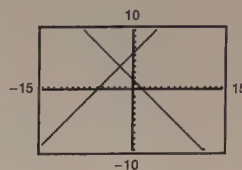


The lines do not appear perpendicular.

The lines are perpendicular because their slopes 1 and -1 are negative reciprocals of each other.

You must use a square setting in order for perpendicular lines to appear perpendicular.

(b)



The lines appear perpendicular.

59. $4x - 2y = 3$

$$y = 2x - \frac{3}{2}$$

$$m = 2$$

(a) $y - 1 = 2(x - 2)$

$$y - 1 = 2x - 4$$

$$2x - y - 3 = 0$$

(b) $y - 1 = -\frac{1}{2}(x - 2)$

$$2y - 2 = -x + 2$$

$$x + 2y - 4 = 0$$

61. $5x - 3y = 0$

$$y = \frac{5}{3}x$$

$$m = \frac{5}{3}$$

(a) $y - \frac{7}{8} = \frac{5}{3}(x - \frac{3}{4})$

$$24y - 21 = 40x - 30$$

$$24y - 40x + 9 = 0$$

(b) $y - \frac{7}{8} = -\frac{3}{5}(x - \frac{3}{4})$

$$40y - 35 = -24x + 18$$

$$40y + 24x - 53 = 0$$

63. The given line is vertical.

(a) $x = 2 \Rightarrow x - 2 = 0$

(b) $y = 5 \Rightarrow y - 5 = 0$

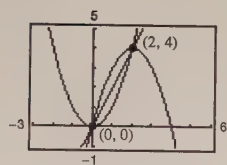
65. The slope is 125. Hence, $V = 125(t - 1) + 2540$

$$= 125t + 2415$$

67. The slope is -2000 . Hence, $V = -2000(t - 1) + 20,400$

$$= -2000t + 22,400$$

69.



You can use the graphing utility to determine that the points of intersection are $(0, 0)$ and $(2, 4)$. Analytically,

$$x^2 = 4x - x^2$$

$$2x^2 - 4x = 0$$

$$2x(x - 2) = 0$$

$$x = 0 \Rightarrow y = 0 \Rightarrow (0, 0)$$

$$x = 2 \Rightarrow y = 4 \Rightarrow (2, 4).$$

The slope of the line joining $(0, 0)$ and $(2, 4)$ is $m = (4 - 0)/(2 - 0) = 2$. Hence, an equation of the line is

$$y - 0 = 2(x - 0)$$

$$y = 2x.$$

$$71. m_1 = \frac{1-0}{-2-(-1)} = -1$$

$$m_2 = \frac{-2-0}{2-(-1)} = -\frac{2}{3}$$

$$m_1 \neq m_2$$

The points are not collinear.

73. Equations of perpendicular bisectors:

$$y - \frac{c}{2} = \frac{a-b}{c} \left(x - \frac{a+b}{2} \right)$$

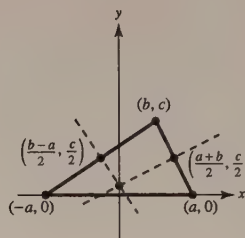
$$y - \frac{c}{2} = \frac{a+b}{-c} \left(x - \frac{b-a}{2} \right)$$

Setting the right-hand sides of the two equations equal and solving for x yields $x = 0$.

Letting $x = 0$ in either equation gives the point of intersection:

$$\left(0, \frac{-a^2 + b^2 + c^2}{2c} \right).$$

This point lies on the third perpendicular bisector, $x = 0$.



75. Equations of altitudes:

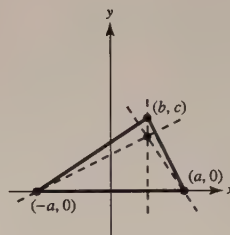
$$y = \frac{a-b}{c}(x+a)$$

$$x = b$$

$$y = -\frac{a+b}{c}(x-a)$$

Solving simultaneously, the point of intersection is

$$\left(b, \frac{a^2 - b^2}{c} \right).$$



77. Find the equation of the line through the points (0, 32) and (100, 212).

$$m = \frac{180}{100} = \frac{9}{5}$$

$$F - 32 = \frac{9}{5}(C - 0)$$

$$F = \frac{9}{5}C + 32$$

or

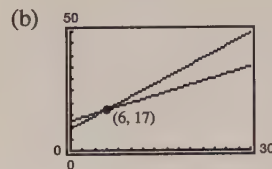
$$C = \frac{1}{9}(5F - 160)$$

$$5F - 9C - 160 = 0$$

For $F = 72^\circ$, $C \approx 22.2^\circ$.

79. (a) $W_1 = 0.75x + 12.50$

$$W_2 = 1.30x + 9.20$$



Using a graphing utility, the point of intersection is (6, 17). Analytically,

$$0.75x + 12.50 = 1.30x + 9.20$$

$$3.3 = 0.55x \Rightarrow x = 6$$

$$y = 0.75(6) + 12.50 = 17.$$

(c) Both jobs pay \$17 per hour if 6 units are produced. For someone who can produce more than 6 units per hour, the second offer would pay more. For a worker who produces less than 6 units per hour, the first offer pays more.

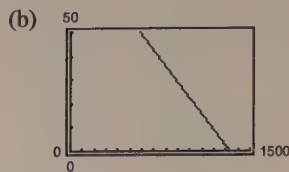
81. (a) Two points are (50, 580) and (47, 625). The slope is

$$m = \frac{625 - 580}{47 - 50} = -15.$$

$$p - 580 = -15(x - 50)$$

$$p = -15x + 750 + 580 = -15x + 1330$$

$$\text{or } x = \frac{1}{15}(1330 - p)$$



$$\text{If } p = 655, x = \frac{1}{15}(1330 - 655) = 45 \text{ units.}$$

$$(c) \text{ If } p = 595, x = \frac{1}{15}(1330 - 595) = 49 \text{ units.}$$

$$83. 4x + 3y - 10 = 0 \Rightarrow d = \frac{|4(0) + 3(0) - 10|}{\sqrt{4^2 + 3^2}} = \frac{10}{5} = 2$$

$$85. x - y - 2 = 0 \Rightarrow d = \frac{|1(-2) + (-1)(1) - 2|}{\sqrt{1^2 + 1^2}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

87. A point on the line
- $x + y = 1$
- is (0, 1). The distance from the point (0, 1) to
- $x + y - 5 = 0$
- is

$$d = \frac{|1(0) + 1(1) - 5|}{\sqrt{1^2 + 1^2}} = \frac{|1 - 5|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

89. If
- $A = 0$
- , then
- $By + C = 0$
- is the horizontal line
- $y = -C/B$
- . The distance to
- (x_1, y_1)
- is

$$d = \left| y_1 - \left(\frac{-C}{B} \right) \right| = \frac{|By_1 + C|}{|B|} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

If $B = 0$, then $Ax + C = 0$ is the vertical line $x = -C/A$. The distance to (x_1, y_1) is

$$d = \left| x_1 - \left(\frac{-C}{A} \right) \right| = \frac{|Ax_1 + C|}{|A|} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

(Note that A and B cannot both be zero.)

The slope of the line $Ax + By + C = 0$ is $-A/B$. The equation of the line through (x_1, y_1) perpendicular to $Ax + By + C = 0$ is:

$$y - y_1 = \frac{B}{A}(x - x_1)$$

$$Ay - Ay_1 = Bx - Bx_1$$

$$Bx_1 - Ay_1 = Bx - Ay$$

The point of intersection of these two lines is:

$$Ax + By = -C \quad \Rightarrow \quad A^2x + AB y = -AC \quad (1)$$

$$Bx - Ay = Bx_1 - Ay_1 \Rightarrow \frac{B^2x - AB y}{A^2 + B^2} = \frac{B^2x_1 - AB y_1}{A^2 + B^2} \quad (2)$$

$$(A^2 + B^2)x = -AC + B^2x_1 - AB y_1 \quad (\text{By adding equations (1) and (2)})$$

$$x = \frac{-AC + B^2x_1 - AB y_1}{A^2 + B^2}$$

$$Ax + By = -C \quad \Rightarrow \quad ABx + B^2y = -BC \quad (3)$$

$$Bx - Ay = Bx_1 - Ay_1 \Rightarrow \frac{-ABx + A^2y}{A^2 + B^2} = \frac{-ABx_1 + A^2y_1}{A^2 + B^2} \quad (4)$$

$$(A^2 + B^2)y = -BC - ABx_1 + A^2y_1 \quad (\text{By adding equations (3) and (4)})$$

$$y = \frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2}$$

89. —CONTINUED—

$$\left(\frac{-AC + B^2x_1 - AB y_1}{A^2 + B^2}, \frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2} \right) \text{ point of intersection}$$

The distance between (x_1, y_1) and this point gives us the distance between (x_1, y_1) and the line $Ax + By + C = 0$.

$$\begin{aligned} d &= \sqrt{\left[\frac{-AC + B^2x_1 - AB y_1}{A^2 + B^2} - x_1 \right]^2 + \left[\frac{-BC - ABx_1 + A^2y_1}{A^2 + B^2} - y_1 \right]^2} \\ &= \sqrt{\left[\frac{-AC - AB y_1 - A^2x_1}{A^2 + B^2} \right]^2 + \left[\frac{-BC - ABx_1 - B^2y_1}{A^2 + B^2} \right]^2} \\ &= \sqrt{\left[\frac{-A(C + B y_1 + A x_1)}{A^2 + B^2} \right]^2 + \left[\frac{-B(C + A x_1 + B y_1)}{A^2 + B^2} \right]^2} \\ &= \sqrt{\frac{(A^2 + B^2)(C + A x_1 + B y_1)^2}{(A^2 + B^2)^2}} \\ &= \frac{|A x_1 + B y_1 + C|}{\sqrt{A^2 + B^2}} \end{aligned}$$

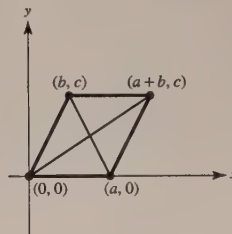
91. For simplicity, let the vertices of the rhombus be $(0, 0)$, $(a, 0)$, (b, c) , and $(a + b, c)$, as shown in the figure. The slopes of the diagonals are then

$$m_1 = \frac{c}{a + b} \text{ and } m_2 = \frac{c}{b - a}.$$

Since the sides of the Rhombus are equal, $a^2 = b^2 + c^2$, and we have

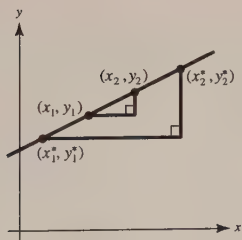
$$m_1 m_2 = \frac{c}{a + b} \cdot \frac{c}{b - a} = \frac{c^2}{b^2 - a^2} = \frac{c^2}{-c^2} = -1.$$

Therefore, the diagonals are perpendicular.



93. Consider the figure below in which the four points are collinear. Since the triangles are similar, the result immediately follows.

$$\frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}$$



95. True.

$$ax + by = c_1 \Rightarrow y = -\frac{a}{b}x + \frac{c_1}{b} \Rightarrow m_1 = -\frac{a}{b}$$

$$bx - ay = c_2 \Rightarrow y = \frac{b}{a}x - \frac{c_2}{a} \Rightarrow m_2 = \frac{b}{a}$$

$$m_2 = -\frac{1}{m_1}$$

Section P.3 Functions and Their Graphs

1. (a) $f(0) = 2(0) - 3 = -3$

(b) $f(-3) = 2(-3) - 3 = -9$

(c) $f(b) = 2b - 3$

(d) $f(x - 1) = 2(x - 1) - 3 = 2x - 5$

3. (a) $g(0) = 3 - 0^2 = 3$

(b) $g(\sqrt{3}) = 3 - (\sqrt{3})^2 = 3 - 3 = 0$

(c) $g(-2) = 3 - (-2)^2 = 3 - 4 = -1$

(d) $g(t - 1) = 3 - (t - 1)^2 = -t^2 + 2t + 2$

5. (a) $f(0) = \cos(2(0)) = \cos 0 = 1$

(c) $f\left(\frac{\pi}{3}\right) = \cos\left(2\left(\frac{\pi}{3}\right)\right) = \cos\frac{2\pi}{3} = -\frac{1}{2}$

(b) $f\left(-\frac{\pi}{4}\right) = \cos\left(2\left(-\frac{\pi}{4}\right)\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

7.
$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2, \Delta x \neq 0$$

9.
$$\begin{aligned} \frac{f(x) - f(2)}{x - 2} &= \frac{(1/\sqrt{x-1} - 1)}{x - 2} \\ &= \frac{1 - \sqrt{x-1}}{(x-2)\sqrt{x-1}} \cdot \frac{1 + \sqrt{x-1}}{1 + \sqrt{x-1}} = \frac{2 - x}{(x-2)\sqrt{x-1}(1 + \sqrt{x-1})} = \frac{-1}{\sqrt{x-1}(1 + \sqrt{x-1})}, x \neq 2 \end{aligned}$$

11. $h(x) = -\sqrt{x+3}$

Domain: $x + 3 \geq 0 \Rightarrow [-3, \infty)$

Range: $(-\infty, 0]$

13. $f(t) = \sec \frac{\pi t}{4}$

$$\frac{\pi t}{4} \neq \frac{(2k+1)\pi}{2} \Rightarrow t \neq 4k + 2$$

Domain: all $t \neq 4k + 2$, k an integer

Range: $(-\infty, -1], [1, \infty)$

15. $f(x) = \frac{1}{x}$

Domain: $(-\infty, 0), (0, \infty)$

Range: $(-\infty, 0), (0, \infty)$

17. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$

(a) $f(-1) = 2(-1) + 1 = -1$

(b) $f(0) = 2(0) + 2 = 2$

(c) $f(2) = 2(2) + 2 = 6$

(d) $f(t^2 + 1) = 2(t^2 + 1) + 2 = 2t^2 + 4$

(Note: $t^2 + 1 \geq 0$ for all t)

Domain: $(-\infty, \infty)$

Range: $(-\infty, 1), [2, \infty)$

19. $f(x) = \begin{cases} |x| + 1, & x < 1 \\ -x + 1, & x \geq 1 \end{cases}$

(a) $f(-3) = |-3| + 1 = 4$

(b) $f(1) = -1 + 1 = 0$

(c) $f(3) = -3 + 1 = -2$

(d) $f(b^2 + 1) = -(b^2 + 1) + 1 = -b^2$

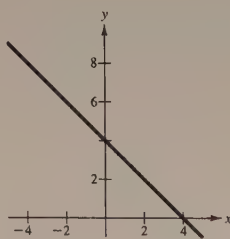
Domain: $(-\infty, \infty)$

Range: $(-\infty, 0] \cup [1, \infty)$

21. $f(x) = 4 - x$

Domain: $(-\infty, \infty)$

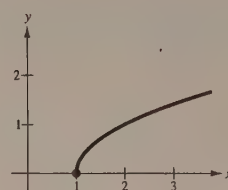
Range: $(-\infty, \infty)$



23. $h(x) = \sqrt{x - 1}$

Domain: $[1, \infty)$

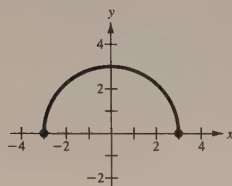
Range: $[0, \infty)$



25. $f(x) = \sqrt{9 - x^2}$

Domain: $[-3, 3]$

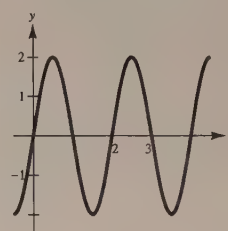
Range: $[0, 3]$



27. $g(t) = 2 \sin \pi t$

Domain: $(-\infty, \infty)$

Range: $[-2, 2]$



29. $x - y^2 = 0 \Rightarrow y = \pm \sqrt{x}$

y is not a function of x . Some vertical lines intersect the graph twice.

31. y is a function of x . Vertical lines intersect the graph at most once.

33. $x^2 + y^2 = 4 \Rightarrow y = \pm \sqrt{4 - x^2}$

y is not a function of x since there are two values of y for some x .

35. $y^2 = x^2 - 1 \Rightarrow y = \pm \sqrt{x^2 - 1}$

y is not a function of x since there are two values of y for some x .

37. $f(x) = |x| + |x - 2|$

If $x < 0$, then $f(x) = -x - (x - 2) = -2x + 2 = 2(1 - x)$.

If $0 \leq x < 2$, then $f(x) = x - (x - 2) = 2$.

If $x \geq 2$, then $f(x) = x + (x - 2) = 2x - 2 = 2(x - 1)$.

Thus,

$$f(x) = \begin{cases} 2(1 - x), & x < 0 \\ 2, & 0 \leq x < 2 \\ 2(x - 1), & x \geq 2. \end{cases}$$

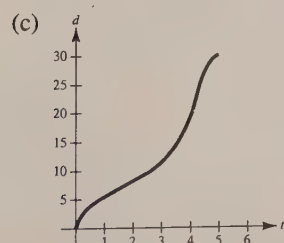
39. The function is $g(x) = cx^2$. Since $(1, -2)$ satisfies the equation, $c = -2$. Thus, $g(x) = -2x^2$.

41. The function is $r(x) = c/x$, since it must be undefined at $x = 0$. Since $(1, 32)$ satisfies the equation, $c = 32$. Thus, $r(x) = 32/x$.

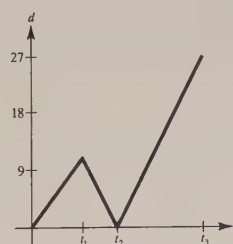
43. (a) For each time t , there corresponds a depth d .

(b) Domain: $0 \leq t \leq 5$

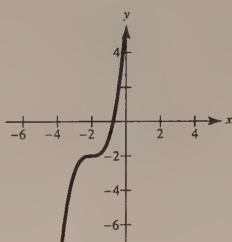
Range: $0 \leq d \leq 30$



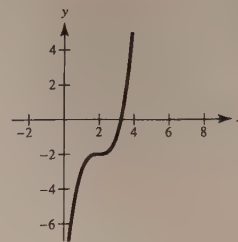
45.



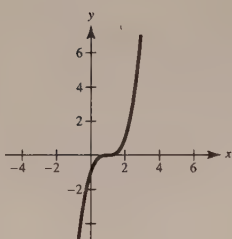
47. (a) The graph is shifted 3 units to the left.



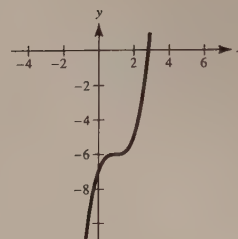
- (b) The graph is shifted 1 unit to the right.



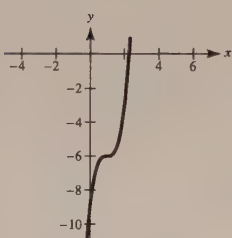
- (c) The graph is shifted 2 units upward.



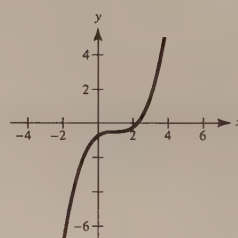
- (d) The graph is shifted 4 units downward.



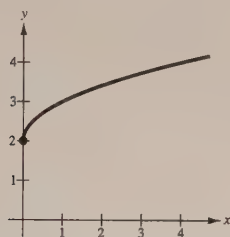
- (e) The graph is stretched vertically by a factor of 3.



- (f) The graph is stretched vertically by a factor of $\frac{1}{4}$.

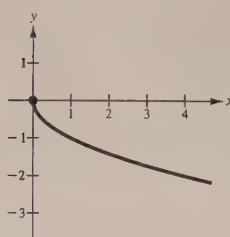


49. (a) $y = \sqrt{x} + 2$



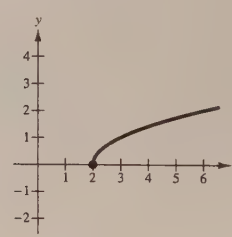
Vertical shift 2 units upward

(b) $y = -\sqrt{x}$



Reflection about the x -axis

(c) $y = \sqrt{x - 2}$



Horizontal shift 2 units to the right

51. (a) $T(4) = 16^\circ$, $T(15) \approx 23^\circ$

(b) If $H(t) = T(t - 1)$, then the program would turn on (and off) one hour later.

(c) If $H(t) = T(t) - 1$, then the overall temperature would be reduced 1 degree.

53. $f(x) = x^2$, $g(x) = \sqrt{x}$

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x, \quad x \geq 0$$

Domain: $[0, \infty)$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$$

Domain: $(-\infty, \infty)$

No. Their domains are different. $(f \circ g) = (g \circ f)$ for $x \geq 0$.

55. $f(x) = \frac{3}{x}$, $g(x) = x^2 - 1$

$$(f \circ g)(x) = f(g(x)) = f(x^2 - 1) = \frac{3}{x^2 - 1}$$

Domain: all $x \neq \pm 1$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{3}{x}\right) = \left(\frac{3}{x}\right)^2 - 1 = \frac{9}{x^2} - 1 = \frac{9 - x^2}{x^2}$$

Domain: all $x \neq 0$

No, $f \circ g \neq g \circ f$.

$$57. (A \circ r)(t) = A(r(t)) = A(0.6t) = \pi(0.6t)^2 = 0.36\pi t^2$$

$(A \circ r)(t)$ represents the area of the circle at time t .

$$61. f(-x) = (-x) \cos(-x) = -x \cos x = -f(x)$$

Odd

$$63. (a) \text{ If } f \text{ is even, then } \left(\frac{3}{2}, 4\right) \text{ is on the graph.}$$

$$\begin{aligned} 65. f(-x) &= a_{2n+1}(-x)^{2n+1} + \cdots + a_3(-x)^3 + a_1(-x) \\ &= -[a_{2n+1}x^{2n+1} + \cdots + a_3x^3 + a_1x] \\ &= -f(x) \end{aligned}$$

Odd

$$67. \text{ Let } F(x) = f(x)g(x) \text{ where } f \text{ and } g \text{ are even. Then}$$

$$F(-x) = f(-x)g(-x) = f(x)g(x) = F(x).$$

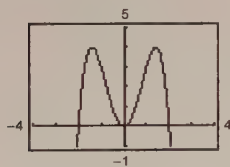
Thus, $F(x)$ is even. Let $F(x) = f(x)g(x)$ where f and g are odd. Then

$$F(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = F(x).$$

Thus, $F(x)$ is even.

$$69. f(x) = x^2 + 1 \text{ and } g(x) = x^4 \text{ are even.}$$

$$f(x)g(x) = (x^2 + 1)(x^4) = x^6 + x^4 \text{ is even.}$$



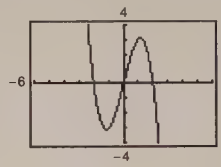
$$59. f(-x) = (-x)^2(4 - (-x)^2) = x^2(4 - x^2) = f(x)$$

Even

$$(b) \text{ If } f \text{ is odd, then } \left(\frac{3}{2}, -4\right) \text{ is on the graph.}$$

$$f(x) = x^3 - x \text{ is odd and } g(x) = x^2 \text{ is even.}$$

$$f(x)g(x) = (x^3 - x)(x^2) = x^5 - x^3 \text{ is odd.}$$



71. (a)

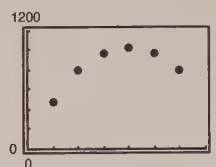
x	length and width	volume V
1	$24 - 2(1)$	484
2	$24 - 2(2)$	800
3	$24 - 2(3)$	972
4	$24 - 2(4)$	1024
5	$24 - 2(5)$	980
6	$24 - 2(6)$	864

The maximum volume appears to be 1024 cm^3 .

$$(c) V = x(24 - 2x)^2 = 4x(12 - x)^2$$

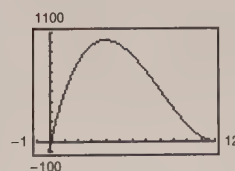
Domain: $0 < x < 12$

(b)



Yes, V is a function of x .

(d)



Maximum volume is $V = 1024 \text{ cm}^3$ for box having dimensions $4 \times 16 \times 16 \text{ cm}$.

$$73. \text{ False; let } f(x) = x^2.$$

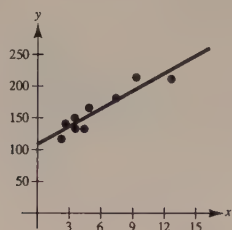
$$\text{Then } f(-3) = f(3) = 9, \text{ but } -3 \neq 3.$$

$$75. \text{ True, the function is even.}$$

Section P.4 Fitting Models to Data

1. Quadratic function

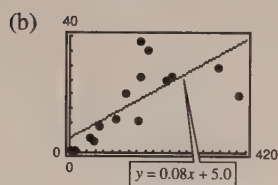
5. (a), (b)



Yes. The cancer mortality increases linearly with increased exposure to the carcinogenic substance.

(c) If $x = 3$, then $y \approx 136$.

9. (a) Let x = per capita energy usage (in millions of Btu)
 y = per capita gross national product (in thousands)
 $y = 0.0764x + 4.9985 \approx 0.08x + 5.0$
 $r = 0.7052$



(c) Denmark, Japan, and Canada

(d) Deleting the data for the three countries above,

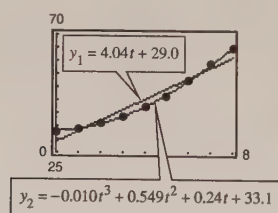
$$y = 0.0959x + 1.0539$$

($r = 0.9202$ is much closer to 1.)

13. (a) $y_1 = 4.0367t + 28.9644$

$$y_2 = -0.0099t^3 + 0.5488t^2 + 0.2399t + 33.1414$$

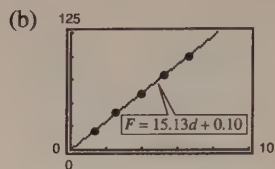
(b)



(c) The cubic model is better.

3. Linear function

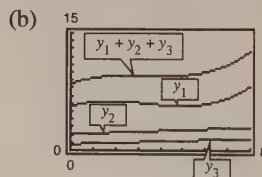
7. (a) $d = 0.066F$ or $F = 15.1d + 0.1$



The model fits well.

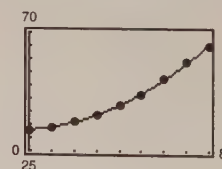
(c) If $F = 55$, then $d \approx 0.066(55) = 3.63$ cm.

11. (a) $y_1 = 0.0343t^3 - 0.3451t^2 + 0.8837t + 5.6061$
 $y_2 = 0.1095t + 2.0667$
 $y_3 = 0.0917t + 0.7917$



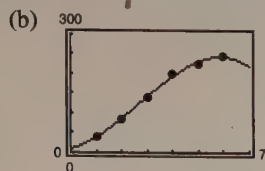
For 2002, $t = 12$ and $y_1 + y_2 + y_3 \approx 31.06$ cents/mile

- (d) $y_3 = 0.4297t^2 + 0.5994t + 32.9745$



- (e) The slope represents the average increase per year in the number of people (in millions) in HMOs.
 (f) For 2000, $t = 10$, and $y_1 \approx 69.3$ million. (linear)
 $y_2 \approx 80.5$ million (cubic)

15. (a) $y = -1.81x^3 + 14.58x^2 + 16.39x + 10$



(c) If $x = 4.5$, $y \approx 214$ horsepower.

17. (a) Yes, y is a function of t . At each time t , there is one and only one displacement y .

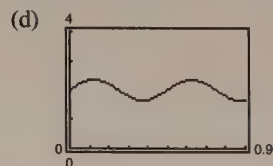
(b) The amplitude is approximately

$$(2.35 - 1.65)/2 = 0.35.$$

The period is approximately

$$2(0.375 - 0.125) = 0.5.$$

(c) One model is $y = 0.35 \sin(4\pi t) + 2$.



19. Answers will vary.

Review Exercises for Chapter P

1. $y = 2x - 3$

$$x = 0 \Rightarrow y = 2(0) - 3 = -3 \Rightarrow (0, -3) \quad \text{y-intercept}$$

$$y = 0 \Rightarrow 0 = 2x - 3 \Rightarrow x = \frac{3}{2} \Rightarrow \left(\frac{3}{2}, 0\right) \quad \text{x-intercept}$$

3. $y = \frac{x-1}{x-2}$

$$x = 0 \Rightarrow y = \frac{0-1}{0-2} = \frac{1}{2} \Rightarrow \left(0, \frac{1}{2}\right) \quad \text{y-intercept}$$

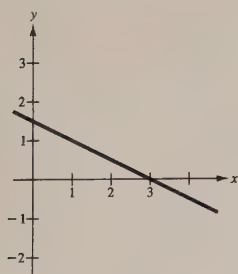
$$y = 0 \Rightarrow 0 = \frac{x-1}{x-2} \Rightarrow x = 1 \Rightarrow (1, 0) \quad \text{x-intercept}$$

5. Symmetric with respect to y-axis since

$$(-x)^2y - (-x)^2 + 4y = 0$$

$$x^2y - x^2 + 4y = 0.$$

7. $y = -\frac{1}{2}x + \frac{3}{2}$



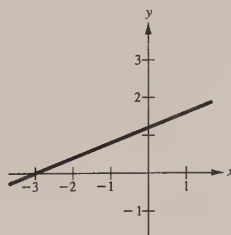
9. $-\frac{1}{3}x + \frac{5}{6}y = 1$

$$-\frac{2}{5}x + y = \frac{6}{5}$$

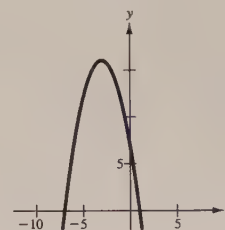
$$y = \frac{2}{5}x + \frac{6}{5}$$

Slope: $\frac{2}{5}$

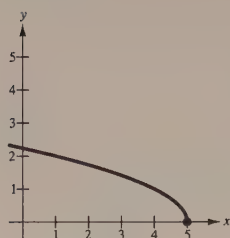
y-intercept: $\frac{6}{5}$



11. $y = 7 - 6x - x^2$



13. $y = \sqrt{5-x}$

Domain: $(-\infty, 5]$ 

15. $y = 4x^2 - 25$

Xmin = -5
Xmax = 5
Xscl = 1
Ymin = -30
Ymax = 10
Yscl = 5

17. $3x - 4y = 8$

$4x + 4y = 20$

$7x = 28$

$x = 4$

$y = 1$

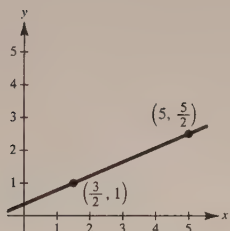
Point: (4, 1)

19. You need factors $(x + 2)$ and $(x - 2)$. Multiply by x to obtain origin symmetry

$y = x(x + 2)(x - 2)$

$= x^3 - 4x$

21.



Slope $= \frac{(5/2) - 1}{5 - (3/2)} = \frac{3/2}{7/2} = \frac{3}{7}$

23. $\frac{1-t}{1-0} = \frac{1-5}{1-(-2)}$

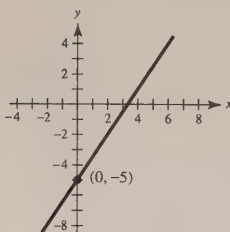
$1-t = -\frac{4}{3}$

$t = \frac{7}{3}$

25. $y - (-5) = \frac{3}{2}(x - 0)$

$y = \frac{3}{2}x - 5$

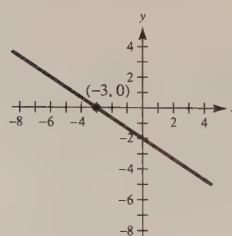
$2y - 3x + 10 = 0$



27. $y - 0 = -\frac{2}{3}(x - (-3))$

$y = -\frac{2}{3}x - 2$

$3y + 2x + 6 = 0$



29. (a) $y - 4 = \frac{7}{16}(x + 2)$

$$16y - 64 = 7x + 14$$

$$0 = 7x - 16y + 78$$

(c) $m = \frac{4 - 0}{-2 - 0} = -2$

$$y = -2x$$

$$2x + y = 0$$

(b) Slope of line is $\frac{5}{3}$.

$$y - 4 = \frac{5}{3}(x + 2)$$

$$3y - 12 = 5x + 10$$

$$0 = 5x - 3y + 22$$

(d) $x = -2$

$$x + 2 = 0$$

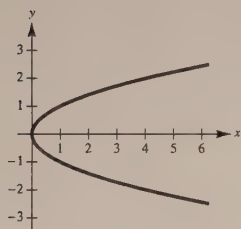
31. The slope is -850 . $V = -850t + 12,500$.

$$V(3) = -850(3) + 12,500 = \$9950$$

33. $x - y^2 = 0$

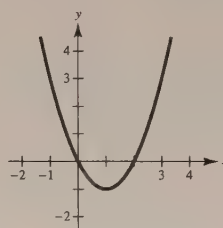
$$y = \pm\sqrt{x}$$

Not a function of x since there are two values of y for some x .



35. $y = x^2 - 2x$

Function of x since there is one value of y for each x .



37. $f(x) = \frac{1}{x}$

(a) $f(0)$ does not exist.

(b)
$$\frac{f(1 + \Delta x) - f(1)}{\Delta x} = \frac{\frac{1}{1 + \Delta x} - \frac{1}{1}}{\Delta x} = \frac{1 - 1 - \Delta x}{(1 + \Delta x)\Delta x}$$

$$= \frac{-1}{1 + \Delta x}, \Delta x \neq -1, 0$$

39. (a) Domain: $36 - x^2 \geq 0 \Rightarrow -6 \leq x \leq 6$ or $[-6, 6]$

Range: $[0, 6]$

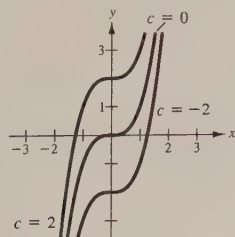
(b) Domain: all $x \neq 5$ or $(-\infty, 5), (5, \infty)$

Range: all $y \neq 0$ or $(-\infty, 0), (0, \infty)$

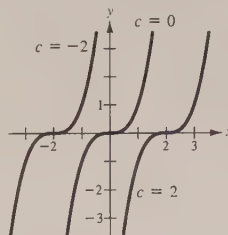
(c) Domain: all x or $(-\infty, \infty)$

Range: all y or $(-\infty, \infty)$

41. (a) $f(x) = x^3 + c, c = -2, 0, 2$

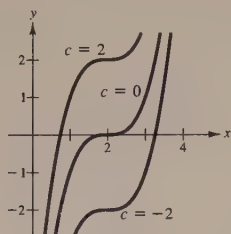


(b) $f(x) = (x - c)^3, c = -2, 0, 2$

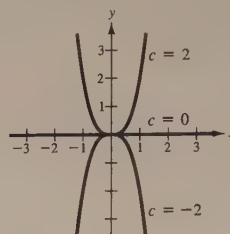


41. —CONTINUED—

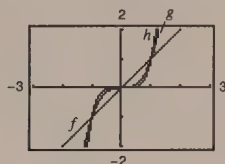
(c) $f(x) = (x - 2)^3 + c$, $c = -2, 0, 2$



(d) $f(x) = cx^3$, $c = -2, 0, 2$



43. (a) Odd powers: $f(x) = x$, $g(x) = x^3$, $h(x) = x^5$

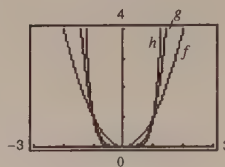


The graphs of f , g , and h all rise to the right and fall to the left. As the degree increases, the graph rises and falls more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

(b) $y = x^7$ will look like $h(x) = x^5$, but rise and fall even more steeply.

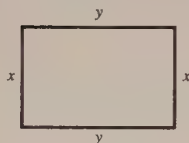
$y = x^8$ will look like $h(x) = x^6$, but rise even more steeply.

Even powers: $f(x) = x^2$, $g(x) = x^4$, $h(x) = x^6$



The graphs of f , g , and h all rise to the left and to the right. As the degree increases, the graph rises more steeply. All three graphs pass through the points $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

45. (a)

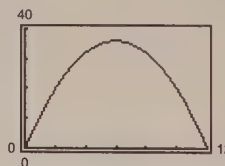


$2x + 2y = 24$

$y = 12 - x$

$A = xy = x(12 - x) = 12x - x^2$

(b) Domain: $0 < x < 12$



(c) Maximum area is $A = 36$. In general, the maximum area is attained when the rectangle is a square. In this case, $x = 6$.

47. (a) 3 (cubic), negative leading coefficient

(b) 4 (quartic), positive leading coefficient

(c) 2 (quadratic), negative leading coefficient

(d) 5, positive leading coefficient

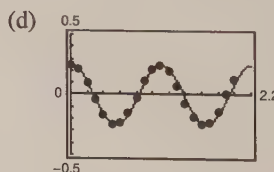
49. (a) Yes, y is a function of t . At each time t , there is one and only one displacement y .

(b) The amplitude is approximately

$(0.25 - (-0.25))/2 = 0.25.$

The period is approximately 1.1.

(c) One model is $y = \frac{1}{4} \cos\left(\frac{2\pi}{1.1}t\right) \approx \frac{1}{4} \cos(5.7t)$



Problem Solving for Chapter P

1. (a) $x^2 - 6x + y^2 - 8y = 0$

$$(x^2 - 6x + 9) + (y^2 - 8y + 16) = 9 + 16$$

$$(x - 3)^2 + (y - 4)^2 = 25$$

Center: (3, 4) Radius: 5

(c) Slope of line from (6, 0) to (3, 4) is $\frac{4 - 0}{3 - 6} = -\frac{4}{3}$.

Slope of tangent line is $\frac{3}{4}$. Hence,

$$y - 0 = \frac{3}{4}(x - 6) \Rightarrow y = \frac{3}{4}x - \frac{9}{2} \quad \text{Tangent line}$$

(b) Slope of line from (0, 0) to (3, 4) is $\frac{4}{3}$. Slope of tangent line is $-\frac{3}{4}$. Hence,

$$y - 0 = -\frac{3}{4}(x - 0) \Rightarrow y = -\frac{3}{4}x \quad \text{Tangent line}$$

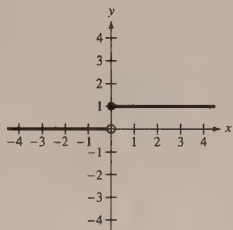
(d) $-\frac{3}{4}x = \frac{3}{4}x - \frac{9}{2}$

$$\frac{3}{2}x = \frac{9}{2}$$

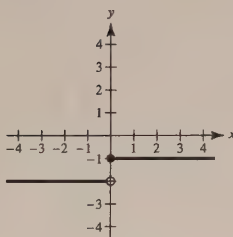
$$x = 3$$

Intersection: $\left(3, -\frac{9}{4}\right)$

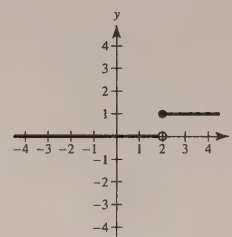
3. $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$



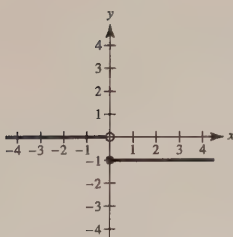
(a) $H(x) - 2$



(b) $H(x - 2)$



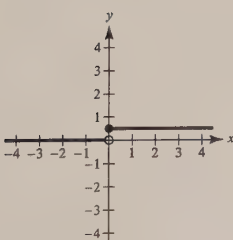
(c) $-H(x)$



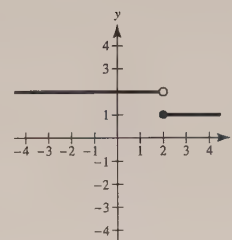
(d) $H(-x)$



(e) $\frac{1}{2}H(x)$



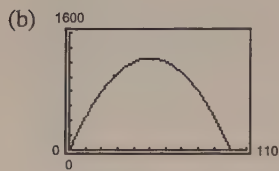
(f) $-H(x - 2) + 2$



5. (a) $x + 2y = 100 \Rightarrow y = \frac{100 - x}{2}$

$$A(x) = xy = x\left(\frac{100 - x}{2}\right) = -\frac{x^2}{2} + 50x$$

Domain: $0 < x < 100$



Maximum of 1250 m^2 at $x = 50 \text{ m}$, $y = 25 \text{ m}$.

$$\begin{aligned} \text{(c)} \quad A(x) &= -\frac{1}{2}(x^2 - 100x) \\ &= -\frac{1}{2}(x^2 - 100x + 2500) + 1250 \\ &= -\frac{1}{2}(x - 50)^2 + 1250 \end{aligned}$$

$A(50) = 1250 \text{ m}^2$ is the maximum. $x = 50 \text{ m}$, $y = 25 \text{ m}$.

9. (a) Slope $= \frac{9 - 4}{3 - 2} = 5$. Slope of tangent line is less than 5.

(b) Slope $= \frac{4 - 1}{2 - 1} = 3$. Slope of tangent line is greater than 3.

(c) Slope $= \frac{4.41 - 4}{2.1 - 2} = 4.1$. Slope of tangent line is less than 4.1.

$$\begin{aligned} \text{(d)} \quad \text{Slope} &= \frac{f(2 + h) - f(2)}{(2 + h) - 2} \\ &= \frac{(2 + h)^2 - 4}{h} \\ &= \frac{4h + h^2}{h} \\ &= 4 + h, h \neq 0 \end{aligned}$$

(e) Letting h get closer and closer to 0, the slope approaches 4. Hence, the slope at $(2, 4)$ is 4.

11. (a) $\frac{I}{x^2} = \frac{2I}{(x - 3)^2}$

$$x^2 - 6x + 9 = 2x^2$$

$$x^2 + 6x - 9 = 0$$

$$x = \frac{-6 \pm \sqrt{36 + 36}}{2} = -3 \pm \sqrt{18} \approx 1.2426, -7.2426$$



(b) $\frac{I}{x^2 + y^2} = \frac{2I}{(x - 3)^2 + y^2}$

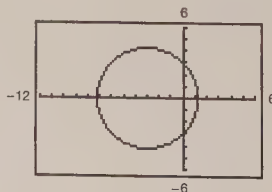
$$(x - 3)^2 + y^2 = 2(x^2 + y^2)$$

$$x^2 - 6x + 9 + y^2 = 2x^2 + 2y^2$$

$$x^2 + y^2 + 6x - 9 = 0$$

$$(x + 3)^2 + y^2 = 18$$

Circle of radius $\sqrt{18}$ and center $(-3, 0)$.



7. The length of the trip in the water is $\sqrt{2^2 + x^2}$, and the length of the trip over land is $\sqrt{1 + (3 - x)^2}$. Hence, the total time is

$$T = \frac{\sqrt{4 + x^2}}{2} + \frac{\sqrt{1 + (3 - x)^2}}{4} \text{ hours.}$$

13.

$$d_1 d_2 = 1$$

$$[(x+1)^2 + y^2][(x-1)^2 + y^2] = 1$$

$$(x+1)^2(x-1)^2 + y^2[(x+1)^2 + (x-1)^2] + y^4 = 1$$

$$(x^2 - 1)^2 + y^2[2x^2 + 2] + y^4 = 1$$

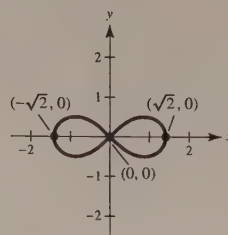
$$x^4 - 2x^2 + 1 + 2x^2y^2 + 2y^2 + y^4 = 1$$

$$(x^4 + 2x^2y^2 + y^4) - 2x^2 + 2y^2 = 0$$

$$(x^2 + y^2)^2 = 2(x^2 - y^2)$$

Let $y = 0$. Then $x^4 = 2x^2 \Rightarrow x = 0$ or $x^2 = 2$.

Thus, $(0, 0)$, $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ are on the curve.



CHAPTER 1

Limits and Their Properties

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CHAPTER 1

Limits and Their Properties

Section 1.1 A Preview of Calculus

Solutions to Odd-Numbered Exercises

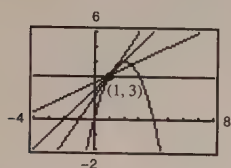
1. Precalculus: $(20 \text{ ft/sec})(15 \text{ seconds}) = 300 \text{ feet}$

3. Calculus required: slope of tangent line at $x = 2$ is rate of change, and equals about 0.16.

5. Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(3) = \frac{15}{2} \text{ sq. units}$

7. Precalculus: Volume $= (2)(4)(3) = 24 \text{ cubic units}$

9. (a)



(b) The graphs of y_2 are approximations to the tangent line to y_1 at $x = 1$.

(c) The slope is approximately 2. For a better approximation make the list numbers smaller:
 $\{0.2, 0.1, 0.01, 0.001\}$

11. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$
 $\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

Section 1.2 Finding Limits Graphically and Numerically

1.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.3448	0.3344	0.3334	0.3332	0.3322	0.3226

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2-x-2} \approx 0.3333 \quad (\text{Actual limit is } \frac{1}{3}.)$$

3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.2911	0.2889	0.2887	0.2887	0.2884	0.2863

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+3} - \sqrt{3}}{x} \approx 0.2887 \quad (\text{Actual limit is } 1/(2\sqrt{3}).)$$

5.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	-0.0641	-0.0627	-0.0625	-0.0625	-0.0623	-0.0610

$$\lim_{x \rightarrow 3} \frac{[1/(x+1)] - (1/4)}{x-3} \approx -0.0625 \quad (\text{Actual limit is } -\frac{1}{16}.)$$

7.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \approx 1.0000 \quad (\text{Actual limit is } 1.) \quad (\text{Make sure you use radian mode.})$$

9. $\lim_{x \rightarrow 3} (4 - x) = 1$

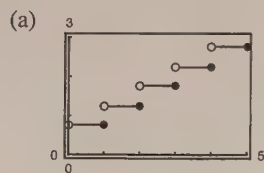
11. $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (4 - x) = 2$

13. $\lim_{x \rightarrow 5} \frac{|x-5|}{x-5}$ does not exist. For values of x to the left of 5, $|x-5|/(x-5)$ equals -1 , whereas for values of x to the right of 5, $|x-5|/(x-5)$ equals 1 .

15. $\lim_{x \rightarrow \pi/2} \tan x$ does not exist since the function increases and decreases without bound as x approaches $\pi/2$.

17. $\lim_{x \rightarrow 0} \cos(1/x)$ does not exist since the function oscillates between -1 and 1 as x approaches 0 .

19. $C(t) = 0.75 - 0.50\lceil -(t-1) \rceil$



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	1.75	2.25	2.25	2.25	2.25	2.25	2.25

$$\lim_{t \rightarrow 3.5} C(t) = 2.25$$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	1.25	1.75	1.75	1.75	2.25	2.25	2.25

$$\lim_{t \rightarrow 3} C(t) \text{ does not exist. The values of } C \text{ jump from } 1.75 \text{ to } 2.25 \text{ at } t = 3.$$

21. You need to find δ such that $0 < |x-1| < \delta$ implies

$$|f(x) - 1| = \left| \frac{1}{x} - 1 \right| < 0.1. \text{ That is,}$$

$$-0.1 < \frac{1}{x} - 1 < 0.1$$

$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$

$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$

$$\frac{10}{9} > x > \frac{10}{11}$$

$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$

$$\frac{1}{9} > x - 1 > -\frac{1}{11}.$$

So take $\delta = \frac{1}{11}$. Then $0 < |x-1| < \delta$ implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$

$$-\frac{1}{11} < x - 1 < \frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 1| = \left| \frac{1}{x} - 1 \right| < 0.1.$$

$$23. \lim_{x \rightarrow 2} (3x + 2) = 8 = L$$

$$|(3x + 2) - 8| < 0.01$$

$$|3x - 6| < 0.01$$

$$3|x - 2| < 0.01$$

$$0 < |x - 2| < \frac{0.01}{3} \approx 0.0033 = \delta$$

Hence, if $0 < |x - 2| < \delta = \frac{0.01}{3}$, you have

$$3|x - 2| < 0.01$$

$$|3x - 6| < 0.01$$

$$|(3x + 2) - 8| < 0.01$$

$$|f(x) - L| < 0.01$$

$$27. \lim_{x \rightarrow 2} (x + 3) = 5$$

Given $\epsilon > 0$:

$$|(x + 3) - 5| < \epsilon$$

$$|x - 2| < \epsilon = \delta$$

Hence, let $\delta = \epsilon$.

Hence, if $0 < |x - 2| < \delta = \epsilon$, you have

$$|x - 2| < \epsilon$$

$$|(x + 3) - 5| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$31. \lim_{x \rightarrow 6} 3 = 3$$

Given $\epsilon > 0$:

$$|3 - 3| < \epsilon$$

$$0 < \epsilon$$

Hence, any $\delta > 0$ will work.

Hence, for any $\delta > 0$, you have

$$|3 - 3| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$25. \lim_{x \rightarrow 2} (x^2 - 3) = 1 = L$$

$$|(x^2 - 3) - 1| < 0.01$$

$$|x^2 - 4| < 0.01$$

$$|(x + 2)(x - 2)| < 0.01$$

$$|x + 2| |x - 2| < 0.01$$

$$|x - 2| < \frac{0.01}{|x + 2|}$$

If we assume $1 < x < 3$, then $\delta = 0.01/5 = 0.002$.

Hence, if $0 < |x - 2| < \delta = 0.002$, you have

$$|x - 2| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{|x + 2|}(0.01)$$

$$|x + 2| |x - 2| < 0.01$$

$$|x^2 - 4| < 0.01$$

$$|(x^2 - 3) - 1| < 0.01$$

$$|f(x) - L| < 0.01$$

$$29. \lim_{x \rightarrow -4} \left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$$

Given $\epsilon > 0$:

$$\left|\left(\frac{1}{2}x - 1\right) - (-3)\right| < \epsilon$$

$$\left|\frac{1}{2}x + 2\right| < \epsilon$$

$$\frac{1}{2}|x - (-4)| < \epsilon$$

$$|x - (-4)| < 2\epsilon$$

Hence, let $\delta = 2\epsilon$.

Hence, if $0 < |x - (-4)| < \delta = 2\epsilon$, you have

$$|x - (-4)| < 2\epsilon$$

$$\left|\frac{1}{2}x + 2\right| < \epsilon$$

$$\left|\left(\frac{1}{2}x - 1\right) + 3\right| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$33. \lim_{x \rightarrow 0} \sqrt[3]{x} = 0$$

$$\text{Given } \epsilon > 0: \left|\sqrt[3]{x} - 0\right| < \epsilon$$

$$\left|\sqrt[3]{x}\right| < \epsilon$$

$$|x| < \epsilon^3 = \delta$$

Hence, let $\delta = \epsilon^3$.

Hence for $0 < |x - 0| < \delta = \epsilon^3$, you have

$$|x| < \epsilon^3$$

$$\left|\sqrt[3]{x}\right| < \epsilon$$

$$\left|\sqrt[3]{x} - 0\right| < \epsilon$$

$$|f(x) - L| < \epsilon$$

$$35. \lim_{x \rightarrow -2} |x - 2| = |(-2) - 2| = 4$$

Given $\epsilon > 0$:

$$||x - 2| - 4| < \epsilon$$

$$|-(x - 2) - 4| < \epsilon \quad (x - 2 < 0)$$

$$|-x - 2| = |x + 2| = |x - (-2)| < \epsilon$$

Hence, $\delta = \epsilon$.

Hence for $0 < |x - (-2)| < \delta = \epsilon$, you have

$$|x + 2| < \epsilon$$

$$|-(x + 2)| < \epsilon$$

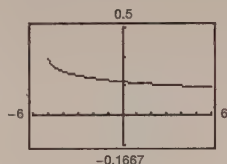
$$|-(x - 2) - 4| < \epsilon$$

$$||x - 2| - 4| < \epsilon \quad (\text{because } x - 2 < 0)$$

$$|f(x) - L| < \epsilon$$

$$39. f(x) = \frac{\sqrt{x+5} - 3}{x-4}$$

$$\lim_{x \rightarrow 4} f(x) = \frac{1}{6}$$



The domain is $[-5, 4) \cup (4, \infty)$.

The graphing utility does not show the hole at $(4, \frac{1}{6})$.

$$37. \lim_{x \rightarrow 1} (x^2 + 1) = 2$$

Given $\epsilon > 0$:

$$|(x^2 + 1) - 2| < \epsilon$$

$$|x^2 - 1| < \epsilon$$

$$|(x + 1)(x - 1)| < \epsilon$$

$$|x - 1| < \frac{\epsilon}{|x + 1|}$$

If we assume $0 < x < 2$, then $\delta = \epsilon/3$.

Hence for $0 < |x - 1| < \delta = \frac{\epsilon}{3}$, you have

$$|x - 1| < \frac{1}{3}\epsilon < \frac{1}{|x + 1|}\epsilon$$

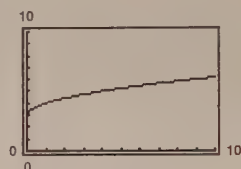
$$|x^2 - 1| < \epsilon$$

$$|(x^2 + 1) - 2| < \epsilon$$

$$|f(x) - 2| < \epsilon$$

$$41. f(x) = \frac{x-9}{\sqrt{x}-3}$$

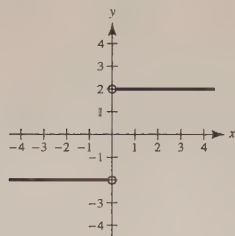
$$\lim_{x \rightarrow 9} f(x) = 6$$



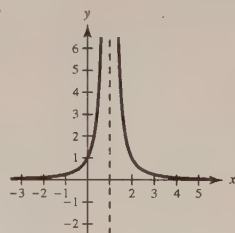
The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

43. $\lim_{x \rightarrow 8} f(x) = 25$ means that the values of f approach 25 as x gets closer and closer to 8.

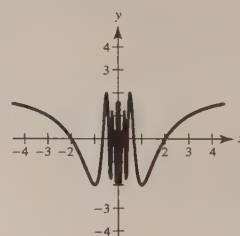
45. (i) The values of f approach different numbers as x approaches c from different sides of c :



(ii) The values of f increase without bound as x approaches c :

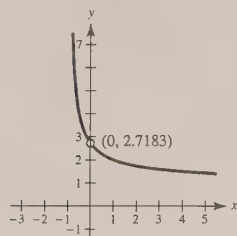


(iii) The values of f oscillate between two fixed numbers as x approaches c :



$$47. f(x) = (1 + x)^{1/x}$$

$$\lim_{x \rightarrow 0} (1 + x)^{1/x} = e \approx 2.71828$$



x	$f(x)$	x	$f(x)$
-0.1	2.867972	0.1	2.593742
-0.01	2.731999	0.01	2.704814
-0.001	2.719642	0.001	2.716942
-0.0001	2.718418	0.0001	2.718146
-0.00001	2.718295	0.00001	2.718268
-0.000001	2.718283	0.000001	2.718280

49. False; $f(x) = (\sin x)/x$ is undefined when $x = 0$.
From Exercise 7, we have

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

51. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}$$

$$f(4) = 10$$

$$\lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 4} (x^2 - 4x) = 0 \neq 10$$

53. Answers will vary.

55. If $\lim_{x \rightarrow c} f(x) = L_1$ and $\lim_{x \rightarrow c} f(x) = L_2$, then for every $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \epsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \epsilon$. Let δ equal the smaller of δ_1 and δ_2 . Then for $|x - c| < \delta$, we have
- $$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \epsilon + \epsilon.$$

Therefore, $|L_1 - L_2| < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

57. $\lim_{x \rightarrow c} [f(x) - L] = 0$ means that for every $\epsilon > 0$ there exists $\delta > 0$ such that if

$$0 < |x - c| < \delta,$$

then

$$|(f(x) - L) - 0| < \epsilon.$$

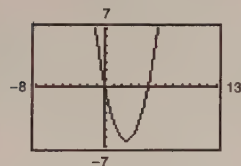
This means the same as $|f(x) - L| < \epsilon$ when

$$0 < |x - c| < \delta.$$

Thus, $\lim_{x \rightarrow c} f(x) = L$.

Section 1.3 Evaluating Limits Analytically

1.

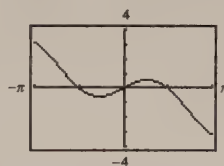


$$h(x) = x^2 - 5x$$

$$(a) \lim_{x \rightarrow 5} h(x) = 0$$

$$(b) \lim_{x \rightarrow -1} h(x) = 6$$

3.



$$f(x) = x \cos x$$

$$(a) \lim_{x \rightarrow 0} f(x) = 0$$

$$(b) \lim_{x \rightarrow \pi/3} f(x) \approx 0.524$$

$$\left(= \frac{\pi}{6} \right)$$

$$5. \lim_{x \rightarrow 2} x^4 = 2^4 = 16$$

$$7. \lim_{x \rightarrow 0} (2x - 1) = 2(0) - 1 = -1$$

$$9. \lim_{x \rightarrow -3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$$

$$11. \lim_{x \rightarrow -3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 = 18 - 12 + 1 = 7$$

$$13. \lim_{x \rightarrow 2} \frac{1}{x} = \frac{1}{2}$$

$$15. \lim_{x \rightarrow 1} \frac{x - 3}{x^2 + 4} = \frac{1 - 3}{1^2 + 4} = \frac{-2}{5} = -\frac{2}{5}$$

$$17. \lim_{x \rightarrow 7} \frac{5x}{\sqrt{x} + 2} = \frac{5(7)}{\sqrt{7} + 2} = \frac{35}{\sqrt{9}} = \frac{35}{3}$$

$$19. \lim_{x \rightarrow 3} \sqrt{x + 1} = \sqrt{3 + 1} = 2$$

$$21. \lim_{x \rightarrow -4} (x + 3)^2 = (-4 + 3)^2 = 1$$

$$25. (a) \lim_{x \rightarrow 1} f(x) = 4 - 1 = 3$$

$$(b) \lim_{x \rightarrow 3} g(x) = \sqrt{3 + 1} = 2$$

$$(c) \lim_{x \rightarrow 1} g(f(x)) = g(3) = 2$$

$$29. \lim_{x \rightarrow 2} \cos \frac{\pi x}{3} = \cos \frac{\pi 2}{3} = -\frac{1}{2}$$

$$33. \lim_{x \rightarrow 5\pi/6} \sin x = \sin \frac{5\pi}{6} = \frac{1}{2}$$

$$37. (a) \lim_{x \rightarrow c} [5g(x)] = 5 \lim_{x \rightarrow c} g(x) = 5(3) = 15$$

$$(b) \lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) = 2 + 3 = 5$$

$$(c) \lim_{x \rightarrow c} [f(x)g(x)] = \left[\lim_{x \rightarrow c} f(x) \right] \left[\lim_{x \rightarrow c} g(x) \right] = (2)(3) = 6$$

$$(d) \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \frac{2}{3}$$

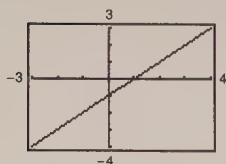
$$41. f(x) = -2x + 1 \text{ and } g(x) = \frac{-2x^2 + x}{x} \text{ agree except at } x = 0.$$

$$(a) \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} f(x) = 1$$

$$(b) \lim_{x \rightarrow -1} g(x) = \lim_{x \rightarrow -1} f(x) = 3$$

$$45. f(x) = \frac{x^2 - 1}{x + 1} \text{ and } g(x) = x - 1 \text{ agree except at } x = -1.$$

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} g(x) = -2$$



$$\begin{aligned} 49. \lim_{x \rightarrow 5} \frac{x - 5}{x^2 - 25} &= \lim_{x \rightarrow 5} \frac{x - 5}{(x + 5)(x - 5)} \\ &= \lim_{x \rightarrow 5} \frac{1}{x + 5} = \frac{1}{10} \end{aligned}$$

$$23. (a) \lim_{x \rightarrow 1} f(x) = 5 - 1 = 4$$

$$(b) \lim_{x \rightarrow 4} g(x) = 4^3 = 64$$

$$(c) \lim_{x \rightarrow 1} g(f(x)) = g(f(1)) = g(4) = 64$$

$$27. \lim_{x \rightarrow \pi/2} \sin x = \sin \frac{\pi}{2} = 1$$

$$31. \lim_{x \rightarrow 0} \sec 2x = \sec 0 = 1$$

$$35. \lim_{x \rightarrow 3} \tan\left(\frac{\pi x}{4}\right) = \tan \frac{3\pi}{4} = -1$$

$$39. (a) \lim_{x \rightarrow c} [f(x)]^3 = \left[\lim_{x \rightarrow c} f(x) \right]^3 = (4)^3 = 64$$

$$(b) \lim_{x \rightarrow c} \sqrt{f(x)} = \sqrt{\lim_{x \rightarrow c} f(x)} = \sqrt{4} = 2$$

$$(c) \lim_{x \rightarrow c} [3f(x)] = 3 \lim_{x \rightarrow c} f(x) = 3(4) = 12$$

$$(d) \lim_{x \rightarrow c} [f(x)]^{3/2} = \left[\lim_{x \rightarrow c} f(x) \right]^{3/2} = (4)^{3/2} = 8$$

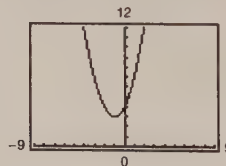
$$43. f(x) = x(x + 1) \text{ and } g(x) = \frac{x^3 - x}{x - 1} \text{ agree except at } x = 1.$$

$$(a) \lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} f(x) = 2$$

$$(b) \lim_{x \rightarrow -1} g(x) = \lim_{x \rightarrow -1} f(x) = 0$$

$$47. f(x) = \frac{x^3 - 8}{x - 2} \text{ and } g(x) = x^2 + 2x + 4 \text{ agree except at } x = 2.$$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} g(x) = 12$$



$$\begin{aligned} 51. \lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x^2 - 9} &= \lim_{x \rightarrow -3} \frac{(x + 3)(x - 2)}{(x + 3)(x - 3)} \\ &= \lim_{x \rightarrow -3} \frac{x - 2}{x - 3} = \frac{-5}{-6} = \frac{5}{6} \end{aligned}$$

$$\begin{aligned}
 53. \lim_{x \rightarrow 0} \frac{\sqrt{x+5} - \sqrt{5}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+5} - \sqrt{5}}{x} \cdot \frac{\sqrt{x+5} + \sqrt{5}}{\sqrt{x+5} + \sqrt{5}} \\
 &= \lim_{x \rightarrow 0} \frac{(x+5) - 5}{x(\sqrt{x+5} + \sqrt{5})} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+5} + \sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}
 \end{aligned}$$

$$\begin{aligned}
 55. \lim_{x \rightarrow 4} \frac{\sqrt{x+5} - 3}{x-4} &= \lim_{x \rightarrow 4} \frac{\sqrt{x+5} - 3}{x-4} \cdot \frac{\sqrt{x+5} + 3}{\sqrt{x+5} + 3} \\
 &= \lim_{x \rightarrow 4} \frac{(x+5) - 9}{(x-4)(\sqrt{x+5} + 3)} = \lim_{x \rightarrow 4} \frac{1}{\sqrt{x+5} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}
 \end{aligned}$$

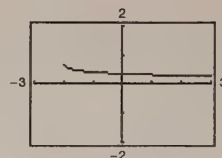
$$57. \lim_{x \rightarrow 0} \frac{\frac{1}{3+x} - \frac{1}{3}}{x} = \lim_{x \rightarrow 0} \frac{\frac{3 - (3+x)}{3(3+x)}}{x} = \lim_{x \rightarrow 0} \frac{-1}{3(3+x)} = -\frac{1}{9}$$

$$59. \lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x) - 2x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x + 2\Delta x - 2x}{\Delta x} = \lim_{\Delta x \rightarrow 0} 2 = 2$$

$$\begin{aligned}
 61. \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2) = 2x - 2
 \end{aligned}$$

$$63. \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \approx 0.354$$

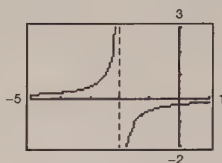
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.345	?	0.354	0.353	0.349



$$\begin{aligned}
 \text{Analytically, } \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \cdot \frac{\sqrt{x+2} + \sqrt{2}}{\sqrt{x+2} + \sqrt{2}} \\
 &= \lim_{x \rightarrow 0} \frac{x+2-2}{x(\sqrt{x+2} + \sqrt{2})} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354
 \end{aligned}$$

$$65. \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = -\frac{1}{4}$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.263	-0.251	-0.250	?	-0.250	-0.249	-0.238



$$\text{Analytically, } \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \lim_{x \rightarrow 0} \frac{2 - (2+x)}{2(2+x)} \cdot \frac{1}{x} = \lim_{x \rightarrow 0} \frac{-x}{2(2+x)} \cdot \frac{1}{x} = \lim_{x \rightarrow 0} \frac{-1}{2(2+x)} = -\frac{1}{4}$$

$$67. \lim_{x \rightarrow 0} \frac{\sin x}{5x} = \lim_{x \rightarrow 0} \left[\left(\frac{\sin x}{x} \right) \left(\frac{1}{5} \right) \right] = (1) \left(\frac{1}{5} \right) = \frac{1}{5}$$

$$69. \lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)}{2x^2} = \lim_{x \rightarrow 0} \left[\frac{1}{2} \cdot \frac{\sin x}{x} \cdot \frac{1 - \cos x}{x} \right] \\ = \frac{1}{2}(1)(0) = 0$$

$$71. \lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \sin x \right] = (1) \sin 0 = 0$$

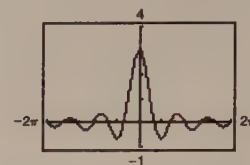
$$73. \lim_{h \rightarrow 0} \frac{(1 - \cos h)^2}{h} = \lim_{h \rightarrow 0} \left[\frac{1 - \cos h}{h} (1 - \cos h) \right] \\ = (0)(0) = 0$$

$$75. \lim_{x \rightarrow \pi/2} \frac{\cos x}{\cot x} = \lim_{x \rightarrow \pi/2} \sin x = 1$$

$$77. \lim_{t \rightarrow 0} \frac{\sin 3t}{2t} = \lim_{t \rightarrow 0} \left(\frac{\sin 3t}{3t} \right) \left(\frac{3}{2} \right) = (1) \left(\frac{3}{2} \right) = \frac{3}{2}$$

$$79. f(t) = \frac{\sin 3t}{t}$$

t	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

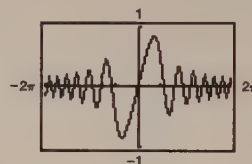


The limit appears to equal 3.

$$\text{Analytically, } \lim_{t \rightarrow 0} \frac{\sin 3t}{t} = \lim_{t \rightarrow 0} 3 \left(\frac{\sin 3t}{3t} \right) = 3(1) = 3.$$

$$81. f(x) = \frac{\sin x^2}{x}$$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.099998	-0.01	-0.001	?	0.001	0.01	0.099998



$$\text{Analytically, } \lim_{x \rightarrow 0} \frac{\sin x^2}{x} = \lim_{x \rightarrow 0} x \left(\frac{\sin x^2}{x^2} \right) = 0(1) = 0.$$

$$83. \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x) + 3 - (2x + 3)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x + 2\Delta x + 3 - 2x - 3}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2\Delta x}{\Delta x} = 2$$

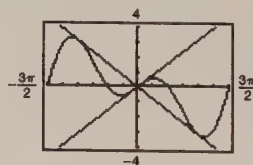
$$85. \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{4}{x + \Delta x} - \frac{4}{x}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{4x - 4(x + \Delta x)}{(x + \Delta x)x\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-4}{(x + \Delta x)x} = \frac{-4}{x^2}$$

$$87. \lim_{x \rightarrow 0} (4 - x^2) \leq \lim_{x \rightarrow 0} f(x) \leq \lim_{x \rightarrow 0} (4 + x^2)$$

$$4 \leq \lim_{x \rightarrow 0} f(x) \leq 4$$

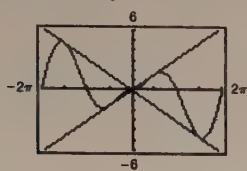
$$\text{Therefore, } \lim_{x \rightarrow 0} f(x) = 4.$$

$$89. f(x) = x \cos x$$



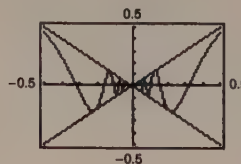
$$\lim_{x \rightarrow 0} (x \cos x) = 0$$

91. $f(x) = |x| \sin x$



$$\lim_{x \rightarrow 0} |x| \sin x = 0$$

93. $f(x) = x \sin \frac{1}{x}$



$$\lim_{x \rightarrow 0} \left(x \sin \frac{1}{x} \right) = 0$$

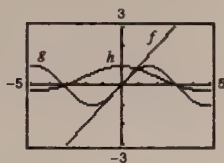
95. We say that two functions f and g agree at all but one point (on an open interval) if $f(x) = g(x)$ for all x in the interval except for $x = c$, where c is in the interval.

97. An indeterminate form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$$

$$\text{for which } \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$$

99. $f(x) = x, g(x) = \sin x, h(x) = \frac{\sin x}{x}$



When you are “close to” 0 the magnitude of f is approximately equal to the magnitude of g .
Thus, $|g|/|f| \approx 1$ when x is “close to” 0.

101. $s(t) = -16t^2 + 1000$

$$\lim_{t \rightarrow 5} \frac{s(5) - s(t)}{5 - t} = \lim_{t \rightarrow 5} \frac{600 - (-16t^2 + 1000)}{5 - t} = \lim_{t \rightarrow 5} \frac{16(t + 5)(t - 5)}{-(t - 5)} = \lim_{t \rightarrow 5} -16(t + 5) = -160 \text{ ft/sec.}$$

Speed = 160 ft/sec

103. $s(t) = -4.9t^2 + 150$

$$\begin{aligned} \lim_{t \rightarrow 3} \frac{s(3) - s(t)}{3 - t} &= \lim_{t \rightarrow 3} \frac{-4.9(3^2) + 150 - (-4.9t^2 + 150)}{3 - t} = \lim_{t \rightarrow 3} \frac{-4.9(9 - t^2)}{3 - t} \\ &= \lim_{t \rightarrow 3} \frac{-4.9(3 - t)(3 + t)}{3 - t} = \lim_{t \rightarrow 3} -4.9(3 + t) = -29.4 \text{ m/sec} \end{aligned}$$

105. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ do not exist.

$$\lim_{x \rightarrow 0} [f(x) + g(x)] = \lim_{x \rightarrow 0} \left[\frac{1}{x} + \left(-\frac{1}{x} \right) \right] = \lim_{x \rightarrow 0} [0] = 0$$

107. Given $f(x) = b$, show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \epsilon$ whenever $|x - c| < \delta$. Since $|f(x) - b| = |b - b| = 0 < \epsilon$ for any $\epsilon > 0$, then any value of $\delta > 0$ will work.

109. If $b = 0$, then the property is true because both sides are equal to 0. If $b \neq 0$, let $\epsilon > 0$ be given. Since $\lim_{x \rightarrow c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon/|b|$ whenever $0 < |x - c| < \delta$. Hence, wherever $0 < |x - c| < \delta$, we have

$$|b||f(x) - L| < \epsilon \quad \text{or} \quad |bf(x) - bL| < \epsilon$$

which implies that $\lim_{x \rightarrow c} [bf(x)] = bL$.

111. $-M|f(x)| \leq f(x)g(x) \leq M|f(x)|$

$$\lim_{x \rightarrow c} (-M|f(x)|) \leq \lim_{x \rightarrow c} f(x)g(x) \leq \lim_{x \rightarrow c} (M|f(x)|)$$

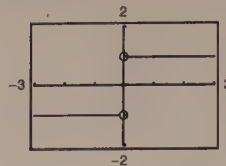
$$-M(0) \leq \lim_{x \rightarrow c} f(x)g(x) \leq M(0)$$

$$0 \leq \lim_{x \rightarrow c} f(x)g(x) \leq 0$$

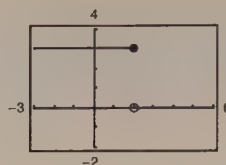
Therefore, $\lim_{x \rightarrow c} f(x)g(x) = 0$.

115. True.

113. False. As x approaches 0 from the left, $\frac{|x|}{x} = -1$.



117. False. The limit does not exist.



119. Let

$$f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0} |f(x)| = \lim_{x \rightarrow 0} 4 = 4.$$

$\lim_{x \rightarrow 0} f(x)$ does not exist since for $x < 0$, $f(x) = -4$ and for $x \geq 0$, $f(x) = 4$.

121. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$$

$\lim_{x \rightarrow 0} f(x)$ does not exist.

No matter how "close to" 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x \rightarrow 0} f(x)$ does not exist.

$$\lim_{x \rightarrow 0} g(x) = 0.$$

When x is "close to" 0, both parts of the function are "close to" 0.

123. (a) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x}$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1 + \cos x}$$

$$= (1) \left(\frac{1}{2} \right) = \frac{1}{2}$$

(b) Thus, $\frac{1 - \cos x}{x^2} \approx \frac{1}{2} \Rightarrow 1 - \cos x \approx \frac{1}{2}x^2$

$$\Rightarrow \cos x \approx 1 - \frac{1}{2}x^2 \text{ for } x \approx 0.$$

(c) $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).

Section 1.4 Continuity and One-Sided Limits

1. (a) $\lim_{x \rightarrow 3^+} f(x) = 1$

(b) $\lim_{x \rightarrow 3^-} f(x) = 1$

(c) $\lim_{x \rightarrow 3} f(x) = 1$

The function is continuous at $x = 3$.

3. (a) $\lim_{x \rightarrow 3^+} f(x) = 0$

(b) $\lim_{x \rightarrow 3^-} f(x) = 0$

(c) $\lim_{x \rightarrow 3} f(x) = 0$

The function is NOT continuous at $x = 3$.

5. (a) $\lim_{x \rightarrow 4^+} f(x) = 2$

(b) $\lim_{x \rightarrow 4^-} f(x) = -2$

(c) $\lim_{x \rightarrow 4} f(x)$ does not exist

The function is NOT continuous at $x = 4$.

7. $\lim_{x \rightarrow 5^+} \frac{x-5}{x^2-25} = \lim_{x \rightarrow 5^+} \frac{1}{x+5} = \frac{1}{10}$

9. $\lim_{x \rightarrow -3^-} \frac{x}{\sqrt{x^2-9}}$ does not exist because $\frac{x}{\sqrt{x^2-9}}$ grows without bound as $x \rightarrow -3^-$.

11. $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$

$$\begin{aligned} 13. \lim_{\Delta x \rightarrow 0^-} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} &= \lim_{\Delta x \rightarrow 0^-} \frac{x - (x + \Delta x)}{x(x + \Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{-\Delta x}{x(x + \Delta x)} \cdot \frac{1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0^-} \frac{-1}{x(x + \Delta x)} \\ &= \frac{-1}{x(x + 0)} = -\frac{1}{x^2} \end{aligned}$$

15. $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \frac{x+2}{2} = \frac{5}{2}$

17. $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x+1) = 2$

$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^3 + 1) = 2$

$\lim_{x \rightarrow 1} f(x) = 2$

19. $\lim_{x \rightarrow \pi} \cot x$ does not exist since

$\lim_{x \rightarrow \pi^-} \cot x$ and $\lim_{x \rightarrow \pi^+} \cot x$ do not exist.

21. $\lim_{x \rightarrow 4^-} (3\llbracket x \rrbracket - 5) = 3(3) - 5 = 4$
($\llbracket x \rrbracket = 3$ for $3 < x < 4$)

23. $\lim_{x \rightarrow 3} (2 - \llbracket -x \rrbracket)$ does not exist

because

$\lim_{x \rightarrow 3^-} (2 - \llbracket -x \rrbracket) = 2 - (-3) = 5$

and

$\lim_{x \rightarrow 3^+} (2 - \llbracket -x \rrbracket) = 2 - (-4) = 6$

25. $f(x) = \frac{1}{x^2 - 4}$

has discontinuities at $x = -2$ and $x = 2$ since $f(-2)$ and $f(2)$ are not defined.

27. $f(x) = \frac{\llbracket x \rrbracket}{2} + x$

has discontinuities at each integer k since $\lim_{x \rightarrow k^-} f(x) \neq \lim_{x \rightarrow k^+} f(x)$.

29. $g(x) = \sqrt{25 - x^2}$ is continuous on $[-5, 5]$.

31. $\lim_{x \rightarrow 0^-} f(x) = 3 = \lim_{x \rightarrow 0^+} f(x)$.
 f is continuous on $[-1, 4]$.

33. $f(x) = x^2 - 2x + 1$ is continuous for all real x .

35. $f(x) = 3x - \cos x$ is continuous for all real x .

37. $f(x) = \frac{x}{x^2 - x}$ is not continuous at $x = 0, 1$. Since

$\frac{x}{x^2 - x} = \frac{1}{x - 1}$ for $x \neq 0, x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

39. $f(x) = \frac{x}{x^2 + 1}$ is continuous for all real x .

41. $f(x) = \frac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ since $\lim_{x \rightarrow 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ since

$$\lim_{x \rightarrow -2} f(x) = \lim_{x \rightarrow -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

43. $f(x) = \frac{|x + 2|}{x + 2}$ has a nonremovable discontinuity at $x = -2$ since $\lim_{x \rightarrow -2} f(x)$ does not exist.

$$45. f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$$

has a **possible** discontinuity at $x = 1$.

$$1. f(1) = 1$$

$$2. \left. \begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} x = 1 \\ \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} x^2 = 1 \end{aligned} \right\} \lim_{x \rightarrow 1} f(x) = 1$$

$$3. f(1) = \lim_{x \rightarrow 1} f(x)$$

f is continuous at $x = 1$, therefore, f is continuous for all real x .

47. $f(x) = \begin{cases} \frac{x}{2} + 1, & x \leq 2 \\ 3 - x, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

$$1. f(2) = \frac{2}{2} + 1 = 2$$

$$2. \left. \begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} \left(\frac{x}{2} + 1 \right) = 2 \\ \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (3 - x) = 1 \end{aligned} \right\} \lim_{x \rightarrow 2} f(x) \text{ does not exist.}$$

Therefore, f has a nonremovable discontinuity at $x = 2$.

49. $f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \geq 1 \end{cases} = \begin{cases} \tan \frac{\pi x}{4}, & -1 < x < 1 \\ x, & x \leq -1 \text{ or } x \geq 1 \end{cases}$ has **possible** discontinuities at $x = -1, x = 1$.

$$1. f(-1) = -1 \qquad f(1) = 1$$

$$2. \lim_{x \rightarrow -1} f(x) = -1 \qquad \lim_{x \rightarrow 1} f(x) = 1$$

$$3. f(-1) = \lim_{x \rightarrow -1} f(x) \qquad f(1) = \lim_{x \rightarrow 1} f(x)$$

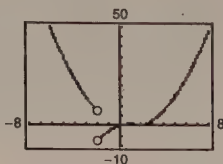
f is continuous at $x = \pm 1$, therefore, f is continuous for all real x .

51. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

55. $\lim_{x \rightarrow 0^+} f(x) = 0$

$\lim_{x \rightarrow 0^-} f(x) = 0$

f is not continuous at $x = -2$.



53. $f(x) = \llbracket x - 1 \rrbracket$ has nonremovable discontinuities at each integer k .

57. $f(2) = 8$

Find a so that $\lim_{x \rightarrow 2^+} ax^2 = 8 \Rightarrow a = \frac{8}{2^2} = 2$.

59. Find a and b such that $\lim_{x \rightarrow -1^+} (ax + b) = -a + b = 2$ and $\lim_{x \rightarrow 3^-} (ax + b) = 3a + b = -2$.

$$a - b = -2$$

$$(+)\ 3a + b = -2$$

$$4a = -4$$

$$a = -1$$

$$b = 2 + (-1) = 1$$

$$f(x) = \begin{cases} 2, & x \leq -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \geq 3 \end{cases}$$

61. $f(g(x)) = (x - 1)^2$

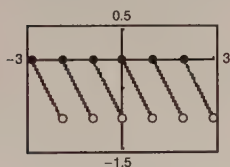
Continuous for all real x .

63. $f(g(x)) = \frac{1}{(x^2 + 5) - 6} = \frac{1}{x^2 - 1}$

Nonremovable discontinuities at $x = \pm 1$

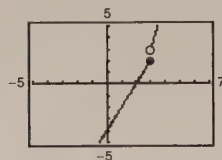
65. $y = \llbracket x \rrbracket - x$

Nonremovable discontinuity at each integer



67. $f(x) = \begin{cases} 2x - 4, & x \leq 3 \\ x^2 - 2x, & x > 3 \end{cases}$

Nonremovable discontinuity at $x = 3$



69. $f(x) = \frac{x}{x^2 + 1}$

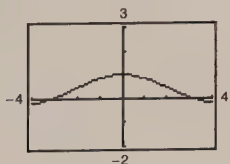
Continuous on $(-\infty, \infty)$

71. $f(x) = \sec \frac{\pi x}{4}$

Continuous on:

$\dots, (-6, -2), (-2, 2), (2, 6), (6, 10), \dots$

73. $f(x) = \frac{\sin x}{x}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(0)$ is not defined, we know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

75. $f(x) = \frac{1}{16}x^4 - x^3 + 3$ is continuous on $[1, 2]$.

$f(1) = \frac{33}{16}$ and $f(2) = -4$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 1 and 2.

77. $f(x) = x^2 - 2 - \cos x$ is continuous on $[0, \pi]$.

$f(0) = -3$ and $f(\pi) = \pi^2 - 1 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for the least one value of c between 0 and π .

81. $g(t) = 2 \cos t - 3t$

g is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(t) = 0$ for at least one value c between 0 and 1. Using a graphing utility, we find that $t \approx 0.5636$.

85. $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$-2 < 4 < 19$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$

$$x^3 - x^2 + x - 6 = 0$$

$$(x - 2)(x^2 + x + 3) = 0$$

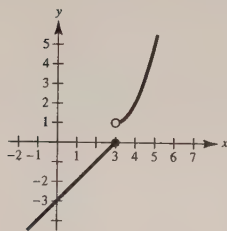
$$x = 2$$

$(x^2 + x + 3)$ has no real solution.)

$$c = 2$$

Thus, $f(2) = 4$.

89.



The function is not continuous at $x = 3$ because $\lim_{x \rightarrow 3^+} f(x) = 1 \neq 0 = \lim_{x \rightarrow 3^-} f(x)$.

79. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.6823$.

83. $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$-1 < 11 < 29$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$

$$x^2 + x - 12 = 0$$

$$(x + 4)(x - 3) = 0$$

$$x = -4 \text{ or } x = 3$$

$c = 3$ ($x = -4$ is not in the interval.)

Thus, $f(3) = 11$.

87. (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

91. The functions agree for integer values of x :

$$\left. \begin{aligned} g(x) &= 3 - \lfloor -x \rfloor = 3 - (-x) = 3 + x \\ f(x) &= 3 + \lfloor x \rfloor = 3 + x \end{aligned} \right\} \text{ for } x \text{ an integer}$$

However, for non-integer values of x , the functions differ by 1.

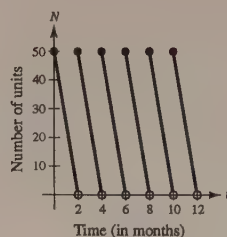
$$f(x) = 3 + \lfloor x \rfloor = g(x) - 1 = 2 - \lfloor -x \rfloor.$$

For example, $f(\frac{1}{2}) = 3 + 0 = 3$, $g(\frac{1}{2}) = 3 - (-1) = 4$.

$$93. N(t) = 25 \left(2 \left\lceil \frac{t+2}{2} \right\rceil - t \right)$$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

Discontinuous at every positive even integer. The company replenishes its inventory every two months.



95. Let $V = \frac{4}{3}\pi r^3$ be the volume of a sphere of radius r . V is continuous on $[1, 5]$.

$$V(1) = \frac{4}{3}\pi \approx 4.19$$

$$V(5) = \frac{4}{3}\pi(5^3) \approx 523.6$$

Since $4.19 < 275 < 523.6$, the Intermediate Value Theorem implies that there is at least one value r between 1 and 5 such that $V(r) = 275$. (In fact, $r \approx 4.0341$.)

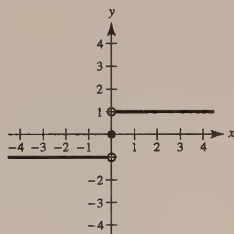
97. Let c be any real number. Then $\lim_{x \rightarrow c} f(x)$ does not exist since there are both rational and irrational numbers arbitrarily close to c . Therefore, f is not continuous at c .

$$99. \operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$

$$(a) \lim_{x \rightarrow 0^-} \operatorname{sgn}(x) = -1$$

$$(b) \lim_{x \rightarrow 0^+} \operatorname{sgn}(x) = 1$$

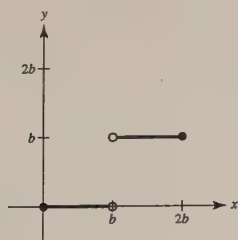
$$(c) \lim_{x \rightarrow 0} \operatorname{sgn}(x) \text{ does not exist.}$$



101. True; if $f(x) = g(x)$, $x \neq c$, then $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x)$ (if they exist) and at least one of these limits then does not equal the corresponding function value at $x = c$.

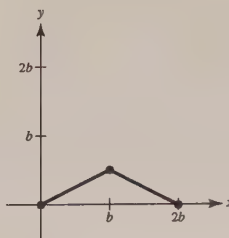
103. False; $f(1)$ is not defined and $\lim_{x \rightarrow 1} f(x)$ does not exist.

$$105. (a) f(x) = \begin{cases} 0 & 0 \leq x < b \\ b & b < x \leq 2b \end{cases}$$



NOT continuous at $x = b$.

$$(b) g(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq b \\ b - \frac{x}{2} & b < x \leq 2b \end{cases}$$



Continuous on $[0, 2b]$.

107. $f(x) = \frac{\sqrt{x+c^2} - c}{x}, c > 0$

Domain: $x + c^2 \geq 0 \Rightarrow x \geq -c^2$ and $x \neq 0, [-c^2, 0) \cup (0, \infty)$

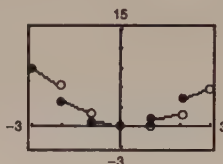
$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x+c^2} - c}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{x+c^2} - c}{x} \cdot \frac{\sqrt{x+c^2} + c}{\sqrt{x+c^2} + c} \\ &= \lim_{x \rightarrow 0} \frac{(x+c^2) - c^2}{x[\sqrt{x+c^2} + c]} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+c^2} + c} = \frac{1}{2c}\end{aligned}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

109. $h(x) = x\llbracket x \rrbracket$

h has nonremovable discontinuities at

$$x = \pm 1, \pm 2, \pm 3, \dots$$



Section 1.5 Infinite Limits

1. $\lim_{x \rightarrow -2^+} 2 \left| \frac{x}{x^2 - 4} \right| = \infty$

3. $\lim_{x \rightarrow -2^+} \tan \frac{\pi x}{4} = -\infty$

$\lim_{x \rightarrow -2^-} 2 \left| \frac{x}{x^2 - 4} \right| = \infty$

$\lim_{x \rightarrow -2^-} \tan \frac{\pi x}{4} = \infty$

5. $f(x) = \frac{1}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	0.308	1.639	16.64	166.6	-166.7	-16.69	-1.695	-0.364

$\lim_{x \rightarrow -3^-} f(x) = \infty$

$\lim_{x \rightarrow -3^+} f(x) = -\infty$

7. $f(x) = \frac{x^2}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	3.769	15.75	150.8	1501	-1499	-149.3	-14.25	-2.273

$\lim_{x \rightarrow -3^-} f(x) = \infty$

$\lim_{x \rightarrow -3^+} f(x) = -\infty$

$$9. \lim_{x \rightarrow 0^+} \frac{1}{x^2} = \infty = \lim_{x \rightarrow 0^-} \frac{1}{x^2}$$

Therefore, $x = 0$ is a vertical asymptote.

$$13. \lim_{x \rightarrow -2^-} \frac{x^2}{x^2 - 4} = \infty \text{ and } \lim_{x \rightarrow -2^+} \frac{x^2}{x^2 - 4} = -\infty$$

Therefore, $x = -2$ is a vertical asymptote.

$$\lim_{x \rightarrow 2^-} \frac{x^2}{x^2 - 4} = -\infty \text{ and } \lim_{x \rightarrow 2^+} \frac{x^2}{x^2 - 4} = \infty$$

Therefore, $x = 2$ is a vertical asymptote.

$$17. f(x) = \tan 2x = \frac{\sin 2x}{\cos 2x} \text{ has vertical asymptotes at}$$

$$x = \frac{(2n+1)\pi}{4} = \frac{\pi}{4} + \frac{n\pi}{2}, n \text{ any integer.}$$

$$21. \lim_{x \rightarrow -2^+} \frac{x}{(x+2)(x-1)} = \infty$$

$$\lim_{x \rightarrow -2^-} \frac{x}{(x+2)(x-1)} = -\infty$$

Therefore, $x = -2$ is a vertical asymptote.

$$\lim_{x \rightarrow 1^+} \frac{x}{(x+2)(x-1)} = \infty$$

$$\lim_{x \rightarrow 1^-} \frac{x}{(x+2)(x-1)} = -\infty$$

Therefore, $x = 1$ is a vertical asymptote.

$$25. f(x) = \frac{(x-5)(x+3)}{(x-5)(x^2+1)} = \frac{x+3}{x^2+1}, x \neq 5$$

No vertical asymptotes. The graph has a hole at $x = 5$.

$$11. \lim_{x \rightarrow 2^+} \frac{x^2 - 2}{(x-2)(x+1)} = \infty$$

$$\lim_{x \rightarrow 2^-} \frac{x^2 - 2}{(x-2)(x+1)} = -\infty$$

Therefore, $x = 2$ is a vertical asymptote.

$$\lim_{x \rightarrow -1^+} \frac{x^2 - 2}{(x-2)(x+1)} = \infty$$

$$\lim_{x \rightarrow -1^-} \frac{x^2 - 2}{(x-2)(x+1)} = -\infty$$

Therefore, $x = -1$ is a vertical asymptote.

15. No vertical asymptote since the denominator is never zero.

$$19. \lim_{t \rightarrow 0^+} \left(1 - \frac{4}{t^2}\right) = -\infty = \lim_{t \rightarrow 0^-} \left(1 - \frac{4}{t^2}\right)$$

Therefore, $t = 0$ is a vertical asymptote.

$$23. f(x) = \frac{x^3 + 1}{x + 1} = \frac{(x+1)(x^2 - x + 1)}{x + 1}$$

has no vertical asymptote since

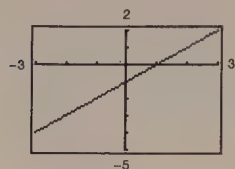
$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} (x^2 - x + 1) = 3$$

$$27. s(t) = \frac{t}{\sin t} \text{ has vertical asymptotes at } t = n\pi, n$$

a nonzero integer. There is no vertical asymptote at $t = 0$ since

$$\lim_{t \rightarrow 0} \frac{t}{\sin t} = 1.$$

$$29. \lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1} = \lim_{x \rightarrow -1} (x - 1) = -2$$



Removable discontinuity at $x = -1$

$$33. \lim_{x \rightarrow 2^+} \frac{x - 3}{x - 2} = -\infty$$

$$37. \lim_{x \rightarrow -3^-} \frac{x^2 + 2x - 3}{x^2 + x - 6} = \lim_{x \rightarrow -3^-} \frac{(x - 1)(x + 3)}{(x - 2)(x + 3)} = \lim_{x \rightarrow -3^-} \frac{x - 1}{x - 2} = \frac{4}{5}$$

$$39. \lim_{x \rightarrow 1} \frac{x^2 - x}{(x^2 + 1)(x - 1)} = \lim_{x \rightarrow 1} \frac{x}{x^2 + 1} = \frac{1}{2}$$

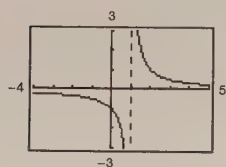
$$43. \lim_{x \rightarrow 0^+} \frac{2}{\sin x} = \infty$$

$$47. \lim_{x \rightarrow (1/2)^-} x \sec(\pi x) = \infty \text{ and } \lim_{x \rightarrow (1/2)^+} x \sec(\pi x) = -\infty.$$

Therefore, $\lim_{x \rightarrow (1/2)} x \sec(\pi x)$ does not exist.

$$49. f(x) = \frac{x^2 + x + 1}{x^3 - 1}$$

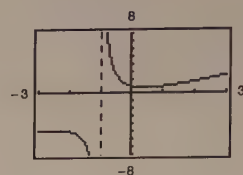
$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{1}{x - 1} = \infty$$



$$31. \lim_{x \rightarrow -1^+} \frac{x^2 + 1}{x + 1} = \infty$$

$$\lim_{x \rightarrow -1^-} \frac{x^2 + 1}{x + 1} = -\infty$$

Vertical asymptote at $x = -1$



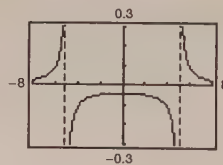
$$35. \lim_{x \rightarrow 3^+} \frac{x^2}{(x - 3)(x + 3)} = \infty$$

$$41. \lim_{x \rightarrow 0^-} \left(1 + \frac{1}{x}\right) = -\infty$$

$$45. \lim_{x \rightarrow \pi} \frac{\sqrt{x}}{\csc x} = \lim_{x \rightarrow \pi} (\sqrt{x} \sin x) = 0$$

$$51. f(x) = \frac{1}{x^2 - 25}$$

$$\lim_{x \rightarrow 5^-} f(x) = -\infty$$



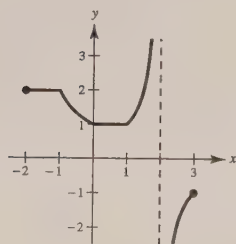
53. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. ∞ is not a number. Rather, the symbol

$$\lim_{x \rightarrow c} f(x) = \infty$$

says how the limit fails to exist.

$$55. \text{One answer is } f(x) = \frac{x - 3}{(x - 6)(x + 2)} = \frac{x - 3}{x^2 - 4x - 12}.$$

57.



$$59. S = \frac{k}{1 - r}, 0 < |r| < 1. \text{ Assume } k \neq 0.$$

$$\lim_{r \rightarrow 1^-} S = \lim_{r \rightarrow 1^-} \frac{k}{1 - r} = \infty \quad (\text{or } -\infty \text{ if } k < 0)$$

61. $C = \frac{528x}{100 - x}, 0 \leq x < 100$

(a) $C(25) = \$176$ million

(b) $C(50) = \$528$ million

(c) $C(75) = \$1584$ million

(d) $\lim_{x \rightarrow 100^-} \frac{528}{100 - x} = \infty$ Thus, it is not possible.

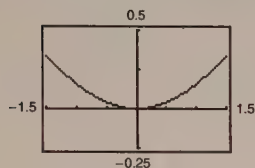
63. (a) $r = \frac{2(7)}{\sqrt{625 - 49}} = \frac{7}{12}$ ft/sec

(b) $r = \frac{2(15)}{\sqrt{625 - 225}} = \frac{3}{2}$ ft/sec

(c) $\lim_{x \rightarrow 25^-} \frac{2x}{\sqrt{625 - x^2}} = \infty$

65. (a)

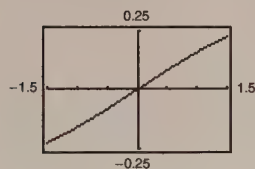
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0411	0.0067	0.0017	≈ 0	≈ 0	≈ 0



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x} = 0$$

(b)

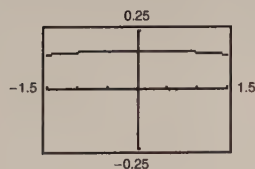
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0823	0.0333	0.0167	0.0017	≈ 0	≈ 0



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^2} = 0$$

(c)

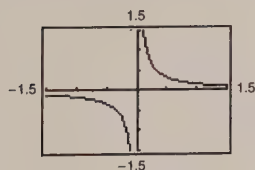
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.1646	0.1663	0.1666	0.1667	0.1667	0.1667



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^3} = 0.1167 \text{ (1/6)}$$

(d)

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.3292	0.8317	1.6658	16.67	166.7	1667.0



$$\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^4} = \infty$$

For $n \geq 3$, $\lim_{x \rightarrow 0^+} \frac{x - \sin x}{x^n} = \infty$.

67. (a) Because the circumference of the motor is half that of the saw arbor, the saw makes $1700/2 = 850$ revolutions per minute.

- (c) $2(20 \cot \phi) + 2(10 \cot \phi)$: straight sections.
The angle subtended in each circle is

$$2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi.$$

Thus, the length of the belt around the pulleys is

$$20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi).$$

$$\text{Total length} = 60 \cot \phi + 30(\pi + 2\phi)$$

$$\text{Domain: } \left(0, \frac{\pi}{2}\right)$$

69. False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ or}$$

$$g(x) = \frac{x}{x^2 + 1}.$$

73. Given $\lim_{x \rightarrow c} f(x) = \infty$ and $\lim_{x \rightarrow c} g(x) = L$:

- (2) Product:

If $L > 0$, then for $\epsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_1$. Thus, $L/2 < g(x) < 3L/2$. Since $\lim_{x \rightarrow c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $|x - c| < \delta_2$. Let δ be the smaller of δ_1 and δ_2 . Then for $0 < |x - c| < \delta$, we have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim_{x \rightarrow c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

- (3) Quotient: Let $\epsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\epsilon$ whenever $0 < |x - c| < \delta_1$ and there exists $\delta_2 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let δ be the smaller of δ_1 and δ_2 . Then for $0 < |x - c| < \delta$, we have

$$\left| \frac{g(x)}{f(x)} \right| < \frac{3L/2}{3L/2\epsilon} = \epsilon.$$

$$\text{Therefore, } \lim_{x \rightarrow c} \frac{g(x)}{f(x)} = 0.$$

75. Given $\lim_{x \rightarrow c} \frac{1}{f(x)} = 0$.

Suppose $\lim_{x \rightarrow c} f(x)$ exists and equals L . Then,

$$\lim_{x \rightarrow c} \frac{1}{f(x)} = \frac{\lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} f(x)} = \frac{1}{L} = 0.$$

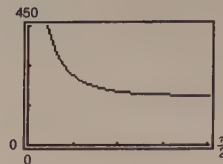
This is not possible. Thus, $\lim_{x \rightarrow c} f(x)$ does not exist.

- (b) The direction of rotation is reversed.

(d)

ϕ	0.3	0.6	0.9	1.2	1.5
L	306.2	217.9	195.9	189.6	188.5

(e)



$$(f) \lim_{\phi \rightarrow (\pi/2)^-} L = 60\pi \approx 188.5$$

(All the belts are around pulleys.)

$$(g) \lim_{\phi \rightarrow 0^+} L = \infty$$

71. False; let

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of f has a vertical asymptote at $x = 0$, but $f(0) = 3$.

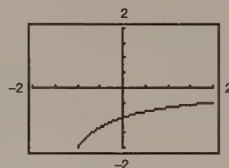
Review Exercises for Chapter 1

1. Calculus required. Using a graphing utility, you can estimate the length to be 8.3.
Or, the length is slightly longer than the distance between the two points, 8.25.

3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-1.05	-1.01	-1.00	-0.9995	-0.995	-0.952

$$\lim_{x \rightarrow 0} f(x) \approx -1$$



5. $h(x) = \frac{x^2 - 2x}{x}$

(a) $\lim_{x \rightarrow 0} h(x) = -2$

(b) $\lim_{x \rightarrow -1} h(x) = -3$

7. $\lim_{x \rightarrow 1} (3 - x) = 3 - 1 = 2$

Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. Then for

$$0 < |x - 1| < \delta = \epsilon, \text{ you have}$$

$$|x - 1| < \epsilon$$

$$|1 - x| < \epsilon$$

$$|(3 - x) - 2| < \epsilon$$

$$|f(x) - L| < \epsilon$$

9. $\lim_{x \rightarrow 2} (x^2 - 3) = 1$

Let $\epsilon > 0$ be given. We need $|x^2 - 3 - 1| < \epsilon \Rightarrow |x^2 - 4| = |(x - 2)(x + 2)| < \epsilon \Rightarrow |x - 2| < \frac{1}{|x + 2|} \epsilon$.

Assuming, $1 < x < 3$, you can choose $\delta = \epsilon/5$. Hence, for $0 < |x - 2| < \delta = \epsilon/5$ you have

$$|x - 2| < \frac{\epsilon}{5} < \frac{1}{|x + 2|} \epsilon$$

$$|x - 2||x + 2| < \epsilon$$

$$|x^2 - 4| < \epsilon$$

$$|(x^2 - 3) - 1| < \epsilon$$

$$|f(x) - L| < \epsilon$$

11. $\lim_{t \rightarrow 4} \sqrt{t + 2} = \sqrt{4 + 2} = \sqrt{6} \approx 2.45$

13. $\lim_{t \rightarrow -2} \frac{t + 2}{t^2 - 4} = \lim_{t \rightarrow -2} \frac{1}{t - 2} = -\frac{1}{4}$

15. $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x \rightarrow 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$

17. $\lim_{x \rightarrow 0} \frac{[1/(x + 1)] - 1}{x} = \lim_{x \rightarrow 0} \frac{1 - (x + 1)}{x(x + 1)} = \lim_{x \rightarrow 0} \frac{-1}{x + 1} = -1$

19. $\lim_{x \rightarrow -5} \frac{x^3 + 125}{x + 5} = \lim_{x \rightarrow -5} \frac{(x + 5)(x^2 - 5x + 25)}{x + 5} = \lim_{x \rightarrow -5} (x^2 - 5x + 25) = 75$

21. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x} = \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right) \left(\frac{1 - \cos x}{x} \right) = (1)(0) = 0$

$$\begin{aligned}
 23. \lim_{\Delta x \rightarrow 0} \frac{\sin[(\pi/6) + \Delta x] - (1/2)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{\sin(\pi/6) \cos \Delta x + \cos(\pi/6) \sin \Delta x - (1/2)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{2} \cdot \frac{(\cos \Delta x - 1)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{\sqrt{3}}{2} \cdot \frac{\sin \Delta x}{\Delta x} \\
 &= 0 + \frac{\sqrt{3}}{2}(1) = \frac{\sqrt{3}}{2}
 \end{aligned}$$

$$25. \lim_{x \rightarrow c} [f(x) \cdot g(x)] = \left(-\frac{3}{4}\right)\left(\frac{2}{3}\right) = -\frac{1}{2}$$

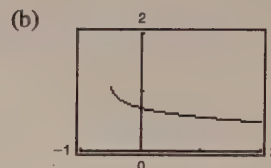
$$27. f(x) = \frac{\sqrt{2x+1} - \sqrt{3}}{x-1}$$

(a)

x	1.1	1.01	1.001	1.0001
$f(x)$	0.5680	0.5764	0.5773	0.5773

$$\lim_{x \rightarrow 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1} \approx 0.577 \quad (\text{Actual limit is } \sqrt{3}/3.)$$

$$\begin{aligned}
 (c) \lim_{x \rightarrow 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1} &= \lim_{x \rightarrow 1^+} \frac{\sqrt{2x+1} - \sqrt{3}}{x-1} \cdot \frac{\sqrt{2x+1} + \sqrt{3}}{\sqrt{2x+1} + \sqrt{3}} \\
 &= \lim_{x \rightarrow 1^+} \frac{(2x+1) - 3}{(x-1)(\sqrt{2x+1} + \sqrt{3})} \\
 &= \lim_{x \rightarrow 1^+} \frac{2}{\sqrt{2x+1} + \sqrt{3}} \\
 &= \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}
 \end{aligned}$$



$$\begin{aligned}
 29. \lim_{t \rightarrow a} \frac{s(a) - s(t)}{a - t} &= \lim_{t \rightarrow 4} \frac{(-4.9(4)^2 + 200) - (-4.9t^2 + 200)}{4 - t} \\
 &= \lim_{t \rightarrow 4} \frac{4.9(t-4)(t+4)}{4 - t} \\
 &= \lim_{t \rightarrow 4} -4.9(t+4) = -39.2 \text{ m/sec}
 \end{aligned}$$

$$31. \lim_{x \rightarrow 3^-} \frac{|x-3|}{x-3} = \lim_{x \rightarrow 3^-} \frac{-(x-3)}{x-3} = -1$$

$$33. \lim_{x \rightarrow 2} f(x) = 0$$

$$\begin{aligned}
 35. \lim_{t \rightarrow 1} h(t) \text{ does not exist because } \lim_{t \rightarrow 1^-} h(t) &= 1 + 1 = 2 \text{ and} \\
 \lim_{t \rightarrow 1^+} h(t) &= \frac{1}{2}(1 + 1) = 1.
 \end{aligned}$$

$$37. f(x) = \llbracket x + 3 \rrbracket$$

$$\lim_{x \rightarrow k^+} \llbracket x + 3 \rrbracket = k + 3 \text{ where } k \text{ is an integer.}$$

$$\lim_{x \rightarrow k^-} \llbracket x + 3 \rrbracket = k + 2 \text{ where } k \text{ is an integer.}$$

Nonremovable discontinuity at each integer k

Continuous on $(k, k+1)$ for all integers k

$$39. f(x) = \frac{3x^2 - x - 2}{x-1} = \frac{(3x+2)(x-1)}{x-1}$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (3x+2) = 5$$

Removable discontinuity at $x = 1$

Continuous on $(-\infty, 1) \cup (1, \infty)$

$$41. f(x) = \frac{1}{(x-2)^2}$$

$$\lim_{x \rightarrow 2} \frac{1}{(x-2)^2} = \infty$$

Nonremovable discontinuity at $x = 2$

Continuous on $(-\infty, 2) \cup (2, \infty)$

$$43. f(x) = \frac{3}{x+1}$$

$$\lim_{x \rightarrow 1^-} f(x) = -\infty$$

$$\lim_{x \rightarrow 1^+} f(x) = \infty$$

Nonremovable discontinuity at $x = -1$

Continuous on $(-\infty, -1) \cup (-1, \infty)$

$$45. f(x) = \csc \frac{\pi x}{2}$$

Nonremovable discontinuities at each even integer.

Continuous on

$$(2k, 2k + 2)$$

for all integers k .

49. f is continuous on $[1, 2]$. $f(1) = -1 < 0$ and $f(2) = 13 > 0$. Therefore by the Intermediate Value Theorem, there is at least one value c in $(1, 2)$ such that $2c^3 - 3 = 0$.

$$53. g(x) = 1 + \frac{2}{x}$$

Vertical asymptote at $x = 0$

$$57. \lim_{x \rightarrow -2^-} \frac{2x^2 + x + 1}{x + 2} = -\infty$$

$$61. \lim_{x \rightarrow 1^-} \frac{x^2 + 2x + 1}{x - 1} = -\infty$$

$$65. \lim_{x \rightarrow 0^+} \frac{\sin 4x}{5x} = \lim_{x \rightarrow 0^+} \left[\frac{4}{5} \left(\frac{\sin 4x}{4x} \right) \right] = \frac{4}{5}$$

$$69. C = \frac{80,000p}{100 - p}, 0 \leq 0 < 100$$

- (a) $C(15) \approx \$14,117.65$ (b) $C(50) = \$80,000$
 (c) $C(90) = \$720,000$ (d) $\lim_{p \rightarrow 100^-} \frac{80,000p}{100 - p} = \infty$

$$47. f(2) = 5$$

Find c so that $\lim_{x \rightarrow 2^+} (cx + 6) = 5$.

$$c(2) + 6 = 5$$

$$2c = -1$$

$$c = -\frac{1}{2}$$

$$51. f(x) = \frac{x^2 - 4}{|x - 2|} = (x + 2) \left[\frac{x - 2}{|x - 2|} \right]$$

$$(a) \lim_{x \rightarrow 2^-} f(x) = -4$$

$$(b) \lim_{x \rightarrow 2^+} f(x) = 4$$

$$(c) \lim_{x \rightarrow 2} f(x) \text{ does not exist.}$$

$$55. f(x) = \frac{8}{(x - 10)^2}$$

Vertical asymptote at $x = 10$

$$59. \lim_{x \rightarrow -1^+} \frac{x + 1}{x^3 + 1} = \lim_{x \rightarrow -1^+} \frac{1}{x^2 - x + 1} = \frac{1}{3}$$

$$63. \lim_{x \rightarrow 0^+} \left(x - \frac{1}{x^3} \right) = -\infty$$

$$67. \lim_{x \rightarrow 0^+} \frac{\csc 2x}{x} = \lim_{x \rightarrow 0^+} \frac{1}{x \sin 2x} = \infty$$

Problem Solving for Chapter 1

$$1. (a) \text{ Perimeter } \triangle PAO = \sqrt{x^2 + (y - 1)^2} + \sqrt{x^2 + y^2} + 1$$

$$= \sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1$$

$$\text{Perimeter } \triangle PBO = \sqrt{(x - 1)^2 + y^2} + \sqrt{x^2 + y^2} + 1$$

$$= \sqrt{(x - 1)^2 + x^4} + \sqrt{x^2 + x^4} + 1$$

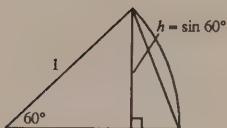
$$(b) r(x) = \frac{\sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1}{\sqrt{(x - 1)^2 + x^4} + \sqrt{x^2 + x^4} + 1}$$

x	4	2	1	0.1	0.01
Perimeter $\triangle PAO$	33.02	9.08	3.41	2.10	2.01
Perimeter $\triangle PBO$	33.77	9.60	3.41	2.00	2.00
$r(x)$	0.98	0.95	1	1.05	1.005

$$(c) \lim_{x \rightarrow 0^+} r(x) = \frac{1 + 0 + 1}{1 + 0 + 1} = \frac{2}{2} = 1$$

3. (a) There are 6 triangles, each with a central angle of $60^\circ = \pi/3$. Hence,

$$\begin{aligned}\text{Area hexagon} &= 6 \left[\frac{1}{2}bh \right] = 6 \left[\frac{1}{2}(1) \sin \frac{\pi}{3} \right] \\ &= \frac{3\sqrt{3}}{2} \approx 2.598.\end{aligned}$$



$$\text{Error: } \pi - \frac{3\sqrt{3}}{2} \approx 0.5435.$$

- (b) There are n triangles, each with central angle of $\theta = 2\pi/n$. Hence,

$$A_n = n \left[\frac{1}{2}bh \right] = n \left[\frac{1}{2}(1) \sin \frac{2\pi}{n} \right] = \frac{n \sin(2\pi/n)}{2}.$$

(c)

n	6	12	24	48	96
A_n	2.598	3	3.106	3.133	3.139

- (d) As n gets larger and larger, $2\pi/n$ approaches 0.

Letting $x = 2\pi/n$,

$$A_n = \frac{\sin(2\pi/n)}{2/n} = \frac{\sin(2\pi/n)}{(2\pi/n)} \pi = \frac{\sin x}{x} \pi$$

which approaches $(1)\pi = \pi$.

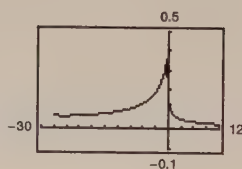
7. (a) $3 + x^{1/3} \geq 0$

$$x^{1/3} \geq -3$$

$$x \geq -27$$

Domain: $x \geq -27, x \neq 1$

(b)



$$\begin{aligned}\text{(d) } \lim_{x \rightarrow 1} f(x) &= \lim_{x \rightarrow 1} \frac{\sqrt{3 + x^{1/3}} - 2}{x - 1} \cdot \frac{\sqrt{3 + x^{1/3}} + 2}{\sqrt{3 + x^{1/3}} + 2} \\ &= \lim_{x \rightarrow 1} \frac{3 + x^{1/3} - 4}{(x - 1)(\sqrt{3 + x^{1/3}} + 2)} \\ &= \lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{(x^{1/3} - 1)(x^{2/3} + x^{1/3} + 1)(\sqrt{3 + x^{1/3}} + 2)} \\ &= \lim_{x \rightarrow 1} \frac{1}{(x^{2/3} + x^{1/3} + 1)(\sqrt{3 + x^{1/3}} + 2)} \\ &= \frac{1}{(1 + 1 + 1)(2 + 2)} = \frac{1}{12}\end{aligned}$$

9. (a) $\lim_{x \rightarrow 2} f(x) = 3$: g_1, g_4

(b) f continuous at 2: g_1

- (c) $\lim_{x \rightarrow 2} f(x) = 3$: g_1, g_3, g_4

5. (a) Slope $= -\frac{12}{5}$

(b) Slope of tangent line is $\frac{5}{12}$.

$$y + 12 = \frac{5}{12}(x - 5)$$

$$y = \frac{5}{12}x - \frac{169}{12} \quad \text{Tangent line}$$

- (c) $Q = (x, y) = (x, -\sqrt{169 - x^2})$

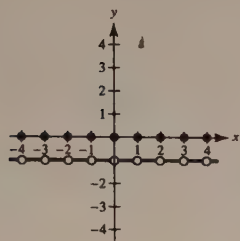
$$m_x = \frac{-\sqrt{169 - x^2} + 12}{x - 5}$$

$$\begin{aligned}\text{(d) } \lim_{x \rightarrow 5} m_x &= \lim_{x \rightarrow 5} \frac{12 - \sqrt{169 - x^2}}{x - 5} \cdot \frac{12 + \sqrt{169 - x^2}}{12 + \sqrt{169 - x^2}} \\ &= \lim_{x \rightarrow 5} \frac{144 - (169 - x^2)}{(x - 5)(12 + \sqrt{169 - x^2})} \\ &= \lim_{x \rightarrow 5} \frac{x^2 - 25}{(x - 5)(12 + \sqrt{169 - x^2})} \\ &= \lim_{x \rightarrow 5} \frac{(x + 5)}{12 + \sqrt{169 - x^2}} \\ &= \frac{10}{12 + 12} = \frac{5}{12}\end{aligned}$$

This is the same slope as part (b).

$$\begin{aligned}\text{(c) } \lim_{x \rightarrow -27^+} f(x) &= \frac{\sqrt{3 + (-27)^{1/3}} - 2}{-27 - 1} \\ &= \frac{-2}{-28} = \frac{1}{14} \approx 0.0714\end{aligned}$$

11.



$$(a) \quad f(1) = \llbracket 1 \rrbracket + \llbracket -1 \rrbracket = 1 + (-1) = 0$$

$$f(0) = 0$$

$$f\left(\frac{1}{2}\right) = 0 + (-1) = -1$$

$$f(-2.7) = -3 + 2 = -1$$

$$(b) \quad \lim_{x \rightarrow 1^-} f(x) = -1$$

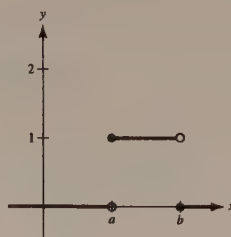
$$\lim_{x \rightarrow 1^+} f(x) = -1$$

$$\lim_{x \rightarrow 1/2} f(x) = -1$$

(c) f is continuous for all real numbers except

$$x = 0, \pm 1, \pm 2, \pm 3, \dots$$

13. (a)



$$(b) \quad (i) \quad \lim_{x \rightarrow a^+} P_{a,b}(x) = 1$$

$$(ii) \quad \lim_{x \rightarrow a^-} P_{a,b}(x) = 0$$

$$(iii) \quad \lim_{x \rightarrow b^+} P_{a,b}(x) = 0$$

$$(iv) \quad \lim_{x \rightarrow b^-} P_{a,b}(x) = 1$$

(c) $P_{a,b}$ is continuous for all positive real numbers except $x = a, b$.

(d) The area under the graph of u , and above the x -axis, is 1.

CHAPTER 2

Differentiation

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CHAPTER 2

Differentiation

Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Odd-Numbered Exercises

1. (a) $m = 0$
(b) $m = -3$

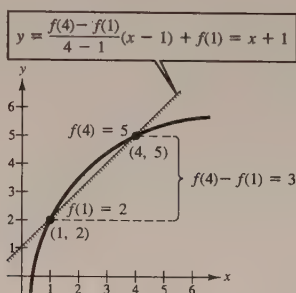
3. (a), (b)

$$(c) y = \frac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$$

$$= \frac{3}{3}(x - 1) + 2$$

$$= 1(x - 1) + 2$$

$$= x + 1$$



5. $f(x) = 3 - 2x$ is a line. Slope $= -2$

$$7. \text{Slope at } (1, -3) = \lim_{\Delta x \rightarrow 0} \frac{g(1 + \Delta x) - g(1)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{(1 + \Delta x)^2 - 4 - (-3)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{1 + 2(\Delta x) + (\Delta x)^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} [2 + 2(\Delta x)] = 2$$

$$\begin{aligned} 9. \text{Slope at } (0, 0) &= \lim_{\Delta t \rightarrow 0} \frac{f(0 + \Delta t) - f(0)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{3(\Delta t) - (\Delta t)^2 - 0}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} (3 - \Delta t) = 3 \end{aligned}$$

$$11. f(x) = 3$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{3 - 3}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} 0 = 0$$

$$13. f(x) = -5x$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-5(x + \Delta x) - (-5x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} -5 = -5 \end{aligned}$$

$$15. h(s) = 3 + \frac{2}{3}s$$

$$h'(s) = \lim_{\Delta s \rightarrow 0} \frac{h(s + \Delta s) - h(s)}{\Delta s}$$

$$= \lim_{\Delta s \rightarrow 0} \frac{3 + \frac{2}{3}(s + \Delta s) - \left(3 + \frac{2}{3}s\right)}{\Delta s}$$

$$= \lim_{\Delta s \rightarrow 0} \frac{\frac{2}{3}\Delta s}{\Delta s} = \frac{2}{3}$$

17. $f(x) = 2x^2 + x - 1$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[2(x + \Delta x)^2 + (x + \Delta x) - 1] - [2x^2 + x - 1]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(2x^2 + 4x\Delta x + 2(\Delta x)^2 + x + \Delta x - 1) - (2x^2 + x - 1)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{4x\Delta x + 2(\Delta x)^2 + \Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} (4x + 2\Delta x + 1) = 4x + 1
 \end{aligned}$$

19. $f(x) = x^3 - 12x$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 - 12(x + \Delta x)] - [x^3 - 12x]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12x - 12\Delta x - x^3 + 12x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12\Delta x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2 - 12) = 3x^2 - 12
 \end{aligned}$$

21. $f(x) = \frac{1}{x - 1}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x - 1} - \frac{1}{x - 1}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x - 1) - (x + \Delta x - 1)}{\Delta x(x + \Delta x - 1)(x - 1)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-\Delta x}{\Delta x(x + \Delta x - 1)(x - 1)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-1}{(x + \Delta x - 1)(x - 1)} \\
 &= -\frac{1}{(x - 1)^2}
 \end{aligned}$$

23. $f(x) = \sqrt{x + 1}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x + 1} - \sqrt{x + 1}}{\Delta x} \cdot \left(\frac{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}} \right) \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x + 1) - (x + 1)}{\Delta x [\sqrt{x + \Delta x + 1} + \sqrt{x + 1}]} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}} \\
 &= \frac{1}{\sqrt{x + 1} + \sqrt{x + 1}} = \frac{1}{2\sqrt{x + 1}}
 \end{aligned}$$

25. (a) $f(x) = x^2 + 1$

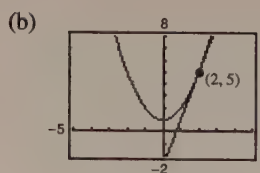
$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 + 1] - [x^2 + 1]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x \end{aligned}$$

At (2, 5), the slope of the tangent line is $m = 2(2) = 4$. The equation of the tangent line is

$$y - 5 = 4(x - 2)$$

$$y - 5 = 4x - 8$$

$$y = 4x - 3.$$



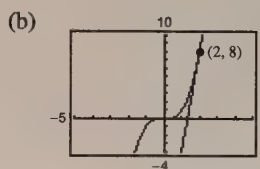
27. (a) $f(x) = x^3$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2 \end{aligned}$$

At (2, 8), the slope of the tangent is $m = 3(2)^2 = 12$. The equation of the tangent line is

$$y - 8 = 12(x - 2)$$

$$y = 12x - 16.$$



29. (a) $f(x) = \sqrt{x}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

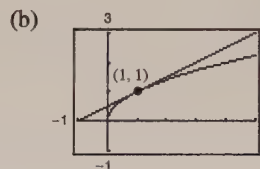
At (1, 1), the slope of the tangent line is

$$m = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

The equation of the tangent line is

$$y - 1 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$



31. (a) $f(x) = x + \frac{4}{x}$

$$\begin{aligned}
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) + \frac{4}{x + \Delta x} - \left(x + \frac{4}{x}\right)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x(x + \Delta x)(x + \Delta x) + 4x - x^2(x + \Delta x) - 4(x + \Delta x)}{x(\Delta x)(x + \Delta x)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 2x^2(\Delta x) + x(\Delta x)^2 - x^3 - x^2(\Delta x) - 4(\Delta x)}{x(\Delta x)(x + \Delta x)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^2(\Delta x) + x(\Delta x)^2 - 4(\Delta x)}{x(\Delta x)(x + \Delta x)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + x(\Delta x) - 4}{x(x + \Delta x)} \\
 &= \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2}
 \end{aligned}$$

At (4, 5), the slope of the tangent line is

$$m = 1 - \frac{4}{16} = \frac{3}{4}.$$

The equation of the tangent line is

$$\begin{aligned}
 y - 5 &= \frac{3}{4}(x - 4) \\
 y &= \frac{3}{4}x + 2.
 \end{aligned}$$

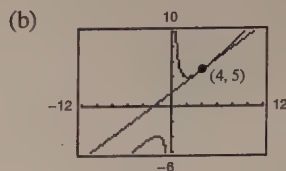
33. From Exercise 27 we know that $f'(x) = 3x^2$. Since the slope of the given line is 3, we have

$$3x^2 = 3$$

$$x = \pm 1.$$

Therefore, at the points (1, 1) and (-1, -1) the tangent lines are parallel to $3x - y + 1 = 0$. These lines have equations

$$\begin{aligned}
 y - 1 &= 3(x - 1) & \text{and} & & y + 1 &= 3(x + 1) \\
 y &= 3x - 2 & & & y &= 3x + 2.
 \end{aligned}$$



35. Using the limit definition of derivative,

$$f'(x) = \frac{-1}{2x\sqrt{x}}.$$

Since the slope of the given line is $-\frac{1}{2}$, we have

$$\begin{aligned}
 -\frac{1}{2x\sqrt{x}} &= -\frac{1}{2} \\
 x &= 1.
 \end{aligned}$$

Therefore, at the point (1, 1) the tangent line is parallel to $x + 2y - 6 = 0$. The equation of this line is

$$\begin{aligned}
 y - 1 &= -\frac{1}{2}(x - 1) \\
 y - 1 &= -\frac{1}{2}x + \frac{1}{2} \\
 y &= -\frac{1}{2}x + \frac{3}{2}.
 \end{aligned}$$

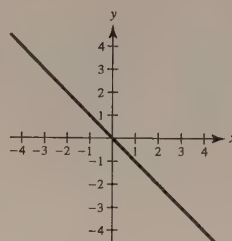
37. $g(5) = 2$ because the tangent line passes through (5, 2).

$$g'(5) = \frac{2 - 0}{5 - 9} = \frac{2}{-4} = -\frac{1}{2}$$

39. $f(x) = x \Rightarrow f'(x) = 1$ matches (b)

41. $f(x) = \sqrt{x} \Rightarrow f'(x)$ matches (a)
(decreasing slope as $x \rightarrow \infty$)

43.



Answers will vary.

 Sample answer: $y = -x$

45. (a) If $f'(c) = 3$ and f is odd, then $f'(-c) = f'(c) = 3$
(b) If $f'(c) = 3$ and f is even, then $f'(-c) = -f'(c) = -3$

47. Let (x_0, y_0) be a point of tangency on the graph of f . By the limit definition for the derivative, $f'(x) = 4 - 2x$. The slope of the line through $(2, 5)$ and (x_0, y_0) equals the derivative of f at x_0 :

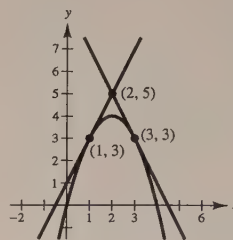
$$\frac{5 - y_0}{2 - x_0} = 4 - 2x_0$$

$$5 - y_0 = (2 - x_0)(4 - 2x_0)$$

$$5 - (4x_0 - x_0^2) = 8 - 8x_0 + 2x_0^2$$

$$0 = x_0^2 - 4x_0 + 3$$

$$0 = (x_0 - 1)(x_0 - 3) \Rightarrow x_0 = 1, 3$$



Therefore, the points of tangency are $(1, 3)$ and $(3, 3)$, and the corresponding slopes are 2 and -2 . The equations of the tangent lines are

$$y - 5 = 2(x - 2) \quad y - 5 = -2(x - 2)$$

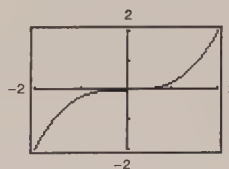
$$y = 2x + 1 \quad y = -2x + 9$$

49. (a) $g'(0) = -3$
(b) $g'(3) = 0$
(c) Because $g'(1) = -\frac{8}{3}$, g is decreasing (falling) at $x = 1$.
(d) Because $g'(-4) = \frac{7}{3}$, g is increasing (rising) at $x = -4$.
(e) Because $g'(4)$ and $g'(6)$ are both positive, $g(6)$ is greater than $g(4)$, and $g(6) - g(4) > 0$.
(f) No, it is not possible. All you can say is that g is decreasing (falling) at $x = 2$.

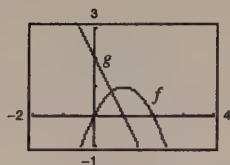
51. $f(x) = \frac{1}{4}x^3$

By the limit definition of the derivative we have $f'(x) = \frac{3}{4}x^2$.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	-2	$-\frac{27}{32}$	$-\frac{1}{4}$	$-\frac{1}{32}$	0	$\frac{1}{32}$	$\frac{1}{4}$	$\frac{27}{32}$	2
$f'(x)$	3	$\frac{27}{16}$	$\frac{3}{4}$	$\frac{3}{16}$	0	$\frac{3}{16}$	$\frac{3}{4}$	$\frac{27}{16}$	3

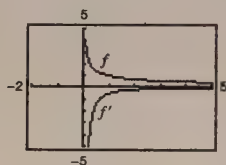


$$\begin{aligned}
 53. \quad g(x) &= \frac{f(x + 0.01) - f(x)}{0.01} \\
 &= [(2(x + 0.01) - (x + 0.01)^2 - 2x + x^2)] \cdot 100 \\
 &= 2 - 2x - 0.01
 \end{aligned}$$



The graph of $g(x)$ is approximately the graph of $f'(x)$.

$$57. \quad f(x) = \frac{1}{\sqrt{x}} \text{ and } f'(x) = \frac{-1}{2x^{3/2}}.$$



As $x \rightarrow \infty$, f is nearly horizontal and thus $f' \approx 0$.

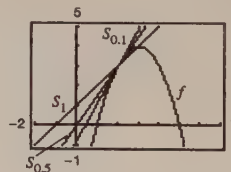
$$59. \quad f(x) = 4 - (x - 3)^2$$

$$\begin{aligned}
 S_{\Delta x}(x) &= \frac{f(2 + \Delta x) - f(2)}{\Delta x} (x - 2) + f(2) \\
 &= \frac{4 - (2 + \Delta x - 3)^2 - 3}{\Delta x} (x - 2) + 3 = \frac{1 - (\Delta x - 1)^2}{\Delta x} (x - 2) + 3 = (-\Delta x + 2)(x - 2) + 3
 \end{aligned}$$

$$(a) \quad \Delta x = 1: S_{\Delta x} = (x - 2) + 3 = x + 1$$

$$\Delta x = 0.5: S_{\Delta x} = \left(\frac{3}{2}\right)(x - 2) + 3 = \frac{3}{2}x$$

$$\Delta x = 0.1: S_{\Delta x} = \left(\frac{19}{10}\right)(x - 2) + 3 = \frac{19}{10}x - \frac{4}{5}$$



(b) As $\Delta x \rightarrow 0$, the line approaches the tangent line to f at $(2, 3)$.

$$61. \quad f(x) = x^2 - 1, c = 2$$

$$f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{(x^2 - 1) - 3}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 4$$

$$63. \quad f(x) = x^3 + 2x^2 + 1, c = -2$$

$$f'(-2) = \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x + 2} = \lim_{x \rightarrow -2} \frac{(x^3 + 2x^2 + 1) - 1}{x + 2} = \lim_{x \rightarrow -2} \frac{x^2(x + 2)}{x + 2} = \lim_{x \rightarrow -2} x^2 = 4$$

$$65. \quad g(x) = \sqrt{|x|}, c = 0$$

$$g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\sqrt{|x|}}{x}. \text{ Does not exist.}$$

$$\text{As } x \rightarrow 0^-, \frac{\sqrt{|x|}}{x} = \frac{-1}{\sqrt{|x|}} \rightarrow -\infty$$

$$\text{As } x \rightarrow 0^+, \frac{\sqrt{|x|}}{x} = \frac{1}{\sqrt{x}} \rightarrow \infty$$

$$55. \quad f(2) = 2(4 - 2) = 4, f(2.1) = 2.1(4 - 2.1) = 3.99$$

$$f'(2) \approx \frac{3.99 - 4}{2.1 - 2} = -0.1 \text{ [Exact: } f'(2) = 0 \text{]}$$

$$67. \quad f(x) = (x - 6)^{2/3}, c = 6$$

$$\begin{aligned}
 f'(6) &= \lim_{x \rightarrow 6} \frac{f(x) - f(6)}{x - 6} \\
 &= \lim_{x \rightarrow 6} \frac{(x - 6)^{2/3} - 0}{x - 6} \\
 &= \lim_{x \rightarrow 6} \frac{1}{(x - 6)^{1/3}}
 \end{aligned}$$

Does not exist.

69. $h(x) = |x + 5|, c = -5$

$$\begin{aligned} h'(-5) &= \lim_{x \rightarrow -5} \frac{h(x) - h(-5)}{x - (-5)} \\ &= \lim_{x \rightarrow -5} \frac{|x + 5| - 0}{x + 5} \\ &= \lim_{x \rightarrow -5} \frac{|x + 5|}{x + 5} \end{aligned}$$

Does not exist.

73. $f(x)$ is differentiable everywhere except at $x = -1$.
(Discontinuity)

77. $f(x)$ is differentiable on the interval $(1, \infty)$.
(At $x = 1$ the tangent line is vertical)

81. $f(x) = |x - 1|$

The derivative from the left is

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{|x - 1| - 0}{x - 1} = -1.$$

The derivative from the right is

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{|x - 1| - 0}{x - 1} = 1.$$

The one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

85. Note that f is continuous at $x = 2$. $f(x) = \begin{cases} x^2 + 1, & x \leq 2 \\ 4x - 3, & x > 2 \end{cases}$

The derivative from the left is $\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{(x^2 + 1) - 5}{x - 2} = \lim_{x \rightarrow 2^-} (x + 2) = 4$.

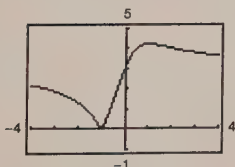
The derivative from the right is $\lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^+} \frac{(4x - 3) - 5}{x - 2} = \lim_{x \rightarrow 2^+} 4 = 4$.

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. ($f'(2) = 4$)

87. (a) The distance from $(3, 1)$ to the line $mx - y + 4 = 0$ is

$$\begin{aligned} d &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \\ &= \frac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}} = \frac{|3m + 3|}{\sqrt{m^2 + 1}} \end{aligned}$$

(b)



The function d is not differentiable at $m = -1$. This corresponds to the line $y = -x + 4$, which passes through the point $(3, 1)$.

71. $f(x)$ is differentiable everywhere except at $x = -3$.
(Sharp turn in the graph.)

75. $f(x)$ is differentiable everywhere except at $x = 3$.
(Sharp turn in the graph.)

79. $f(x)$ is differentiable everywhere except at $x = 0$.
(Discontinuity)

83. $f(x) = \begin{cases} (x - 1)^3, & x \leq 1 \\ (x - 1)^2, & x > 1 \end{cases}$

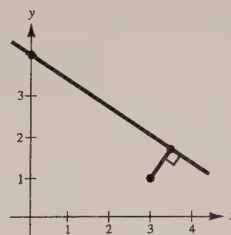
The derivative from the left is

$$\begin{aligned} \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^-} \frac{(x - 1)^3 - 0}{x - 1} \\ &= \lim_{x \rightarrow 1^-} (x - 1)^2 = 0. \end{aligned}$$

The derivative from the right is

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^+} \frac{(x - 1)^2 - 0}{x - 1} \\ &= \lim_{x \rightarrow 1^+} (x - 1) = 0. \end{aligned}$$

These one-sided limits are equal. Therefore, f is differentiable at $x = 1$. ($f'(1) = 0$)



89. False. the slope is $\lim_{\Delta x \rightarrow 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x}$.

91. False. If the derivative from the left of a point does not equal the derivative from the right of a point, then the derivative does not exist at that point. For example, if $f(x) = |x|$, then the derivative from the left at $x = 0$ is -1 and the derivative from the right at $x = 0$ is 1 . At $x = 0$, the derivative does not exist.

$$93. f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem, we have $-|x| \leq x \sin(1/x) \leq |x|$, $x \neq 0$. Thus, $\lim_{x \rightarrow 0} x \sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$. Using the alternative form of the derivative we have

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \rightarrow 0} \left(\sin \frac{1}{x} \right).$$

Since this limit does not exist (it oscillates between -1 and 1), the function is not differentiable at $x = 0$.

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem again we have $-x^2 \leq x^2 \sin(1/x) \leq x^2$, $x \neq 0$. Thus, $\lim_{x \rightarrow 0} x^2 \sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$. Using the alternative form of the derivative again we have

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0.$$

Therefore, g is differentiable at $x = 0$, $g'(0) = 0$.

Section 2.2 Basic Differentiation Rules and Rates of Change

- | | | | |
|---|---|--|---|
| 1. (a) $y = x^{1/2}$
$y' = \frac{1}{2}x^{-1/2}$
$y'(1) = \frac{1}{2}$ | (b) $y = x^{3/2}$
$y' = \frac{3}{2}x^{1/2}$
$y'(1) = \frac{3}{2}$ | (c) $y = x^2$
$y' = 2x$
$y'(1) = 2$ | (d) $y = x^3$
$y' = 3x^2$
$y'(1) = 3$ |
| 3. $y = 8$
$y' = 0$ | 5. $y = x^6$
$y' = 6x^5$ | 7. $y = \frac{1}{x^7} = x^{-7}$
$y' = -7x^{-8} = \frac{-7}{x^8}$ | 9. $y = \sqrt[5]{x} = x^{1/5}$
$y' = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}}$ |
| 11. $f(x) = x + 1$
$f'(x) = 1$ | 13. $f(t) = -2t^2 + 3t - 6$
$f'(t) = -4t + 3$ | 15. $g(x) = x^2 + 4x^3$
$g'(x) = 2x + 12x^2$ | 17. $s(t) = t^3 - 2t + 4$
$s'(t) = 3t^2 - 2$ |
| 19. $y = \frac{\pi}{2} \sin \theta - \cos \theta$
$y' = \frac{\pi}{2} \cos \theta + \sin \theta$ | 21. $y = x^2 - \frac{1}{2} \cos x$
$y' = 2x + \frac{1}{2} \sin x$ | 23. $y = \frac{1}{x} - 3 \sin x$
$y' = -\frac{1}{x^2} - 3 \cos x$ | |

Function	Rewrite	Derivative	Simplify
25. $y = \frac{5}{2x^2}$	$y = \frac{5}{2}x^{-2}$	$y' = -5x^{-3}$	$y' = \frac{-5}{x^3}$
27. $y = \frac{3}{(2x)^3}$	$y = \frac{3}{8}x^{-3}$	$y' = \frac{-9}{8}x^{-4}$	$y' = \frac{-9}{8x^4}$
29. $y = \frac{\sqrt{x}}{x}$	$y = x^{-1/2}$	$y' = -\frac{1}{2}x^{-3/2}$	$y' = -\frac{1}{2x^{3/2}}$

31. $f(x) = \frac{3}{x^2} = 3x^{-2}, (1, 3)$

$$f'(x) = -6x^{-3} = \frac{-6}{x^3}$$

$$f'(1) = -6$$

33. $f(x) = -\frac{1}{2} + \frac{7}{5}x^3, \left(0, -\frac{1}{2}\right)$

$$f'(x) = \frac{21}{5}x^2$$

$$f'(0) = 0$$

35. $y = (2x + 1)^2, (0, 1)$

$$= 4x^2 + 4x + 1$$

$$y' = 8x + 4$$

$$y'(0) = 4$$

37. $f(\theta) = 4 \sin \theta - \theta, (0, 0)$

$$f'(\theta) = 4 \cos \theta - 1$$

$$f'(0) = 4(1) - 1 = 3$$

39. $f(x) = x^2 + 5 - 3x^{-2}$

$$f'(x) = 2x + 6x^{-3} = 2x + \frac{6}{x^3}$$

41. $g(t) = t^2 - \frac{4}{t^3} = t^2 - 4t^{-3}$

$$g'(t) = 2t + 12t^{-4} = 2t + \frac{12}{t^4}$$

43. $f(x) = \frac{x^3 - 3x^2 + 4}{x^2} = x - 3 + 4x^{-2}$

$$f'(x) = 1 - \frac{8}{x^3} = \frac{x^3 - 8}{x^3}$$

45. $y = x(x^2 + 1) = x^3 + x$

$$y' = 3x^2 + 1$$

47. $f(x) = \sqrt{x} - 6\sqrt[3]{x} = x^{1/2} - 6x^{1/3}$

$$f'(x) = \frac{1}{2}x^{-1/2} - 2x^{-2/3} = \frac{1}{2\sqrt{x}} - \frac{2}{x^{2/3}}$$

49. $h(s) = s^{4/5} - s^{2/3}$

$$h'(s) = \frac{4}{5}s^{-1/5} - \frac{2}{3}s^{-1/3} = \frac{4}{5s^{1/5}} - \frac{2}{3s^{1/3}}$$

51. $f(x) = 6\sqrt{x} + 5 \cos x = 6x^{1/2} + 5 \cos x$

$$f'(x) = 3x^{-1/2} - 5 \sin x = \frac{3}{\sqrt{x}} - 5 \sin x$$

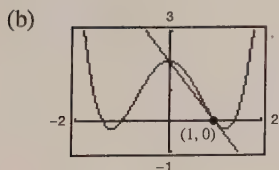
53. (a) $y = x^4 - 3x^2 + 2$

$$y' = 4x^3 - 6x$$

$$\text{At } (1, 0): y' = 4(1)^3 - 6(1) = -2.$$

$$\text{Tangent line: } y - 0 = -2(x - 1)$$

$$2x + y - 2 = 0$$



55. (a) $f(x) = \frac{2}{\sqrt[4]{x^3}} = 2x^{-3/4}$

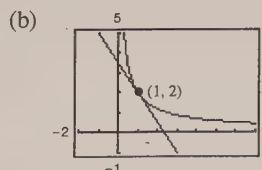
$$f'(x) = \frac{-3}{2}x^{-7/4} = \frac{-3}{2x^{7/4}}$$

$$\text{At } (1, 2), f'(1) = \frac{-3}{2}$$

$$\text{Tangent line: } y - 2 = -\frac{3}{2}(x - 1)$$

$$y = -\frac{3}{2}x + \frac{7}{2}$$

$$3x + 2y - 7 = 0$$



57. $y = x^4 - 8x^2 + 2$

$$y' = 4x^3 - 16x$$

$$= 4x(x^2 - 4)$$

$$= 4x(x - 2)(x + 2)$$

$$y' = 0 \Rightarrow x = 0, \pm 2$$

Horizontal tangents: $(0, 2)$, $(2, -14)$, $(-2, -14)$

61. $y = x + \sin x$, $0 \leq x < 2\pi$

$$y' = 1 + \cos x = 0$$

$$\cos x = -1 \Rightarrow x = \pi$$

At $x = \pi$, $y = \pi$.Horizontal tangent: (π, π)

65. $\frac{k}{x} = -\frac{3}{4}x + 3$ Equate functions

$$-\frac{k}{x^2} = -\frac{3}{4}$$
 Equate derivatives

$$\text{Hence, } k = \frac{3}{4}x^2 \text{ and } \frac{\frac{3}{4}x^2}{x} = -\frac{3}{4}x + 3 \Rightarrow \frac{3}{4}x = -\frac{3}{4}x + 3 \Rightarrow \frac{3}{2}x = 3 \Rightarrow x = 2 \Rightarrow k = 3.$$

67. (a) The slope appears to be steepest between A and B.

(b) The average rate of change between A and B is **greater** than the instantaneous rate of change at B.

59. $y = \frac{1}{x^2} = x^{-2}$

$$y' = -2x^{-3} = \frac{-2}{x^3} \text{ cannot equal zero.}$$

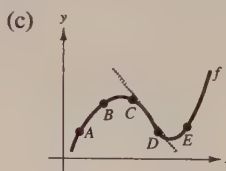
Therefore, there are no horizontal tangents.

63. $x^2 - kx = 4x - 9$ Equate functions

$$2x - k = 4$$
 Equate derivatives

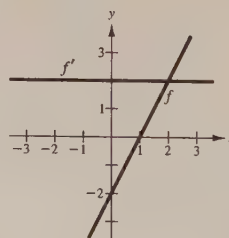
Hence, $k = 2x - 4$ and

$$x^2 - (2x - 4)x = 4x - 9 \Rightarrow -x^2 = -9 \Rightarrow x = \pm 3.$$

For $x = 3$, $k = 2$ and for $x = -3$, $k = -10$.

69. $g(x) = f(x) + 6 \Rightarrow g'(x) = f'(x)$

71.

If f is linear then its derivative is a constant function.

$$f(x) = ax + b$$

$$f'(x) = a$$

73. Let (x_1, y_1) and (x_2, y_2) be the points of tangency on $y = x^2$ and $y = -x^2 + 6x - 5$, respectively. The derivatives of these functions are

$$y' = 2x \Rightarrow m = 2x_1 \quad \text{and} \quad y' = -2x + 6 \Rightarrow m = -2x_2 + 6.$$

$$m = 2x_1 = -2x_2 + 6$$

$$x_1 = -x_2 + 3$$

Since $y_1 = x_1^2$ and $y_2 = -x_2^2 + 6x_2 - 5$,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-x_2^2 + 6x_2 - 5) - (x_1^2)}{x_2 - x_1} = -2x_2 + 6.$$

$$\frac{(-x_2^2 + 6x_2 - 5) - (-x_2 + 3)^2}{x_2 - (-x_2 + 3)} = -2x_2 + 6$$

$$(-x_2^2 + 6x_2 - 5) - (x_2^2 - 6x_2 + 9) = (-2x_2 + 6)(2x_2 - 3)$$

$$-2x_2^2 + 12x_2 - 14 = -4x_2^2 + 18x_2 - 18$$

$$2x_2^2 - 6x_2 + 4 = 0$$

$$2(x_2 - 2)(x_2 - 1) = 0$$

$$x_2 = 1 \text{ or } 2$$

$$x_2 = 1 \Rightarrow y_2 = 0, x_1 = 2 \text{ and } y_1 = 4$$

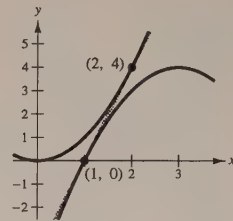
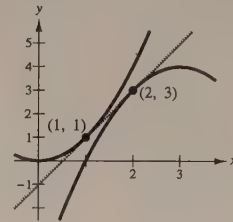
Thus, the tangent line through $(1, 0)$ and $(2, 4)$ is

$$y - 0 = \left(\frac{4 - 0}{2 - 1}\right)(x - 1) \Rightarrow y = 4x - 4.$$

$$x_2 = 2 \Rightarrow y_2 = 3, x_1 = 1 \text{ and } y_1 = 1$$

Thus, the tangent line through $(2, 3)$ and $(1, 1)$ is

$$y - 1 = \left(\frac{3 - 1}{2 - 1}\right)(x - 1) \Rightarrow y = 2x - 1.$$



75. $f(x) = \sqrt{x}, (-4, 0)$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{x}} = \frac{0 - y}{-4 - x}$$

$$4 + x = 2\sqrt{xy}$$

$$4 + x = 2\sqrt{x}\sqrt{x}$$

$$4 + x = 2x$$

$$x = 4, y = 2$$

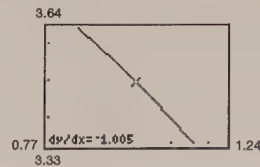
The point $(4, 2)$ is on the graph of f .

$$\text{Tangent line: } y - 2 = \frac{0 - 2}{-4 - 4}(x - 4)$$

$$4y - 8 = x - 4$$

$$0 = x - 4y + 4$$

77. $f'(1) = -1$

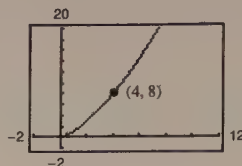


79. (a) One possible secant is between (3.9, 7.7019) and (4, 8):

$$y - 8 = \frac{8 - 7.7019}{4 - 3.9}(x - 4)$$

$$y - 8 = 2.981(x - 4)$$

$$y = S(x) = 2.981x - 3.924$$

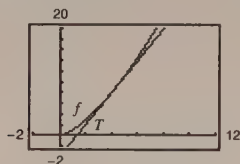


$$(b) f'(x) = \frac{3}{2}x^{1/2} \Rightarrow f'(4) = \frac{3}{2}(2) = 3$$

$$T(x) = 3(x - 4) + 8 = 3x - 4$$

$S(x)$ is an approximation of the tangent line $T(x)$.

- (c) As you move further away from (4, 8), the accuracy of the approximation T gets worse.



Δx	-3	-2	-1	-0.5	-0.1	0	0.1	0.5	1	2	3
$f(4 + \Delta x)$	1	2.828	5.196	6.548	7.702	8	8.302	9.546	11.180	14.697	18.520
$T(4 + \Delta x)$	-1	2	5	6.5	7.7	8	8.3	9.5	11	14	17

81. False. Let $f(x) = x^2$ and $g(x) = x^2 + 4$. Then $f'(x) = g'(x) = 2x$, but $f(x) \neq g(x)$.

83. False. If $y = \pi^2$, then $dy/dx = 0$. (π^2 is a constant.)

85. True. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.

87. $f(t) = 2t + 7, [1, 2]$

$$f'(t) = 2$$

Instantaneous rate of change is the constant 2.

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{[2(2) + 7] - [2(1) + 7]}{1} = 2$$

(These are the same because f is a line of slope 2.)

89. $f(x) = -\frac{1}{x}, [1, 2]$

$$f'(x) = \frac{1}{x^2}$$

Instantaneous rate of change:

$$(1, -1) \Rightarrow f'(1) = 1$$

$$\left(2, -\frac{1}{2}\right) \Rightarrow f'(2) = \frac{1}{4}$$

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{(-1/2) - (-1)}{2 - 1} = \frac{1}{2}$$

91. (a) $s(t) = -\frac{1}{4}16t^2 + 1362$

$$v(t) = -32t$$

(b) $\frac{s(2) - s(1)}{2 - 1} = 1298 - 1346 = -48 \text{ ft/sec}$

(c) $v(t) = s'(t) = -32t$

When $t = 1$: $v(1) = -32 \text{ ft/sec}$.

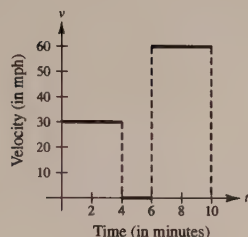
When $t = 2$: $v(2) = -64 \text{ ft/sec}$.

(d) $-16t^2 + 1362 = 0$

$$t^2 = \frac{1362}{16} \Rightarrow t = \frac{\sqrt{1362}}{4} \approx 9.226 \text{ sec}$$

(e) $v\left(\frac{\sqrt{1362}}{4}\right) = -32\left(\frac{\sqrt{1362}}{4}\right)$
 $= -8\sqrt{1362} \approx -295.242 \text{ ft/sec}$

95.



(The velocity has been converted to miles per hour)

99. (a) Using a graphing utility, you obtain

$$R = 0.167v - 0.02.$$

(c) $T = R + B = 0.00586v^2 + 0.1431v + 0.44$

(e) $\frac{dT}{dv} = 0.01172v + 0.1431$

For $v = 40$, $T'(40) \approx 0.612$.

For $v = 80$, $T'(80) \approx 1.081$.

For $v = 100$, $T'(100) \approx 1.315$.

101. $A = s^2$, $\frac{dA}{ds} = 2s$

When $s = 4 \text{ m}$,

$$\frac{dA}{ds} = 8 \text{ square meters per meter change in } s.$$

105. (a) $f'(1.47)$ is the rate of change of the amount of gasoline sold when the price is \$1.47 per gallon.

(b) $f'(1.47)$ is usually negative. As prices go up, sales go down.

93. $s(t) = -4.9t^2 + v_0t + s_0$

$$= -4.9t^2 + 120t$$

$$v(t) = -9.8t + 120$$

$$v(5) = -9.8(5) + 120 = 71 \text{ m/sec}$$

$$v(10) = -9.8(10) + 120 = 22 \text{ m/sec}$$

97. $v = 40 \text{ mph} = \frac{2}{3} \text{ mi/min}$

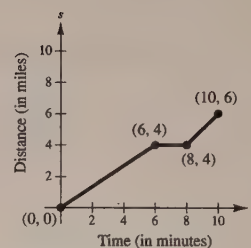
$$\left(\frac{2}{3} \text{ mi/min}\right)(6 \text{ min}) = 4 \text{ mi}$$

$$v = 0 \text{ mph} = 0 \text{ mi/min}$$

$$(0 \text{ mi/min})(2 \text{ min}) = 0 \text{ mi}$$

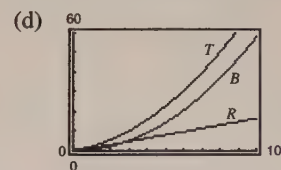
$$v = 60 \text{ mph} = 1 \text{ mi/min}$$

$$(1 \text{ mi/min})(2 \text{ min}) = 2 \text{ mi}$$



(b) Using a graphing utility, you obtain

$$B = 0.00586v^2 - 0.0239v + 0.46.$$



(f) For increasing speeds, the total stopping distance increases.

103. $C = \frac{1,008,000}{Q} + 6.3Q$

$$\frac{dC}{dQ} = -\frac{1,008,000}{Q^2} + 6.3$$

$$C(351) - C(350) \approx 5083.095 - 5085 \approx -\$1.91$$

When $Q = 350$, $\frac{dC}{dQ} \approx -\$1.93$.

107. $y = ax^2 + bx + c$

Since the parabola passes through (0, 1) and (1, 0), we have

$$(0, 1): 1 = a(0)^2 + b(0) + c \Rightarrow c = 1$$

$$(1, 0): 0 = a(1)^2 + b(1) + 1 \Rightarrow b = -a - 1.$$

Thus, $y = ax^2 + (-a - 1)x + 1$. From the tangent line $y = x - 1$, we know that the derivative is 1 at the point (1, 0).

$$y' = 2ax + (-a - 1)$$

$$1 = 2a(1) + (-a - 1)$$

$$1 = a - 1$$

$$a = 2$$

$$b = -a - 1 = -3$$

Therefore, $y = 2x^2 - 3x + 1$.

109. $y = x^3 - 9x$

$$y' = 3x^2 - 9$$

Tangent lines through (1, -9):

$$y + 9 = (3x^2 - 9)(x - 1)$$

$$(x^3 - 9x) + 9 = 3x^3 - 3x^2 - 9x + 9$$

$$0 = 2x^3 - 3x^2 = x^2(2x - 3)$$

$$x = 0 \text{ or } x = \frac{3}{2}$$

The points of tangency are (0, 0) and $(\frac{3}{2}, -\frac{81}{8})$. At (0, 0) the slope is $y'(0) = -9$. At $(\frac{3}{2}, -\frac{81}{8})$ the slope is $y'(\frac{3}{2}) = -\frac{9}{4}$.

Tangent lines:

$$y - 0 = -9(x - 0) \quad \text{and} \quad y + \frac{81}{8} = -\frac{9}{4}(x - \frac{3}{2})$$

$$y = -9x \quad y = -\frac{9}{4}x - \frac{27}{4}$$

$$9x + y = 0$$

$$9x + 4y + 27 = 0$$

111. $f(x) = \begin{cases} ax^3, & x \leq 2 \\ x^2 + b, & x > 2 \end{cases}$

f must be continuous at $x = 2$ to be differentiable at $x = 2$.

$$\left. \begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} ax^3 = 8a \\ \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (x^2 + b) = 4 + b \end{aligned} \right\} \begin{aligned} 8a &= 4 + b \\ 8a - 4 &= b \end{aligned}$$

$$f'(x) = \begin{cases} 3ax^2, & x < 2 \\ 2x, & x > 2 \end{cases}$$

For f to be differentiable at $x = 2$, the left derivative must equal the right derivative.

$$3a(2)^2 = 2(2)$$

$$12a = 4$$

$$a = \frac{1}{3}$$

$$b = 8a - 4 = -\frac{4}{3}$$

113. Let $f(x) = \cos x$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cos x(\cos \Delta x - 1)}{\Delta x} - \lim_{\Delta x \rightarrow 0} \sin x \left(\frac{\sin \Delta x}{\Delta x} \right) \\ &= 0 - \sin x(1) = -\sin x \end{aligned}$$

Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

1. $g(x) = (x^2 + 1)(x^2 - 2x)$

$$\begin{aligned} g'(x) &= (x^2 + 1)(2x - 2) + (x^2 - 2x)(2x) \\ &= 2x^3 - 2x^2 + 2x - 2 + 2x^3 - 4x^2 \\ &= 4x^3 - 6x^2 + 2x - 2 \end{aligned}$$

5. $f(x) = x^3 \cos x$

$$\begin{aligned} f'(x) &= x^3(-\sin x) + \cos x(3x^2) \\ &= 3x^2 \cos x - x^3 \sin x \end{aligned}$$

9. $h(x) = \frac{\sqrt[3]{x}}{x^3 + 1} = \frac{x^{1/3}}{x^3 + 1}$

$$\begin{aligned} h'(x) &= \frac{(x^3 + 1)\frac{1}{3}x^{-2/3} - x^{1/3}(3x^2)}{(x^3 + 1)^2} \\ &= \frac{(x^3 + 1) - x(9x^2)}{3x^{2/3}(x^3 + 1)^2} \\ &= \frac{1 - 8x^3}{3x^{2/3}(x^3 + 1)^2} \end{aligned}$$

13. $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$

$$\begin{aligned} f'(x) &= (x^3 - 3x)(4x + 3) + (2x^2 + 3x + 5)(3x^2 - 3) \\ &= 10x^4 + 12x^3 - 3x^2 - 18x - 15 \\ f'(0) &= -15 \end{aligned}$$

3. $h(t) = \sqrt[3]{t(t^2 + 4)} = t^{1/3}(t^2 + 4)$

$$\begin{aligned} h'(t) &= t^{1/3}(2t) + (t^2 + 4)\frac{1}{3}t^{-2/3} \\ &= 2t^{4/3} + \frac{t^2 + 4}{3t^{2/3}} \\ &= \frac{7t^2 + 4}{3t^{2/3}} \end{aligned}$$

7. $f(x) = \frac{x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

11. $g(x) = \frac{\sin x}{x^2}$

$$g'(x) = \frac{x^2(\cos x) - \sin x(2x)}{(x^2)^2} = \frac{x \cos x - 2 \sin x}{x^3}$$

15. $f(x) = \frac{x^2 - 4}{x - 3}$

$$\begin{aligned} f'(x) &= \frac{(x - 3)(2x) - (x^2 - 4)(1)}{(x - 3)^2} = \frac{2x^2 - 6x - x^2 + 4}{(x - 3)^2} \\ &= \frac{x^2 - 6x + 4}{(x - 3)^2} \end{aligned}$$

$$f'(1) = \frac{1 - 6 + 4}{(1 - 3)^2} = -\frac{1}{4}$$

17. $f(x) = x \cos x$

$$f'(x) = (x)(-\sin x) + (\cos x)(1) = \cos x - x \sin x$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{8}(4 - \pi)$$

<u>Function</u>	<u>Rewrite</u>	<u>Derivative</u>	<u>Simplify</u>
19. $y = \frac{x^2 + 2x}{3}$	$y = \frac{1}{3}x^2 + \frac{2}{3}x$	$y' = \frac{2}{3}x + \frac{2}{3}$	$y' = \frac{2x + 2}{3}$
21. $y = \frac{7}{3x^3}$	$y = \frac{7}{3}x^{-3}$	$y' = -7x^{-4}$	$y' = -\frac{7}{x^4}$
23. $y = \frac{4x^{3/2}}{x}$	$y = 4\sqrt{x}, x > 0$	$y' = 2x^{-1/2}$	$y' = \frac{2}{\sqrt{x}}$
25. $f(x) = \frac{3 - 2x - x^2}{x^2 - 1}$			
	$f'(x) = \frac{(x^2 - 1)(-2 - 2x) - (3 - 2x - x^2)(2x)}{(x^2 - 1)^2}$		
	$= \frac{2x^2 - 4x + 2}{(x^2 - 1)^2} = \frac{2(x - 1)^2}{(x^2 - 1)^2}$		
	$= \frac{2}{(x + 1)^2}, x \neq 1$		
27. $f(x) = x\left(1 - \frac{4}{x + 3}\right) = x - \frac{4x}{x + 3}$		29. $f(x) = \frac{2x + 5}{\sqrt{x}} = 2x^{1/2} + 5x^{-1/2}$	
$f'(x) = 1 - \frac{(x + 3)4 - 4x(1)}{(x + 3)^2} = \frac{(x^2 + 6x + 9) - 12}{(x + 3)^2}$		$f'(x) = x^{-1/2} - \frac{5}{2}x^{-3/2} = x^{-3/2}\left[x - \frac{5}{2}\right]$	
$= \frac{x^2 + 6x - 3}{(x + 3)^2}$		$= \frac{2x - 5}{2x\sqrt{x}} = \frac{2x - 5}{2x^{3/2}}$	
31. $h(s) = (s^3 - 2)^2 = s^6 - 4s^3 + 4$			
$h'(s) = 6s^5 - 12s^2 = 6s^2(s^3 - 2)$			
33. $f(x) = \frac{2 - \frac{1}{x}}{x - 3} = \frac{2x - 1}{x(x - 3)} = \frac{2x - 1}{x^2 - 3x}$			
$f'(x) = \frac{(x^2 - 3x)2 - (2x - 1)(2x - 3)}{(x^2 - 3x)^2} = \frac{2x^2 - 6x - 4x^2 + 8x - 3}{(x^2 - 3x)^2}$			
$= \frac{-2x^2 + 2x - 3}{(x^2 - 3x)^2} = -\frac{2x^2 - 2x + 3}{x^2(x - 3)^2}$			
35. $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$			
$f'(x) = (9x^2 + 4)(x - 5)(x + 1) + (3x^3 + 4x)(1)(x + 1) + (3x^3 + 4x)(x - 5)(1)$			
$= (9x^2 + 4)(x^2 - 4x - 5) + 3x^4 + 3x^3 + 4x^2 + 4x + 3x^4 - 15x^3 + 4x^2 - 20x$			
$= 9x^4 - 36x^3 - 41x^2 - 16x - 20 + 6x^4 - 12x^3 + 8x^2 - 16x$			
$= 15x^4 - 48x^3 - 33x^2 - 32x - 20$			
37. $f(x) = \frac{x^2 + c^2}{x^2 - c^2}$		39. $f(t) = t^2 \sin t$	
$f'(x) = \frac{(x^2 - c^2)(2x) - (x^2 + c^2)(2x)}{(x^2 - c^2)^2}$		$f'(t) = t^2 \cos t + 2t \sin t$	
$= \frac{-4xc^2}{(x^2 - c^2)^2}$		$= t(t \cos t + 2 \sin t)$	

$$41. f(t) = \frac{\cos t}{t}$$

$$f'(t) = \frac{-t \sin t - \cos t}{t^2} = -\frac{t \sin t + \cos t}{t^2}$$

$$45. g(t) = \sqrt[4]{t} + 8 \sec t = t^{1/4} + 8 \sec t$$

$$g'(t) = \frac{1}{4}t^{-3/4} + 8 \sec t \tan t = \frac{1}{4t^{3/4}} + 8 \sec t \tan t$$

$$49. y = -\csc x - \sin x$$

$$y' = \csc x \cot x - \cos x$$

$$= \frac{\cos x}{\sin^2 x} - \cos x$$

$$= \cos x(\csc^2 x - 1)$$

$$= \cos x \cot^2 x$$

$$53. y = 2x \sin x + x^2 \cos x$$

$$y' = 2x \cos x + 2 \sin x + x^2(-\sin x) + 2x \cos x$$

$$= 4x \cos x + 2 \sin x - x^2 \sin x$$

$$57. g(\theta) = \frac{\theta}{1 - \sin \theta}$$

$$g'(\theta) = \frac{1 - \sin \theta + \theta \cos \theta}{(\sin \theta - 1)^2} \quad (\text{form of answer may vary})$$

$$59. y = \frac{1 + \csc x}{1 - \csc x}$$

$$y' = \frac{(1 - \csc x)(-\csc x \cot x) - (1 + \csc x)(\csc x \cot x)}{(1 - \csc x)^2} = \frac{-2 \csc x \cot x}{(1 - \csc x)^2}$$

$$y'\left(\frac{\pi}{6}\right) = \frac{-2(2)(\sqrt{3})}{(1 - 2)^2} = -4\sqrt{3}$$

$$61. h(t) = \frac{\sec t}{t}$$

$$h'(t) = \frac{t(\sec t \tan t) - (\sec t)(1)}{t^2}$$

$$= \frac{\sec t(t \tan t - 1)}{t^2}$$

$$h'(\pi) = \frac{\sec \pi(\pi \tan \pi - 1)}{\pi^2} = \frac{1}{\pi^2}$$

$$43. f(x) = -x + \tan x$$

$$f'(x) = -1 + \sec^2 x = \tan^2 x$$

$$47. y = \frac{3(1 - \sin x)}{2 \cos x} = \frac{3}{2}(\sec x - \tan x)$$

$$y' = \frac{3}{2}(\sec x \tan x - \sec^2 x) = \frac{3}{2}\sec x(\tan x - \sec x)$$

$$= \frac{3}{2}(\sec x \tan x - \tan^2 x - 1)$$

$$51. f(x) = x^2 \tan x$$

$$f'(x) = x^2 \sec^2 x + 2x \tan x$$

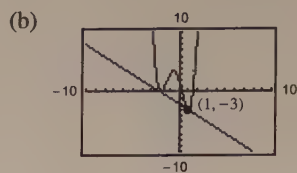
$$= x(x \sec^2 x + 2 \tan x)$$

$$55. g(x) = \left(\frac{x+1}{x+2}\right)(2x-5)$$

$$g'(x) = \frac{2x^2 + 8x - 1}{(x+2)^2} \quad (\text{form of answer may vary})$$

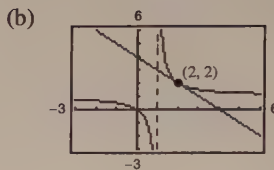
63. (a) $f(x) = (x^3 - 3x + 1)(x + 2)$, $(1, -3)$
 $f'(x) = (x^3 - 3x + 1)(1) + (x + 2)(3x^2 - 3)$
 $= 4x^3 + 6x^2 - 6x - 5$
 $f'(1) = -1 = \text{slope at } (1, -3).$

Tangent line: $y + 3 = -1(x - 1) \Rightarrow y = -x - 2$



65. (a) $f(x) = \frac{x}{x-1}$, $(2, 2)$
 $f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$
 $f'(2) = \frac{-1}{(2-1)^2} = -1 = \text{slope at } (2, 2).$

Tangent line: $y - 2 = -1(x - 2) \Rightarrow y = -x + 4$



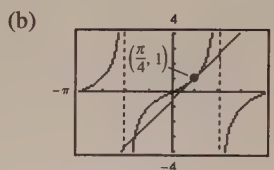
67. (a) $f(x) = \tan x$, $\left(\frac{\pi}{4}, 1\right)$
 $f'(x) = \sec^2 x$
 $f'\left(\frac{\pi}{4}\right) = 2 = \text{slope at } \left(\frac{\pi}{4}, 1\right).$

Tangent line:

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y - 1 = 2x - \frac{\pi}{2}$$

$$4x - 2y - \pi + 2 = 0$$



69. $f(x) = \frac{x^2}{x-1}$
 $f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2}$
 $= \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$
 $f'(x) = 0 \text{ when } x = 0 \text{ or } x = 2.$

Horizontal tangents are at $(0, 0)$ and $(2, 4)$.

71. $f'(x) = \frac{(x+2)3 - 3x(1)}{(x+2)^2} = \frac{6}{(x+2)^2}$
 $g'(x) = \frac{(x+2)5 - (5x+4)(1)}{(x+2)^2} = \frac{6}{(x+2)^2}$
 $g(x) = \frac{5x+4}{(x+2)} = \frac{3x}{(x+2)} + \frac{2x+4}{(x+2)} = f(x) + 2$
 f and g differ by a constant.

73. $f(x) = x^n \sin x$
 $f'(x) = x^n \cos x + nx^{n-1} \sin x$
 $= x^{n-1}(x \cos x + n \sin x)$

When $n = 1$: $f'(x) = x \cos x + \sin x$.

When $n = 2$: $f'(x) = x(x \cos x + 2 \sin x)$.

When $n = 3$: $f'(x) = x^2(x \cos x + 3 \sin x)$.

When $n = 4$: $f'(x) = x^3(x \cos x + 4 \sin x)$.

For general n , $f'(x) = x^{n-1}(x \cos x + n \sin x)$.

75. Area = $A(t) = (2t + 1)\sqrt{t} = 2t^{3/2} + t^{1/2}$
 $A'(t) = 2\left(\frac{3}{2}t^{1/2}\right) + \frac{1}{2}t^{-1/2}$
 $= 3t^{1/2} + \frac{1}{2}t^{-1/2}$
 $= \frac{6t + 1}{2\sqrt{t}} \text{ cm}^2/\text{sec}$

$$77. C = 100\left(\frac{200}{x^2} + \frac{x}{x+30}\right), \quad 1 \leq x$$

$$\frac{dC}{dx} = 100\left(-\frac{400}{x^3} + \frac{30}{(x+30)^2}\right)$$

$$(a) \text{ When } x = 10: \frac{dC}{dx} = -\$38.13.$$

$$(b) \text{ When } x = 15: \frac{dC}{dx} = -\$10.37.$$

$$(c) \text{ When } x = 20: \frac{dC}{dx} = -\$3.80.$$

As the order size increases, the cost per item decreases.

$$81. (a) \quad \sec x = \frac{1}{\cos x}$$

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos x \cos x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

$$(b) \quad \csc x = \frac{1}{\sin x}$$

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = -\frac{\cos x}{\sin x \sin x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

$$(c) \quad \cot x = \frac{\cos x}{\sin x}$$

$$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$83. f(x) = 4x^{3/2}$$

$$f'(x) = 6x^{1/2}$$

$$f''(x) = 3x^{-1/2} = \frac{3}{\sqrt{x}}$$

$$79. P(t) = 500\left[1 + \frac{4t}{50 + t^2}\right]$$

$$P'(t) = 500\left[\frac{(50 + t^2)(4) - (4t)(2t)}{(50 + t^2)^2}\right]$$

$$= 500\left[\frac{200 - 4t^2}{(50 + t^2)^2}\right]$$

$$= 2000\left[\frac{50 - t^2}{(50 + t^2)^2}\right]$$

$$P'(2) \approx 31.55 \text{ bacteria per hour}$$

$$83. f(x) = 4x^{3/2}$$

$$f'(x) = 6x^{1/2}$$

$$f''(x) = 3x^{-1/2} = \frac{3}{\sqrt{x}}$$

$$85. f(x) = \frac{x}{x-1}$$

$$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$$f''(x) = \frac{2}{(x-1)^3}$$

$$87. f(x) = 3 \sin x$$

$$f'(x) = 3 \cos x$$

$$f''(x) = -3 \sin x$$

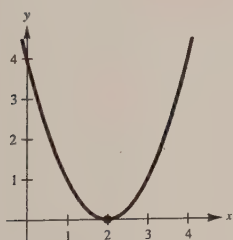
$$89. f'(x) = x^2$$

$$f''(x) = 2x$$

$$91. f'''(x) = 2\sqrt{x}$$

$$f^{(4)}(x) = \frac{1}{2}(2)x^{-1/2} = \frac{1}{\sqrt{x}}$$

93.



$$f(2) = 0$$

One such function is $f(x) = (x-2)^2$.

$$95. f(x) = 2g(x) + h(x)$$

$$f'(x) = 2g'(x) + h'(x)$$

$$f'(2) = 2g'(2) + h'(2)$$

$$= 2(-2) + 4$$

$$= 0$$

$$97. f(x) = \frac{g(x)}{h(x)}$$

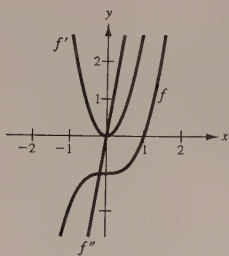
$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(2) = \frac{h(2)g'(2) - g(2)h'(2)}{[h(2)]^2}$$

$$= \frac{(-1)(-2) - (3)(4)}{(-1)^2}$$

$$= -10$$

99.



It appears that f is cubic; so f' would be quadratic and f'' would be linear.

$$103. v(t) = \frac{100t}{2t + 15}$$

$$\begin{aligned} a(t) &= \frac{(2t + 15)(100) - (100t)(2)}{(2t + 15)^2} \\ &= \frac{1500}{(2t + 15)^2} \end{aligned}$$

$$(a) a(5) = \frac{1500}{[2(5) + 15]^2} = 2.4 \text{ ft/sec}^2$$

$$(b) a(10) = \frac{1500}{[2(10) + 15]^2} \approx 1.2 \text{ ft/sec}^2$$

$$(c) a(20) = \frac{1500}{[2(20) + 15]^2} \approx 0.5 \text{ ft/sec}^2$$

$$105. f(x) = g(x)h(x)$$

$$(a) f'(x) = g(x)h'(x) + h(x)g'(x)$$

$$\begin{aligned} f''(x) &= g(x)h''(x) + g'(x)h'(x) + h(x)g''(x) + h'(x)g'(x) \\ &= g(x)h''(x) + 2g'(x)h'(x) + h(x)g''(x) \end{aligned}$$

$$\begin{aligned} f'''(x) &= g(x)h'''(x) + g''(x)h''(x) + 2g'(x)h''(x) + 2g''(x)h'(x) + h(x)g'''(x) + h'(x)g''(x) \\ &= g(x)h'''(x) + 3g''(x)h''(x) + 3g'(x)h''(x) + g'''(x)h(x) \end{aligned}$$

$$\begin{aligned} f^{(4)}(x) &= g(x)h^{(4)}(x) + g'''(x)h'''(x) + 3g''(x)h'''(x) + 3g''(x)h''(x) + 3g'(x)h'''(x) + 3g'''(x)h'(x) \\ &\quad + g'''(x)h'(x) + g^{(4)}(x)h(x) \\ &= g(x)h^{(4)}(x) + 4g''(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x) \end{aligned}$$

$$\begin{aligned} (b) f^{(n)}(x) &= g(x)h^{(n)}(x) + \frac{n(n-1)(n-2) \cdots (2)(1)}{1[(n-1)(n-2) \cdots (2)(1)]} g'(x)h^{(n-1)}(x) + \frac{n(n-1)(n-2) \cdots (2)(1)}{(2)(1)[(n-2)(n-3) \cdots (2)(1)]} g''(x)h^{(n-2)}(x) \\ &\quad + \frac{n(n-1)(n-2) \cdots (2)(1)}{(3)(2)(1)[(n-3)(n-4) \cdots (2)(1)]} g'''(x)h^{(n-3)}(x) + \cdots \\ &\quad + \frac{n(n-1)(n-2) \cdots (2)(1)}{[(n-1)(n-2) \cdots (2)(1)](1)} g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x) \\ &= g(x)h^{(n)}(x) + \frac{n!}{1!(n-1)!} g'(x)h^{(n-1)}(x) + \frac{n!}{2!(n-2)!} g''(x)h^{(n-2)}(x) + \cdots \\ &\quad + \frac{n!}{(n-1)!1!} g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x) \end{aligned}$$

Note: $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ (read “ n factorial.”)

$$101. v(t) = 36 - t^2, 0 \leq t \leq 6$$

$$a(t) = -2t$$

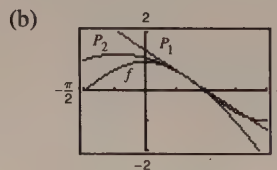
$$v(3) = 27 \text{ m/sec}$$

$$a(3) = -6 \text{ m/sec}^2$$

The speed of the object is decreasing.

$$\begin{aligned}
 107. \quad f(x) &= \cos x & f\left(\frac{\pi}{3}\right) &= \cos \frac{\pi}{3} = \frac{1}{2} \\
 f'(x) &= -\sin x & f'\left(\frac{\pi}{3}\right) &= -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2} \\
 f''(x) &= -\cos x & f''\left(\frac{\pi}{3}\right) &= -\cos \frac{\pi}{3} = -\frac{1}{2} \\
 (a) \quad P_1(x) &= f'(a)(x-a) + f(a) = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2} \\
 P_2(x) &= \frac{1}{2}f''(a)(x-a)^2 + f'(a)(x-a) + f(a) \\
 &= -\frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}
 \end{aligned}$$

(c) P_2 is a better approximation.



(d) The accuracy worsens as you move farther away from $x = a = (\pi/3)$.

109. False. If $y = f(x)g(x)$, then

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x).$$

111. True

$$\begin{aligned}
 h'(c) &= f(c)g'(c) + g(c)f'(c) \\
 &= f(c)(0) + g(c)(0) \\
 &= 0
 \end{aligned}$$

113. True

$$115. \quad f(x) = x|x| = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ -2x, & \text{if } x < 0 \end{cases} = 2|x|$$

$$f''(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0 \end{cases}$$

$f''(0)$ does not exist since the left and right derivatives are not equal.

Section 2.4 The Chain Rule

$$\underline{y = f(g(x))}$$

$$\underline{u = g(x)}$$

$$\underline{y = f(u)}$$

$$1. \quad y = (6x - 5)^4$$

$$u = 6x - 5$$

$$y = u^4$$

$$3. \quad y = \sqrt{x^2 - 1}$$

$$u = x^2 - 1$$

$$y = \sqrt{u}$$

$$5. \quad y = \csc^3 x$$

$$u = \csc x$$

$$y = u^3$$

$$7. \quad y = (2x - 7)^3$$

$$y' = 3(2x - 7)^2(2) = 6(2x - 7)^2$$

$$9. \quad g(x) = 3(4 - 9x)^4$$

$$g'(x) = 12(4 - 9x)^3(-9) = -108(4 - 9x)^3$$

$$11. \quad f(x) = (9 - x^2)^{2/3}$$

$$f'(x) = \frac{2}{3}(9 - x^2)^{-1/3}(-2x) = -\frac{4x}{3(9 - x^2)^{1/3}}$$

$$13. \quad f(t) = (1 - t)^{1/2}$$

$$f'(t) = \frac{1}{2}(1 - t)^{-1/2}(-1) = -\frac{1}{2\sqrt{1 - t}}$$

15. $y = (9x^2 + 4)^{1/3}$

$$y' = \frac{1}{3}(9x^2 + 4)^{-2/3}(18x) = \frac{6x}{(9x^2 + 4)^{2/3}}$$

19. $y = (x - 2)^{-1}$

$$y' = -1(x - 2)^{-2}(1) = \frac{-1}{(x - 2)^2}$$

23. $y = (x + 2)^{-1/2}$

$$\frac{dy}{dx} = -\frac{1}{2}(x + 2)^{-3/2} = -\frac{1}{2(x + 2)^{3/2}}$$

27. $y = x\sqrt{1 - x^2} = x(1 - x^2)^{1/2}$

$$\begin{aligned} y' &= x\left[\frac{1}{2}(1 - x^2)^{-1/2}(-2x)\right] + (1 - x^2)^{1/2}(1) \\ &= -x^2(1 - x^2)^{-1/2} + (1 - x^2)^{1/2} \\ &= (1 - x^2)^{-1/2}[-x^2 + (1 - x^2)] \\ &= \frac{1 - 2x^2}{\sqrt{1 - x^2}} \end{aligned}$$

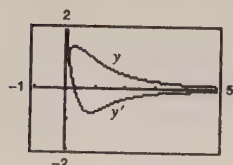
31. $g(x) = \left(\frac{x + 5}{x^2 + 2}\right)^2$

$$\begin{aligned} g'(x) &= 2\left(\frac{x + 5}{x^2 + 2}\right)\left(\frac{(x^2 + 2) - (x + 5)(2x)}{(x^2 + 2)^2}\right) \\ &= \frac{2(x + 5)(2 - 10x - x^2)}{(x^2 + 2)^3} \end{aligned}$$

35. $y = \frac{\sqrt{x} + 1}{x^2 + 1}$

$$y' = \frac{1 - 3x^2 - 4x^{3/2}}{2\sqrt{x}(x^2 + 1)^2}$$

The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



17. $y = 2(4 - x^2)^{1/4}$

$$\begin{aligned} y' &= 2\left(\frac{1}{4}\right)(4 - x^2)^{-3/4}(-2x) \\ &= \frac{-x}{\sqrt[4]{(4 - x^2)^3}} \end{aligned}$$

21. $f(t) = (t - 3)^{-2}$

$$f'(t) = -2(t - 3)^{-3} = \frac{-2}{(t - 3)^3}$$

25. $f(x) = x^2(x - 2)^4$

$$\begin{aligned} f'(x) &= x^2[4(x - 2)^3(1)] + (x - 2)^4(2x) \\ &= 2x(x - 2)^3[2x + (x - 2)] \\ &= 2x(x - 2)^3(3x - 2) \end{aligned}$$

29. $y = \frac{x}{\sqrt{x^2 + 1}} = x(x^2 + 1)^{-1/2}$

$$\begin{aligned} y' &= x\left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x)\right] + (x^2 + 1)^{-1/2}(1) \\ &= -x^2(x^2 + 1)^{-3/2} + (x^2 + 1)^{-1/2} \\ &= (x^2 + 1)^{-3/2}[-x^2 + (x^2 + 1)] \\ &= \frac{1}{(x^2 + 1)^{3/2}} \end{aligned}$$

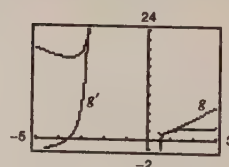
33. $f(v) = \left(\frac{1 - 2v}{1 + v}\right)^3$

$$\begin{aligned} f'(v) &= 3\left(\frac{1 - 2v}{1 + v}\right)^2\left(\frac{(1 + v)(-2) - (1 - 2v)}{(1 + v)^2}\right) \\ &= \frac{-9(1 - 2v)^2}{(1 + v)^4} \end{aligned}$$

37. $g(t) = \frac{3t^2}{\sqrt{t^2 + 2t - 1}}$

$$g'(t) = \frac{3t(t^2 + 3t - 2)}{(t^2 + 2t - 1)^{3/2}}$$

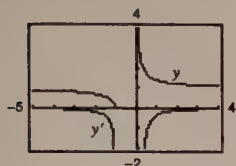
The zeros of g' correspond to the points on the graph of g where the tangent lines are horizontal.



$$39. y = \sqrt{\frac{x+1}{x}}$$

$$y' = -\frac{\sqrt{(x+1)/x}}{2x(x+1)}$$

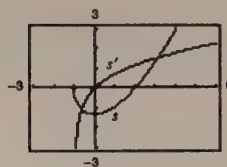
y' has no zeros.



$$41. s(t) = \frac{-2(2-t)\sqrt{1+t}}{3}$$

$$s'(t) = \frac{t}{\sqrt{1+t}}$$

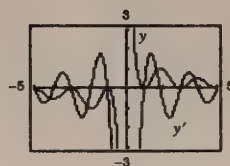
The zero of $s'(t)$ corresponds to the point on the graph of $s(t)$ where the tangent line is horizontal.



$$43. y = \frac{\cos \pi x + 1}{x}$$

$$\frac{dy}{dx} = \frac{-\pi x \sin \pi x - \cos \pi x - 1}{x^2}$$

$$= -\frac{\pi x \sin \pi x + \cos \pi x + 1}{x^2}$$



The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.

$$45. (a) y = \sin x$$

$$y' = \cos x$$

$$y'(0) = 1$$

1 cycle in $[0, 2\pi]$

$$(b) y = \sin 2x$$

$$y' = 2 \cos 2x$$

$$y'(0) = 2$$

2 cycles in $[0, 2\pi]$

The slope of $\sin ax$ at the origin is a .

$$47. y = \cos 3x$$

$$\frac{dy}{dx} = -3 \sin 3x$$

$$49. g(x) = 3 \tan 4x$$

$$g'(x) = 12 \sec^2 4x$$

$$51. y = \sin(\pi x)^2 = \sin(\pi^2 x^2)$$

$$y' = \cos(\pi^2 x^2)[2\pi^2 x] = 2\pi^2 x \cos(\pi^2 x^2)$$

$$53. h(x) = \sin 2x \cos 2x$$

$$h'(x) = \sin 2x(-2 \sin 2x) + \cos 2x(2 \cos 2x)$$

$$= 2 \cos^2 2x - 2 \sin^2 2x$$

$$= 2 \cos 4x.$$

Alternate solution: $h(x) = \frac{1}{2} \sin 4x$

$$h'(x) = \frac{1}{2} \cos 4x(4) = 2 \cos 4x$$

$$55. f(x) = \frac{\cot x}{\sin x} = \frac{\cos x}{\sin^2 x}$$

$$f'(x) = \frac{\sin^2 x(-\sin x) - \cos x(2 \sin x \cos x)}{\sin^4 x}$$

$$= \frac{-\sin^2 x - 2 \cos^2 x}{\sin^3 x} = \frac{-1 - \cos^2 x}{\sin^3 x}$$

57. $y = 4 \sec^2 x$

$$y' = 8 \sec x \cdot \sec x \tan x = 8 \sec^2 x \tan x$$

61. $f(x) = 3 \sec^2(\pi t - 1)$

$$\begin{aligned} f'(t) &= 6 \sec(\pi t - 1) \sec(\pi t - 1) \tan(\pi t - 1)(\pi) \\ &= 6\pi \sec^2(\pi t - 1) \tan(\pi t - 1) = \frac{6\pi \sin(\pi t - 1)}{\cos^3(\pi t - 1)} \end{aligned}$$

65. $y = \sin(\cos x)$

$$\begin{aligned} \frac{dy}{dx} &= \cos(\cos x) \cdot (-\sin x) \\ &= -\sin x \cos(\cos x) \end{aligned}$$

69. $f(x) = \frac{3}{x^3 - 4} = 3(x^3 - 4)^{-1}, \left(-1, -\frac{3}{5}\right)$

$$f'(x) = -3(x^3 - 4)^{-2}(3x^2) = -\frac{9x^2}{(x^3 - 4)^2}$$

$$f'(-1) = -\frac{9}{25}$$

73. $y = 37 - \sec^3(2x), (0, 36)$

$$\begin{aligned} y' &= -3 \sec^2(2x)[2 \sec(2x) \tan(2x)] \\ &= -6 \sec^3(2x) \tan(2x) \\ y'(0) &= 0 \end{aligned}$$

75. (a) $f(x) = \sqrt{3x^2 - 2}, (3, 5)$

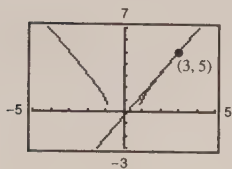
$$\begin{aligned} f'(x) &= \frac{1}{2}(3x^2 - 2)^{-1/2}(6x) \\ &= \frac{3x}{\sqrt{3x^2 - 2}} \end{aligned}$$

$$f'(3) = \frac{9}{5}$$

Tangent line:

$$y - 5 = \frac{9}{5}(x - 3) \Rightarrow 9x - 5y - 2 = 0$$

(b)



59. $f(\theta) = \frac{1}{4} \sin^2 2\theta = \frac{1}{4}(\sin 2\theta)^2$

$$\begin{aligned} f'(\theta) &= 2\left(\frac{1}{4}\right)(\sin 2\theta)(\cos 2\theta)(2) \\ &= \sin 2\theta \cos 2\theta = \frac{1}{2} \sin 4\theta \end{aligned}$$

63. $y = \sqrt{x} + \frac{1}{4} \sin(2x)^2$

$$= \sqrt{x} + \frac{1}{4} \sin(4x^2)$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} + \frac{1}{4} \cos(4x^2)(8x)$$

$$= \frac{1}{2\sqrt{x}} + 2x \cos(2x)^2$$

67. $s(t) = (t^2 + 2t + 8)^{1/2}, (2, 4)$

$$s'(t) = \frac{1}{2}(t^2 + 2t + 8)^{-1/2}(2t + 2)$$

$$= \frac{t + 1}{\sqrt{t^2 + 2t + 8}}$$

$$s'(2) = \frac{3}{4}$$

71. $f(t) = \frac{3t + 2}{t - 1}, (0, -2)$

$$f'(t) = \frac{(t - 1)(3) - (3t + 2)(1)}{(t - 1)^2} = \frac{-5}{(t - 1)^2}$$

$$f'(0) = -5$$

77. (a) $f(x) = \sin 2x, (\pi, 0)$

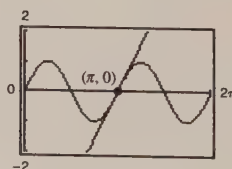
$$f'(x) = 2 \cos 2x$$

$$f'(\pi) = 2$$

Tangent line:

$$y = 2(x - \pi) \Rightarrow 2x - y - 2\pi = 0$$

(b)



79. $f(x) = 2(x^2 - 1)^3$

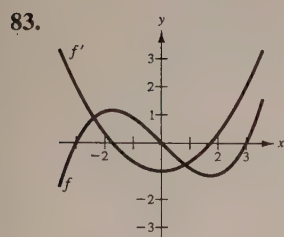
$$f'(x) = 6(x^2 - 1)^2(2x)$$

$$= 12x(x^4 - 2x^2 + 1)$$

$$= 12x^5 - 24x^3 + 12x$$

$$f''(x) = 60x^4 - 72x^2 + 12$$

$$= 12(5x^2 - 1)(x^2 - 1)$$



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

87. $g(x) = f(3x)$

$$g'(x) = f'(3x)(3) \Rightarrow g'(x) = 3f'(3x)$$

89. (a) $f(x) = g(x)h(x)$

$$f'(x) = g(x)h'(x) + g'(x)h(x)$$

$$f'(5) = (-3)(-2) + (6)(3) = 24$$

(c) $f(x) = \frac{g(x)}{h(x)}$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(5) = \frac{(3)(6) - (-3)(-2)}{(3)^2} = \frac{12}{9} = \frac{4}{3}$$

91. (a) $F = 132,400(331 - v)^{-1}$

$$F' = (-1)(132,400)(331 - v)^{-2}(-1)$$

$$= \frac{132,400}{(331 - v)^2}$$

$$\text{When } v = 30, F' \approx 1.461.$$

93. $\theta = 0.2 \cos 8t$

The maximum angular displacement is $\theta = 0.2$ (since $-1 \leq \cos 8t \leq 1$).

$$\frac{d\theta}{dt} = 0.2[-8 \sin 8t] = -1.6 \sin 8t$$

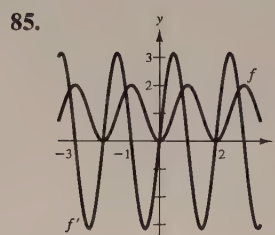
When $t = 3$, $d\theta/dt = -1.6 \sin 24 \approx 1.4489$ radians per second.

81. $f(x) = \sin x^2$

$$f'(x) = 2x \cos x^2$$

$$f''(x) = 2x[2x(-\sin x^2)] + 2 \cos x^2$$

$$= 2[\cos x^2 - 2x^2 \sin x^2]$$



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

(b) $f(x) = g(h(x))$

$$f'(x) = g'(h(x))h'(x)$$

$$f'(5) = g'(3)(-2) = -2g'(3)$$

Need $g'(3)$ to find $f'(5)$.

(d) $f(x) = [g(x)]^3$

$$f'(x) = 3[g(x)]^2 g'(x)$$

$$f'(5) = 3(-3)^2(6) = 162$$

(b) $F = 132,400(331 + v)^{-1}$

$$F' = (-1)(132,400)(331 + v)^{-2}(1)$$

$$= \frac{-132,400}{(331 + v)^2}$$

$$\text{When } v = 30, F' \approx -1.016.$$

95. $S = C(R^2 - r^2)$

$$\frac{dS}{dt} = C\left(2R \frac{dR}{dt} - 2r \frac{dr}{dt}\right)$$

Since r is constant, we have $dr/dt = 0$ and

$$\frac{dS}{dt} = (1.76 \times 10^5)(2)(1.2 \times 10^{-2})(10^{-5})$$

$$= 4.224 \times 10^{-2} = 0.04224.$$

97. (a) $x = -1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6162$

(b) $C = 60x + 1350$

$$= 60(-1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6162) + 1350$$

$$\frac{dC}{dt} = 60(-4.9116t^2 + 38.624t - 0.5082)$$

$$= -294.696t^2 + 2317.44t - 30.492$$

The function $\frac{dC}{dt}$ is quadratic, not linear. The cost function levels off at the end of the day, perhaps due to fatigue.

99. $f(x) = \sin \beta x$

(a) $f'(x) = \beta \cos \beta x$

$$f''(x) = -\beta^2 \sin \beta x$$

$$f'''(x) = -\beta^3 \cos \beta x$$

$$f^{(4)}(x) = \beta^4 \sin \beta x$$

(b) $f''(x) + \beta^2 f(x) = -\beta^2 \sin \beta x + \beta^2 (\sin \beta x) = 0$

(c) $f^{(2k)}(x) = (-1)^k \beta^{2k} \sin \beta x$

$$f^{(2k-1)}(x) = (-1)^{k+1} \beta^{2k-1} \cos \beta x$$

101. (a) $r'(x) = f'(g(x))g'(x)$

$$r'(1) = f'(g(1))g'(1)$$

$$\text{Note that } g(1) = 4 \text{ and } f'(4) = \frac{5-0}{6-2} = \frac{5}{4}$$

$$\text{Also, } g'(1) = 0. \text{ Thus, } r'(1) = 0$$

(b) $s'(x) = g'(f(x))f'(x)$

$$s'(4) = g'(f(4))f'(4)$$

$$\text{Note that } f(4) = \frac{5}{2}, g'\left(\frac{5}{2}\right) = \frac{6-4}{6-2} = \frac{1}{2} \text{ and}$$

$$f'(4) = \frac{5}{4}.$$

$$\text{Thus, } s'(4) = \frac{1}{2} \left(\frac{5}{4} \right) = \frac{5}{8}.$$

103. $g = \sqrt{x(x+n)}$

$$= \sqrt{x^2 + nx}$$

$$\frac{dg}{dx} = \frac{1}{2}(x^2 + nx)^{-1/2}(2x + n)$$

$$= \frac{2x + n}{2\sqrt{x^2 + nx}}$$

$$= \frac{(2x + n)/2}{\sqrt{x(x+n)}}$$

$$= \frac{[x + (x + n)]/2}{\sqrt{x(x+n)}}$$

$$= \frac{a}{g}$$

105. $g(x) = |2x - 3|$

$$g'(x) = 2 \left(\frac{2x - 3}{|2x - 3|} \right), \quad x \neq \frac{3}{2}$$

107. $h(x) = |x| \cos x$

$$h'(x) = -|x| \sin x + \frac{x}{|x|} \cos x, \quad x \neq 0$$

109. (a) $f(x) = \tan \frac{\pi x}{4}$

$f(1) = 1$

$f'(x) = \frac{\pi}{4} \sec^2 \frac{\pi x}{4}$

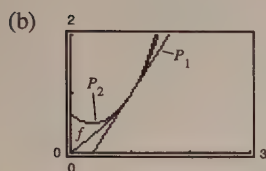
$f'(1) = \frac{\pi}{4}(2) = \frac{\pi}{2}$

$f''(x) = \frac{\pi}{2} \sec^2 \frac{\pi x}{4} \cdot \tan \frac{\pi x}{4} \left(\frac{\pi}{4} \right)$

$f''(1) = \frac{\pi^2}{8}(2)(1) = \frac{\pi^2}{4}$

$P_1(x) = f'(1)(x-1) + f(1) = \frac{\pi}{2}(x-1) + 1$

$P_2(x) = \frac{1}{2} \left(\frac{\pi^2}{4} \right) (x-1)^2 + f'(1)(x-1) + f(1) = \frac{\pi^2}{8}(x-1)^2 + \frac{\pi}{2}(x-1) + 1$

(c) P_2 is a better approximation than P_1 (d) The accuracy worsens as you move away from $x = c = 1$.

111. False. If $y = (1-x)^{1/2}$, then $y' = \frac{1}{2}(1-x)^{-1/2}(-1)$.

113. True

Section 2.5 Implicit Differentiation

1. $x^2 + y^2 = 36$

$2x + 2yy' = 0$

$y' = \frac{-x}{y}$

3. $x^{1/2} + y^{1/2} = 9$

$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$

$y' = -\frac{x^{-1/2}}{y^{-1/2}} = -\sqrt{\frac{y}{x}}$

5. $x^3 - xy + y^2 = 4$

$3x^2 - xy' - y + 2yy' = 0$

$(2y - x)y' = y - 3x^2$

$y' = \frac{y - 3x^2}{2y - x}$

7. $x^3y^3 - y - x = 0$

$3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$

$(3x^3y^2 - 1)y' = 1 - 3x^2y^3$

$y' = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$

9. $x^3 - 3x^2y + 2xy^2 = 12$

$3x^2 - 3x^2y' - 6xy + 4xyy' + 2y^2 = 0$

$(4xy - 3x^2)y' = 6xy - 3x^2 - 2y^2$

$y' = \frac{6xy - 3x^2 - 2y^2}{4xy - 3x^2}$

11. $\sin x + 2\cos 2y = 1$

$\cos x - 4(\sin 2y)y' = 0$

$y' = \frac{\cos x}{4 \sin 2y}$

13. $\sin x = x(1 + \tan y)$

$$\cos x = x(\sec^2 y)y' + (1 + \tan y)(1)$$

$$y' = \frac{\cos x - \tan y - 1}{x \sec^2 y}$$

17. (a) $x^2 + y^2 = 16$

$$y^2 = 16 - x^2$$

$$y = \pm \sqrt{16 - x^2}$$

(c) Explicitly:

$$\begin{aligned} \frac{dy}{dx} &= \pm \frac{1}{2}(16 - x^2)^{-1/2}(-2x) \\ &= \frac{\mp x}{\sqrt{16 - x^2}} = \frac{-x}{\pm \sqrt{16 - x^2}} = \frac{-x}{y} \end{aligned}$$

19. (a) $16y^2 = 144 - 9x^2$

$$y^2 = \frac{1}{16}(144 - 9x^2) = \frac{9}{16}(16 - x^2)$$

$$y = \pm \frac{3}{4}\sqrt{16 - x^2}$$

(c) Explicitly:

$$\begin{aligned} \frac{dy}{dx} &= \pm \frac{3}{8}(16 - x^2)^{-1/2}(-2x) \\ &= \mp \frac{3x}{4\sqrt{16 - x^2}} = \frac{-3x}{4(4/3)y} = \frac{-9x}{16y} \end{aligned}$$

21. $xy = 4$

$$xy' + y(1) = 0$$

$$xy' = -y$$

$$y' = \frac{-y}{x}$$

$$\text{At } (-4, -1): y' = -\frac{1}{4}$$

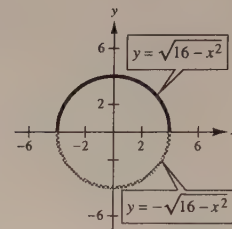
15. $y = \sin(xy)$

$$y' = [xy' + y] \cos(xy)$$

$$y' - x \cos(xy)y' = y \cos(xy)$$

$$y' = \frac{y \cos(xy)}{1 - x \cos(xy)}$$

(b)

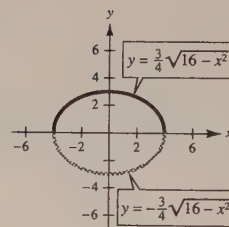


(d) Implicitly:

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

(b)



(d) Implicitly:

$$18x + 32yy' = 0$$

$$y' = \frac{-9x}{16y}$$

23. $y^2 = \frac{x^2 - 4}{x^2 + 4}$

$$2yy' = \frac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2}$$

$$2yy' = \frac{16x}{(x^2 + 4)^2}$$

$$y' = \frac{8x}{y(x^2 + 4)^2}$$

At $(2, 0)$, y' is undefined.

25. $x^{2/3} + y^{2/3} = 5$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}}$$

At (8, 1): $y' = -\frac{1}{2}$.

29. $(x^2 + 4)y = 8$

$$(x^2 + 4)y' + y(2x) = 0$$

$$y' = \frac{-2xy}{x^2 + 4}$$

$$= \frac{-2x[8/(x^2 + 4)]}{x^2 + 4}$$

$$= \frac{-16x}{(x^2 + 4)^2}$$

At (2, 1): $y' = \frac{-32}{64} = -\frac{1}{2}$

(Or, you could just solve for y : $y = \frac{8}{x^2 + 4}$)

33. $\tan y = x$

$$y'\sec^2 y = 1$$

$$y' = \frac{1}{\sec^2 y} = \cos^2 y, -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$y' = \frac{1}{1 + x^2}$$

37. $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

$$x - yy' = 0$$

$$1 - yy'' - (y')^2 = 0$$

$$1 - yy'' - \left(\frac{x}{y}\right)^2 = 0$$

$$y^2 - y^3y'' = x^2$$

$$y'' = \frac{y^2 - x^2}{y^3} = \frac{-16}{y^3}$$

27. $\tan(x + y) = x$

$$(1 + y') \sec^2(x + y) = 1$$

$$\begin{aligned} y' &= \frac{1 - \sec^2(x + y)}{\sec^2(x + y)} \\ &= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1} = -\sin^2(x + y) \\ &= -\frac{x^2}{x^2 + 1} \end{aligned}$$

At (0, 0): $y' = 0$.

31. $(x^2 + y^2)^2 = 4x^2y$

$$2(x^2 + y^2)(2x + 2yy') = 4x^2y' + y(8x)$$

$$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy$$

$$4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2$$

$$4y'(x^2y + y^3 - x^2) = 4(2xy - x^3 - xy^2)$$

$$y' = \frac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}$$

At (1, 1): $y' = 0$.

35. $x^2 + y^2 = 36$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

$$y'' = \frac{y(-1) + xy'}{y^2} = \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} = \frac{-y^2 - x^2}{y^3} = \frac{-36}{y^3}$$

39. $y^2 = x^3$

$$2yy' = 3x^2$$

$$y' = \frac{3x^2}{2y} = \frac{3x^2}{2y} \cdot \frac{xy}{xy} = \frac{3y}{2x} \cdot \frac{x^3}{y^2} = \frac{3y}{2x}$$

$$\begin{aligned} y'' &= \frac{2x(3y') - 3y(2)}{4x^2} \\ &= \frac{2x[3 \cdot (3y/2x)] - 6y}{4x^2} \end{aligned}$$

$$= \frac{3y}{4x^2} = \frac{3x}{4y}$$

41. $\sqrt{x} + \sqrt{y} = 4$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

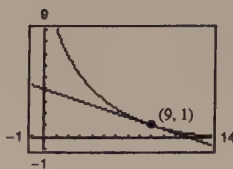
$$y' = \frac{-\sqrt{y}}{\sqrt{x}}$$

$$\text{At } (9, 1), y' = -\frac{1}{3}$$

$$\text{Tangent line: } y - 1 = -\frac{1}{3}(x - 9)$$

$$y = -\frac{1}{3}x + 4$$

$$x + 3y - 12 = 0$$



43. $x^2 + y^2 = 25$

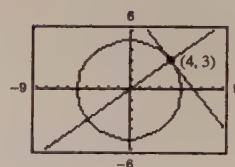
$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

At (4, 3):

$$\text{Tangent line: } y - 3 = \frac{-4}{3}(x - 4) \Rightarrow 4x + 3y - 25 = 0$$

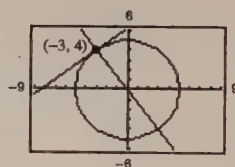
$$\text{Normal line: } y - 3 = \frac{3}{4}(x - 4) \Rightarrow 3x - 4y = 0.$$



At (-3, 4):

$$\text{Tangent line: } y - 4 = \frac{3}{4}(x + 3) \Rightarrow 3x - 4y + 25 = 0$$

$$\text{Normal line: } y - 4 = \frac{-4}{3}(x + 3) \Rightarrow 4x + 3y = 0.$$



45. $x^2 + y^2 = r^2$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y} = \text{slope of tangent line}$$

$$\frac{y}{x} = \text{slope of normal line}$$

Let (x_0, y_0) be a point on the circle. If $x_0 = 0$, then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If $x_0 \neq 0$, then the equation of the normal line is

$$y - y_0 = \frac{y_0}{x_0}(x - x_0)$$

$$y = \frac{y_0}{x_0}x$$

which passes through the origin.

47. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

$$50x + 32yy' + 200 - 160y' = 0$$

$$y' = \frac{200 + 50x}{160 - 32y}$$

 Horizontal tangents occur when $x = -4$:

$$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$$

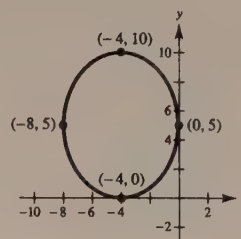
$$y(y - 10) = 0 \Rightarrow y = 0, 10$$

 Horizontal tangents: $(-4, 0), (-4, 10)$.

 Vertical tangents occur when $y = 5$:

$$25x^2 + 400 + 200x - 800 + 400 = 0$$

$$25x(x + 8) = 0 \Rightarrow x = 0, -8$$

 Vertical tangents: $(0, 5), (-8, 5)$.

 49. Find the points of intersection by letting $y^2 = 4x$ in the equation $2x^2 + y^2 = 6$.

$$2x^2 + 4x = 6 \quad \text{and} \quad (x + 3)(x - 1) = 0$$

 The curves intersect at $(1, \pm 2)$.

Ellipse:

$$4x + 2yy' = 0$$

$$y' = -\frac{2x}{y}$$

 At $(1, 2)$, the slopes are:

$$y' = -1$$

 At $(1, -2)$, the slopes are:

$$y' = 1$$

Parabola:

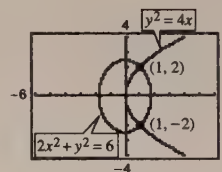
$$2yy' = 4$$

$$y' = \frac{2}{y}$$

$$y' = 1.$$

$$y' = -1.$$

Tangents are perpendicular.


 51. $y = -x$ and $x = \sin y$

 Point of intersection: $(0, 0)$

$$y = -x:$$

$$y' = -1$$

$$x = \sin y:$$

$$1 = y' \cos y$$

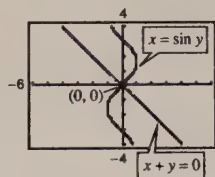
$$y' = \sec y$$

 At $(0, 0)$, the slopes are:

$$y' = -1$$

$$y' = 1.$$

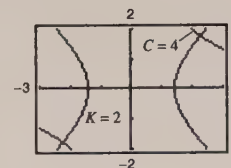
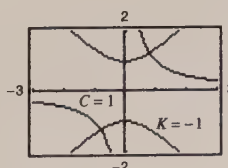
Tangents are perpendicular.


 53. $xy = C$ $x^2 - y^2 = K$

$$xy' + y = 0 \quad 2x - 2yy' = 0$$

$$y' = -\frac{y}{x}$$

$$y' = \frac{x}{y}$$

 At any point of intersection (x, y) the product of the slopes is $(-y/x)(x/y) = -1$. The curves are orthogonal.


55. $2y^2 - 3x^4 = 0$

(a) $4yy' - 12x^3 = 0$

$$4yy' = 12x^3$$

$$y' = \frac{12x^3}{4y} = \frac{3x^3}{y}$$

(b) $4y \frac{dy}{dt} - 12x^3 \frac{dx}{dt} = 0$

$$y \frac{dy}{dt} = 3x^3 \frac{dx}{dt}$$

57. $\cos \pi y - 3 \sin \pi x = 1$

(a) $-\pi \sin(\pi y)y' - 3\pi \cos \pi x = 0$

$$y' = \frac{-3 \cos \pi x}{\sin \pi y}$$

(b) $-\pi \sin(\pi y) \frac{dy}{dt} - 3\pi \cos(\pi x) \frac{dx}{dt} = 0$

$$-\sin(\pi y) \frac{dy}{dt} = 3 \cos(\pi x) \frac{dx}{dt}$$

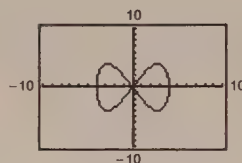
59. A function is in explicit form if y is written as a function of x : $y = f(x)$. For example, $y = x^3$. An implicit equation is not in the form $y = f(x)$. For example, $x^2 + y^2 = 5$.

61. (a) $x^4 = 4(4x^2 - y^2)$

$$4y^2 = 16x^2 - x^4$$

$$y^2 = 4x^2 - \frac{1}{4}x^4$$

$$y = \pm \sqrt{4x^2 - \frac{1}{4}x^4}$$



(b) $y = 3 \Rightarrow 9 = 4x^2 - \frac{1}{4}x^4$

$$36 = 16x^2 - x^4$$

$$x^4 - 16x^2 + 36 = 0$$

$$x^2 = \frac{16 \pm \sqrt{256 - 144}}{2} = 8 \pm \sqrt{28}$$

Note that $x^2 = 8 \pm \sqrt{28} = 8 \pm 2\sqrt{7} = (1 \pm \sqrt{7})^2$.

Hence, there are four values of x :

$$-1 - \sqrt{7}, 1 - \sqrt{7}, -1 + \sqrt{7}, 1 + \sqrt{7}$$

To find the slope, $2yy' = 8x - x^3 \Rightarrow y' = \frac{x(8 - x^2)}{2(3)}$.

For $x = -1 - \sqrt{7}$, $y' = \frac{1}{3}(\sqrt{7} + 7)$, and the line is

$$y_1 = \frac{1}{3}(\sqrt{7} + 7)(x + 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} + 7)x + 8\sqrt{7} + 23].$$

For $x = 1 - \sqrt{7}$, $y' = \frac{1}{3}(\sqrt{7} - 7)$, and the line is

$$y_2 = \frac{1}{3}(\sqrt{7} - 7)(x - 1 + \sqrt{7}) + 3 = \frac{1}{3}[(\sqrt{7} - 7)x + 23 - 8\sqrt{7}].$$

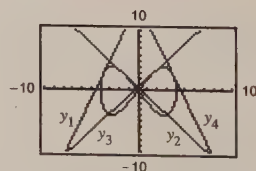
For $x = -1 + \sqrt{7}$, $y' = -\frac{1}{3}(\sqrt{7} - 7)$, and the line is

$$y_3 = -\frac{1}{3}(\sqrt{7} - 7)(x + 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} - 7)x - (23 - 8\sqrt{7})].$$

For $x = 1 + \sqrt{7}$, $y' = -\frac{1}{3}(\sqrt{7} + 7)$, and the line is

$$y_4 = -\frac{1}{3}(\sqrt{7} + 7)(x - 1 - \sqrt{7}) + 3 = -\frac{1}{3}[(\sqrt{7} + 7)x - (8\sqrt{7} + 23)].$$

—CONTINUED—



61. —CONTINUED—

(c) Equating y_3 and y_4 :

$$-\frac{1}{3}(\sqrt{7} - 7)(x + 1 - \sqrt{7}) + 3 = -\frac{1}{3}(\sqrt{7} + 7)(x - 1 - \sqrt{7}) + 3$$

$$(\sqrt{7} - 7)(x + 1 - \sqrt{7}) = (\sqrt{7} + 7)(x - 1 - \sqrt{7})$$

$$\sqrt{7}x + \sqrt{7} - 7 - 7x - 7 + 7\sqrt{7} = \sqrt{7}x - \sqrt{7} - 7 + 7x - 7 - 7\sqrt{7}$$

$$16\sqrt{7} = 14x$$

$$x = \frac{8\sqrt{7}}{7}$$

If $x = \frac{8\sqrt{7}}{7}$, then $y = 5$ and the lines intersect at $\left(\frac{8\sqrt{7}}{7}, 5\right)$.

63. $y = x^{p/q}$

$$y^q = x^p$$

$$qy^{q-1}y' = px^{p-1}$$

$$y' = \frac{p}{q} \frac{x^{p-1}}{y^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}y}{x^p} = \frac{p}{q} x^{p/q-1}$$

Thus, if $y = x^n = x^{p/q}$, $y' = nx^{n-1}$.

Section 2.6 Related Rates

1. $y = \sqrt{x}$

$$\frac{dy}{dt} = \left(\frac{1}{2\sqrt{x}}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = 2\sqrt{x} \frac{dy}{dt}$$

(a) When $x = 4$ and $dx/dt = 3$,

$$\frac{dy}{dt} = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4}.$$

(b) When $x = 25$ and $dy/dt = 2$,

$$\frac{dx}{dt} = 2\sqrt{25}(2) = 20.$$

3. $xy = 4$

$$x \frac{dy}{dt} + y \frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{y}{x}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{x}{y}\right) \frac{dy}{dt}$$

(a) When $x = 8$, $y = 1/2$, and $dx/dt = 10$,

$$\frac{dy}{dt} = -\frac{1/2}{8}(10) = -\frac{5}{8}.$$

(b) When $x = 1$, $y = 4$, and $dy/dt = -6$,

$$\frac{dx}{dt} = -\frac{1}{4}(-6) = \frac{3}{2}.$$

5. $y = x^2 + 1$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = 2x \frac{dx}{dt}$$

(a) When $x = -1$,

$$\frac{dy}{dt} = 2(-1)(2) = -4 \text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 2(0)(2) = 0 \text{ cm/sec.}$$

(c) When $x = 1$,

$$\frac{dy}{dt} = 2(1)(2) = 4 \text{ cm/sec.}$$

9. (a) $\frac{dx}{dt}$ negative $\Rightarrow \frac{dy}{dt}$ positive

(b) $\frac{dy}{dt}$ positive $\Rightarrow \frac{dx}{dt}$ negative

13. $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x^2 + 1)^2} = \sqrt{x^4 + 3x^2 + 1}$

$$\frac{dx}{dt} = 2$$

$$\frac{dD}{dt} = \frac{1}{2}(x^4 + 3x^2 + 1)^{-1/2}(4x^3 + 6x) \frac{dx}{dt} = \frac{2x^3 + 3x}{\sqrt{x^4 + 3x^2 + 1}} \frac{dx}{dt} = \frac{4x^3 + 6x}{\sqrt{x^4 + 3x^2 + 1}}$$

15. $A = \pi r^2$

$$\frac{dr}{dt} = 3$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

(a) When $r = 6$,

$$\frac{dA}{dt} = 2\pi(6)(3) = 36\pi \text{ cm}^2/\text{min.}$$

(b) When $r = 24$,

$$\frac{dA}{dt} = 2\pi(24)(3) = 144\pi \text{ cm}^2/\text{min.}$$

7. $y = \tan x$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \sec^2 x \frac{dx}{dt}$$

(a) When $x = -\pi/3$,

$$\frac{dy}{dt} = (2)^2(2) = 8 \text{ cm/sec.}$$

(b) When $x = -\pi/4$,

$$\frac{dy}{dt} = (\sqrt{2})^2(2) = 4 \text{ cm/sec.}$$

(c) When $x = 0$,

$$\frac{dy}{dt} = (1)^2(2) = 2 \text{ cm/sec.}$$

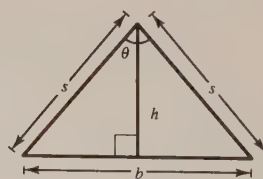
11. Yes, y changes at a constant rate: $\frac{dy}{dt} = a \cdot \frac{dx}{dt}$

No, the rate $\frac{dy}{dt}$ is a multiple of $\frac{dx}{dt}$.

17. (a) $\sin \frac{\theta}{2} = \frac{(1/2)b}{s} \Rightarrow b = 2s \sin \frac{\theta}{2}$

$$\cos \frac{\theta}{2} = \frac{h}{s} \Rightarrow h = s \cos \frac{\theta}{2}$$

$$\begin{aligned} A &= \frac{1}{2}bh = \frac{1}{2}\left(2s \sin \frac{\theta}{2}\right)\left(s \cos \frac{\theta}{2}\right) \\ &= \frac{s^2}{2}\left(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right) = \frac{s^2}{2} \sin \theta \end{aligned}$$



(b) $\frac{dA}{dt} = \frac{s^2}{2} \cos \theta \frac{d\theta}{dt}$ where $\frac{d\theta}{dt} = \frac{1}{2} \text{ rad/min.}$

$$\text{When } \theta = \frac{\pi}{6}, \frac{dA}{dt} = \frac{s^2}{2} \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{2}\right) = \frac{\sqrt{3}s^2}{8}$$

$$\text{When } \theta = \frac{\pi}{3}, \frac{dA}{dt} = \frac{s^2}{2} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = \frac{s^2}{8}$$

(c) If $d\theta/dt$ is constant, dA/dt is proportional to $\cos \theta$.

19. $V = \frac{4}{3}\pi r^3, \frac{dV}{dt} = 800$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \left(\frac{dV}{dt} \right) = \frac{1}{4\pi r^2} (800)$$

(a) When $r = 30$, $\frac{dr}{dt} = \frac{1}{4\pi(30)^2} (800) = \frac{2}{9\pi}$ cm/min.

(b) When $r = 60$, $\frac{dr}{dt} = \frac{1}{4\pi(60)^2} (800) = \frac{1}{18\pi}$ cm/min.

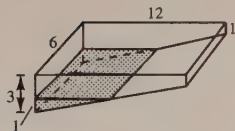
23. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{9}{4}h^2 \right) h$ [since $2r = 3h$]
 $= \frac{3\pi}{4}h^3$

$$\frac{dV}{dt} = 10$$

$$\frac{dV}{dt} = \frac{9\pi}{4}h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4(dV/dt)}{9\pi h^2}$$

When $h = 15$, $\frac{dh}{dt} = \frac{4(10)}{9\pi(15)^2} = \frac{8}{405\pi}$ ft/min.

25.



(a) Total volume of pool $= \frac{1}{2}(2)(12)(6) + (1)(6)(12) = 144 \text{ m}^3$

Volume of 1m. of water $= \frac{1}{2}(1)(6)(6) = 18 \text{ m}^3$

(see similar triangle diagram)

% pool filled $= \frac{18}{144}(100\%) = 12.5\%$

(b) Since for $0 \leq h \leq 2$, $b = 6h$, you have

$$V = \frac{1}{2}bh(6) = 3bh = 3(6h)h = 18h^2$$

$$\frac{dV}{dt} = 36h \frac{dh}{dt} = \frac{1}{4} \Rightarrow \frac{dh}{dt} = \frac{1}{144h} = \frac{1}{144(1)} = \frac{1}{144} \text{ m/min.}$$

21. $s = 6x^2$

$$\frac{dx}{dt} = 3$$

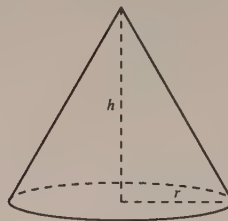
$$\frac{ds}{dt} = 12x \frac{dx}{dt}$$

(a) When $x = 1$,

$$\frac{ds}{dt} = 12(1)(3) = 36 \text{ cm}^2/\text{sec.}$$

(b) When $x = 10$,

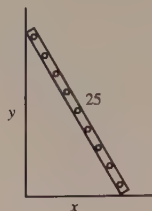
$$\frac{ds}{dt} = 12(10)(3) = 360 \text{ cm}^2/\text{sec.}$$



27. $x^2 + y^2 = 25^2$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt} = \frac{-2x}{y} \text{ since } \frac{dx}{dt} = 2.$$



(a) When $x = 7$, $y = \sqrt{576} = 24$, $\frac{dy}{dt} = \frac{-2(7)}{24} = \frac{-7}{12}$ ft/sec.

When $x = 15$, $y = \sqrt{400} = 20$, $\frac{dy}{dt} = \frac{-2(15)}{20} = \frac{-3}{2}$ ft/sec.

When $x = 24$, $y = 7$, $\frac{dy}{dt} = \frac{-2(24)}{7} = \frac{-48}{7}$ ft/sec.

(b) $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2} \left(x \frac{dy}{dt} + y \frac{dx}{dt} \right)$$

From part (a) we have $x = 7$, $y = 24$, $\frac{dx}{dt} = 2$,

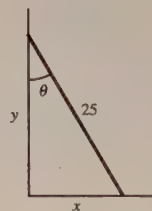
and $\frac{dy}{dt} = -\frac{7}{12}$.

$$\begin{aligned} \text{Thus, } \frac{dA}{dt} &= \frac{1}{2} \left[7 \left(-\frac{7}{12} \right) + 24(2) \right] \\ &= \frac{527}{24} \approx 21.96 \text{ ft}^2/\text{sec}. \end{aligned}$$

(c) $\tan \theta = \frac{x}{y}$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \cos^2 \theta \left[\frac{1}{y} \cdot \frac{dx}{dt} - \frac{x}{y^2} \cdot \frac{dy}{dt} \right]$$



Using $x = 7$, $y = 24$, $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = -\frac{7}{12}$ and $\cos \theta = \frac{24}{25}$, we have $\frac{d\theta}{dt} = \left(\frac{24}{25} \right)^2 \left[\frac{1}{24}(2) - \frac{7}{(24)^2} \left(-\frac{7}{12} \right) \right] = \frac{1}{12}$ rad/sec.

29. When $y = 6$, $x = \sqrt{12^2 - 6^2} = 6\sqrt{3}$, and

$$\begin{aligned} s &= \sqrt{x^2 + (12 - y)^2} \\ &= \sqrt{108 + 36} = 12. \end{aligned}$$

$$x^2 + (12 - y)^2 = s^2$$

$$2x \frac{dx}{dt} + 2(12 - y)(-1) \frac{dy}{dt} = 2s \frac{ds}{dt}$$

$$x \frac{dx}{dt} + (y - 12) \frac{dy}{dt} = s \frac{ds}{dt}$$

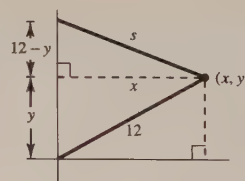
Also, $x^2 + y^2 = 12^2$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}$$

Thus, $x \frac{dx}{dt} + (y - 12) \left(\frac{-x}{y} \frac{dx}{dt} \right) = s \frac{ds}{dt}$

$$\frac{dx}{dt} \left[x - x + \frac{12x}{y} \right] = s \frac{ds}{dt} \Rightarrow \frac{dx}{dt} = \frac{sy}{12x} \cdot \frac{ds}{dt} = \frac{(12)(6)}{(12)(6\sqrt{3})} (-0.2) = \frac{-1}{5\sqrt{3}} = \frac{-\sqrt{3}}{15} \text{ m/sec (horizontal)}$$

$$\frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt} = \frac{-6\sqrt{3}}{6} \cdot \frac{(-\sqrt{3})}{15} = \frac{1}{5} \text{ m/sec (vertical)}.$$



31. (a) $s^2 = x^2 + y^2$

$$\frac{dx}{dt} = -450$$

$$\frac{dy}{dt} = -600$$

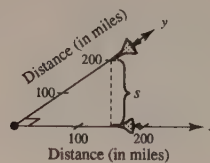
$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x(dx/dt) + y(dy/dt)}{s}$$

When $x = 150$ and $y = 200$, $s = 250$ and

$$\frac{ds}{dt} = \frac{150(-450) + 200(-600)}{250} = -750 \text{ mph.}$$

(b) $t = \frac{250}{750} = \frac{1}{3} \text{ hr} = 20 \text{ min}$



33. $s^2 = 90^2 + x^2$

$$x = 30$$

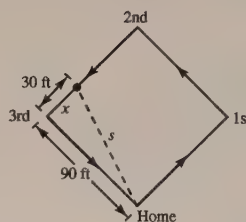
$$\frac{dx}{dt} = -28$$

$$2s \frac{ds}{dt} = 2x \frac{dx}{dt} \Rightarrow \frac{ds}{dt} = \frac{x}{s} \cdot \frac{dx}{dt}$$

When $x = 30$,

$$s = \sqrt{90^2 + 30^2} = 30\sqrt{10}$$

$$\frac{ds}{dt} = \frac{30}{30\sqrt{10}}(-28) = \frac{-28}{\sqrt{10}} \approx -8.85 \text{ ft/sec.}$$

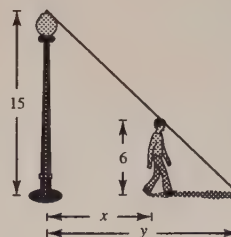


35. (a) $\frac{15}{6} = \frac{y}{y-x} \Rightarrow 15y - 15x = 6y$

$$y = \frac{5}{3}x$$

$$\frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = \frac{5}{3} \cdot \frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3} \text{ ft/sec}$$



(b) $\frac{d(y-x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{25}{3} - 5 = \frac{10}{3} \text{ ft/sec}$

37. $x(t) = \frac{1}{2} \sin \frac{\pi t}{6}, x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi/6} = 12$ seconds

(b) When $x = \frac{1}{2}, y = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ m.

Lowest point: $\left(0, \frac{\sqrt{3}}{2}\right)$

(c) When $x = \frac{1}{4}, y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and $t = 1$

$$\frac{dx}{dt} = \frac{1}{2} \left(\frac{\pi}{6} \right) \cos \frac{\pi t}{6} = \frac{\pi}{12} \cos \frac{\pi t}{6}$$

$$x^2 + y^2 = 1$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}$$

Thus,

$$\begin{aligned} \frac{dy}{dt} &= -\frac{1/4}{\sqrt{15}/4} \cdot \frac{\pi}{12} \cos\left(\frac{\pi}{6}\right) \\ &= \frac{-\pi}{\sqrt{15}} \left(\frac{1}{12}\right) \frac{\sqrt{3}}{2} = \frac{-\pi}{24} \frac{1}{\sqrt{5}} = \frac{-\sqrt{5}\pi}{120} \end{aligned}$$

$$\text{Speed} = \left| \frac{-\sqrt{5}\pi}{120} \right| = \frac{\sqrt{5}\pi}{120} \text{ m/sec}$$

41. $pV^{1.3} = k$

$$1.3 pV^{0.3} \frac{dV}{dt} + V^{1.3} \frac{dp}{dt} = 0$$

$$V^{0.3} \left(1.3p \frac{dV}{dt} + V \frac{dp}{dt} \right) = 0$$

$$1.3p \frac{dV}{dt} = -V \frac{dp}{dt}$$

43. $\tan \theta = \frac{y}{30}$

$$\frac{dy}{dt} = 3 \text{ m/sec.}$$

$$\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{30} \frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{30} \cos^2 \theta \cdot \frac{dy}{dt}$$

When $y = 30$, $\theta = \pi/4$ and $\cos \theta = \sqrt{2}/2$. Thus,

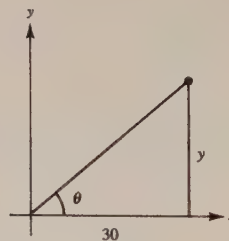
$$\frac{d\theta}{dt} = \frac{1}{30} \left(\frac{1}{2} \right) (3) = \frac{1}{20} \text{ rad/sec.}$$

39. Since the evaporation rate is proportional to the surface area, $dV/dt = k(4\pi r^2)$. However, since $V = (4/3)\pi r^3$, we have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

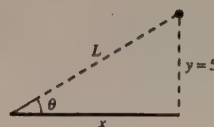
Therefore,

$$k(4\pi r^2) = 4\pi r^2 \frac{dr}{dt} \Rightarrow k = \frac{dr}{dt}$$



45. $\tan \theta = \frac{y}{x}, y = 5$

$$\frac{dx}{dt} = -600 \text{ mi/hr}$$



$$(\sec^2 \theta) \frac{d\theta}{dt} = -\frac{5}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \cos^2 \theta \left(-\frac{5}{x^2} \right) \frac{dx}{dt} = \frac{x^2}{L^2} \left(-\frac{5}{x^2} \right) \frac{dx}{dt}$$

$$= \left(-\frac{5^2}{L^2} \right) \left(\frac{1}{5} \right) \frac{dx}{dt} = (-\sin^2 \theta) \left(\frac{1}{5} \right) (-600) = 120 \sin^2 \theta$$

(a) When $\theta = 30^\circ$, $\frac{d\theta}{dt} = \frac{120}{4} = 30 \text{ rad/hr} = \frac{1}{2} \text{ rad/min}$.

(b) When $\theta = 60^\circ$, $\frac{d\theta}{dt} = 120 \left(\frac{3}{4} \right) = 90 \text{ rad/hr} = \frac{3}{2} \text{ rad/min}$.

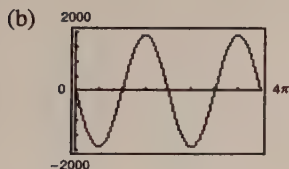
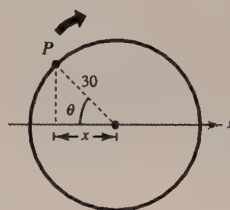
(c) When $\theta = 75^\circ$, $\frac{d\theta}{dt} = 120 \sin^2 75^\circ \approx 111.96 \text{ rad/hr} \approx 1.87 \text{ rad/min}$.

47. $\frac{d\theta}{dt} = (10 \text{ rev/sec})(2\pi \text{ rad/rev}) = 20\pi \text{ rad/sec}$

(a) $\cos \theta = \frac{x}{30}$

$$-\sin \theta \frac{d\theta}{dt} = \frac{1}{30} \frac{dx}{dt}$$

$$\frac{dx}{dt} = -30 \sin \theta \frac{d\theta}{dt} = -30 \sin \theta (20\pi) = -600\pi \sin \theta$$



(c) $|dx/dt| = |-600\pi \sin \theta|$ is greatest when $|\sin \theta| = 1 \Rightarrow \theta = (\pi/2) + n\pi$ (or $90^\circ + n \cdot 180^\circ$).

$|dx/dt|$ is least when $\theta = n\pi$ (or $n \cdot 180^\circ$).

(d) For $\theta = 30^\circ$, $\frac{dx}{dt} = -600\pi \sin(30^\circ) = -600\pi \frac{1}{2} = -300\pi \text{ cm/sec}$.

For $\theta = 60^\circ$, $\frac{dx}{dt} = -600\pi \sin(60^\circ) = -600\pi \frac{\sqrt{3}}{2} = -300\sqrt{3}\pi \text{ cm/sec}$.

49. $\tan \theta = \frac{x}{50} \Rightarrow x = 50 \tan \theta$

$$\frac{dx}{dt} = 50 \sec^2 \theta \frac{d\theta}{dt}$$

$$2 = 50 \sec^2 \theta \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{25} \cos^2 \theta, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

51. $x^2 + y^2 = 25$; acceleration of the top of the ladder $= \frac{d^2y}{dt^2}$

First derivative: $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

$$x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

Second derivative: $x \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{dx}{dt} + y \frac{d^2y}{dt^2} + \frac{dy}{dt} \cdot \frac{dy}{dt} = 0$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{y}\right) \left[-x \frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 \right]$$

When $x = 7$, $y = 24$, $\frac{dy}{dt} = -\frac{7}{12}$, and $\frac{dx}{dt} = 2$ (see Exercise 27). Since $\frac{dx}{dt}$ is constant, $\frac{d^2x}{dt^2} = 0$.

$$\frac{d^2y}{dt^2} = \frac{1}{24} \left[-7(0) - (2)^2 - \left(-\frac{7}{12}\right)^2 \right] = \frac{1}{24} \left[-4 - \frac{49}{144} \right] = \frac{1}{24} \left[-\frac{625}{144} \right] \approx -0.1808 \text{ ft/sec}^2$$

53. (a) Using a graphing utility, you obtain $m(s) = -0.881s^2 + 29.10s - 206.2$

(b) $\frac{dm}{dt} = \frac{dm}{ds} \frac{ds}{dt} = (-1.762s + 29.10) \frac{ds}{dt}$

(c) If $t = 5$ (1995), then $s = 15.5$ and $\frac{ds}{dt} = 1.2$.

Thus, $\frac{dm}{dt} = (-1.762(15.5) + 29.10)(1.2) \approx 2.15$ million.

Review Exercises for Chapter 2

1. $f(x) = x^2 - 2x + 3$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 - 2(x + \Delta x) + 3] - [x^2 - 2x + 3]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + 3) - (x^2 - 2x + 3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x(\Delta x) + (\Delta x)^2 - 2(\Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x - 2) = 2x - 2 \end{aligned}$$

3. $f(x) = \sqrt{x} + 1$

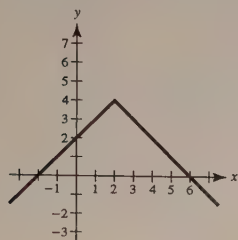
$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(\sqrt{x + \Delta x} + 1) - (\sqrt{x} + 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

5. f is differentiable for all $x \neq -1$.

7. $f(x) = 4 - |x - 2|$

(a) Continuous at $x = 2$.

(b) Not differentiable at $x = 2$ because of the sharp turn in the graph.

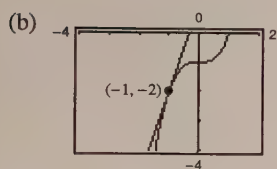


11. (a) Using the limit definition, $f'(x) = 3x^2$.

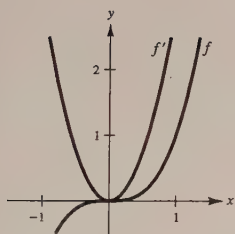
At $x = -1$, $f'(-1) = 3$. The tangent line is

$$y - (-2) = 3(x - (-1))$$

$$y = 3x + 1$$



15.



17. $y = 25$

$$y' = 0$$

19. $f(x) = x^8$

$$f'(x) = 8x^7$$

21. $h(t) = 3t^4$

$$h'(t) = 12t^3$$

23. $f(x) = x^3 - 3x^2$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

25. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x} = 6x^{1/2} + 3x^{1/3}$

$$h'(x) = 3x^{-1/2} + x^{-2/3} = \frac{3}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$$

27. $g(t) = \frac{2}{3}t^{-2}$

$$g'(t) = \frac{-4}{3}t^{-3} = \frac{-4}{3t^3}$$

29. $f(\theta) = 2\theta - 3\sin \theta$

$$f'(\theta) = 2 - 3\cos \theta$$

31. $f(\theta) = 3\cos \theta - \frac{\sin \theta}{4}$

$$f'(\theta) = -3\sin \theta - \frac{\cos \theta}{4}$$

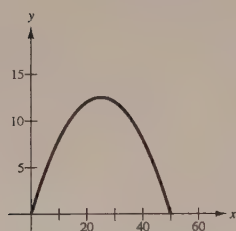
33. $F = 200\sqrt{T}$

$$F'(t) = \frac{100}{\sqrt{T}}$$

(a) When $T = 4$, $F'(4) = 50$ vibrations/sec/lb.

(b) When $T = 9$, $F'(9) = 33\frac{1}{3}$ vibrations/sec/lb.

37. (a)



Total horizontal distance: 50

(b) $0 = x - 0.02x^2$

$$0 = x\left(1 - \frac{x}{50}\right) \text{ implies } x = 50.$$

39. $x(t) = t^2 - 3t + 2 = (t - 2)(t - 1)$

(a) $v(t) = x'(t) = 2t - 3$

(c) $v(t) = 0$ for $t = \frac{3}{2}$.

$$x = \left(\frac{3}{2} - 2\right)\left(\frac{3}{2} - 1\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$$

41. $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$

$$\begin{aligned} f'(x) &= (3x^2 + 7)(2x - 2) + (x^2 - 2x + 3)(6x) \\ &= 2(6x^3 - 9x^2 + 16x - 7) \end{aligned}$$

45. $f(x) = 2x - x^{-2}$

$$\begin{aligned} f'(x) &= 2 + 2x^{-3} = 2\left(1 + \frac{1}{x^3}\right) \\ &= \frac{2(x^3 + 1)}{x^3} \end{aligned}$$

49. $f(x) = (4 - 3x^2)^{-1}$

$$f'(x) = -(4 - 3x^2)^{-2}(-6x) = \frac{6x}{(4 - 3x^2)^2}$$

53. $y = 3x^2 \sec x$

$$y' = 3x^2 \sec x \tan x + 6x \sec x$$

35. $s(t) = -16t^2 + s_0$

$$s(9.2) = -16(9.2)^2 + s_0 = 0$$

$$s_0 = 1354.24$$

The building is approximately 1354 feet high (or 415 m).

(c) Ball reaches maximum height when $x = 25$.

(d) $y = x - 0.02x^2$

$$y' = 1 - 0.04x$$

$$y'(0) = 1$$

$$y'(10) = 0.6$$

$$y'(25) = 0$$

$$y'(30) = -0.2$$

$$y'(50) = -1$$

(e) $y'(25) = 0$

(b) $v(t) < 0$ for $t < \frac{3}{2}$.

(d) $x(t) = 0$ for $t = 1, 2$.

$$|v(1)| = |2(1) - 3| = 1$$

$$|v(2)| = |2(2) - 3| = 1$$

The speed is 1 when the position is 0.

43. $h(x) = \sqrt{x} \sin x = x^{1/2} \sin x$

$$h'(x) = \frac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x$$

47. $f(x) = \frac{x^2 + x - 1}{x^2 - 1}$

$$\begin{aligned} f'(x) &= \frac{(x^2 - 1)(2x + 1) - (x^2 + x - 1)(2x)}{(x^2 - 1)^2} \\ &= \frac{-(x^2 + 1)}{(x^2 - 1)^2} \end{aligned}$$

51. $y = \frac{x^2}{\cos x}$

$$y' = \frac{\cos x (2x) - x^2(-\sin x)}{\cos^2 x} = \frac{2x \cos x + x^2 \sin x}{\cos^2 x}$$

55. $y = -x \tan x$

$$y' = -x \sec^2 x - \tan x$$

57. $y = x \cos x - \sin x$

$$y' = -x \sin x + \cos x - \cos x = -x \sin x$$

61. $f(\theta) = 3 \tan \theta$

$$f'(\theta) = 3 \sec^2 \theta$$

$$f''(\theta) = 6 \sec \theta (\sec \theta \tan \theta) = 6 \sec^2 \theta \tan \theta$$

65. $f(x) = (1 - x^3)^{1/2}$

$$f'(x) = \frac{1}{2}(1 - x^3)^{-1/2}(-3x^2)$$

$$= -\frac{3x^2}{2\sqrt{1-x^3}}$$

69. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

$$f'(s) = (s^2 - 1)^{5/2}(3s^2) + (s^3 + 5)\left(\frac{5}{2}\right)(s^2 - 1)^{3/2}(2s)$$

$$= s(s^2 - 1)^{3/2}[3s(s^2 - 1) + 5(s^3 + 5)]$$

$$= s(s^2 - 1)^{3/2}(8s^3 - 3s + 25)$$

73. $y = \frac{1}{2} \csc 2x$

$$y' = \frac{1}{2}(-\csc 2x \cot 2x)(2)$$

$$= -\csc 2x \cot 2x$$

77. $y = \frac{2}{3} \sin^{3/2} x - \frac{2}{7} \sin^{7/2} x$

$$y' = \sin^{1/2} x \cos x - \sin^{5/2} x \cos x$$

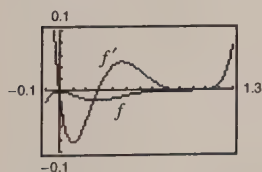
$$= (\cos x) \sqrt{\sin x} (1 - \sin^2 x)$$

$$= (\cos^3 x) \sqrt{\sin x}$$

81. $f(t) = t^2(t - 1)^5$

$$f'(t) = t(t - 1)^4(7t - 2)$$

The zeros of f' correspond to the points on the graph of f where the tangent line is horizontal.



59. $g(t) = t^3 - 3t + 2$

$$g'(t) = 3t^2 - 3$$

$$g''(t) = 6t$$

63. $y = 2 \sin x + 3 \cos x$

$$y' = 2 \cos x - 3 \sin x$$

$$y'' = -2 \sin x - 3 \cos x$$

$$y'' + y = -(2 \sin x + 3 \cos x) + (2 \sin x + 3 \cos x) = 0$$

67. $h(x) = \left(\frac{x-3}{x^2+1}\right)^2$

$$h'(x) = 2\left(\frac{x-3}{x^2+1}\right)\left(\frac{(x^2+1)(1) - (x-3)(2x)}{(x^2+1)^2}\right)$$

$$= \frac{2(x-3)(-x^2+6x+1)}{(x^2+1)^3}$$

71. $y = 3 \cos(3x + 1)$

$$y' = -9 \sin(3x + 1)$$

75. $y = \frac{x}{2} - \frac{\sin 2x}{4}$

$$y' = \frac{1}{2} - \frac{1}{4} \cos 2x(2)$$

$$= \frac{1}{2}(1 - \cos 2x) = \sin^2 x$$

79. $y = \frac{\sin \pi x}{x+2}$

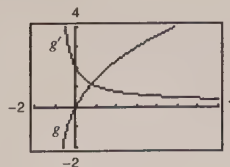
$$y' = \frac{(x+2)\pi \cos \pi x - \sin \pi x}{(x+2)^2}$$

83. $g(x) = 2x(x+1)^{-1/2}$

$$g'(x) = \frac{x+2}{(x+1)^{3/2}}$$

g' does not equal zero for any value of x in the domain.

The graph of g has no horizontal tangent lines.

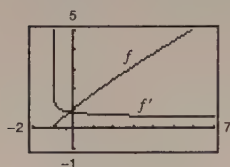


85. $f(t) = \sqrt{t+1} \sqrt[3]{t+1}$

$$f(t) = (t+1)^{1/2}(t+1)^{1/3} = (t+1)^{5/6}$$

$$f'(t) = \frac{5}{6(t+1)^{1/6}}$$

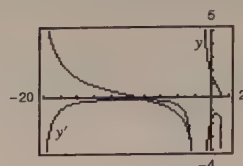
f' does not equal zero for any x in the domain. The graph of f has no horizontal tangent lines.



87. $y = \tan \sqrt{1-x}$

$$y' = -\frac{\sec^2 \sqrt{1-x}}{2\sqrt{1-x}}$$

y' does not equal zero for any x in the domain. The graph has no horizontal tangent lines.



89. $y = 2x^2 + \sin 2x$

$$y' = 4x + 2 \cos 2x$$

$$y'' = 4 - 4 \sin 2x$$

91. $f(x) = \cot x$

$$f'(x) = -\csc^2 x$$

$$\begin{aligned} f''(x) &= -2 \csc x (-\csc x \cdot \cot x) \\ &= 2 \csc^2 x \cot x \end{aligned}$$

93. $f(t) = \frac{t}{(1-t)^2}$

$$f'(t) = \frac{t+1}{(1-t)^3}$$

$$f''(t) = \frac{2(t+2)}{(1-t)^4}$$

95. $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

$$g'(\theta) = 3 \sec^2 3\theta - \cos(\theta - 1)$$

$$g''(\theta) = 18 \sec^2 3\theta \tan 3\theta + \sin(\theta - 1)$$

97. $T = \frac{700}{t^2 + 4t + 10}$

$$T = 700(t^2 + 4t + 10)^{-1}$$

$$T' = \frac{-1400(t+2)}{(t^2 + 4t + 10)^2}$$

(a) When $t = 1$,

$$T' = \frac{-1400(1+2)}{(1+4+10)^2} \approx -18.667 \text{ deg/hr.}$$

(b) When $t = 3$,

$$T' = \frac{-1400(3+2)}{(9+12+10)^2} \approx -7.284 \text{ deg/hr.}$$

(c) When $t = 5$,

$$T' = \frac{-1400(5+2)}{(25+20+10)^2} \approx -3.240 \text{ deg/hr.}$$

(d) When $t = 10$,

$$T' = \frac{-1400(10+2)}{(100+40+10)^2} \approx -0.747 \text{ deg/hr.}$$

99. $x^2 + 3xy + y^3 = 10$

$$2x + 3xy' + 3y + 3y^2y' = 0$$

$$3(x + y^2)y' = -(2x + 3y)$$

$$y' = \frac{-(2x + 3y)}{3(x + y^2)}$$

103.

$$x_i \sin y = y \cos x$$

$$(x \cos y)y' + \sin y = -y \sin x + y' \cos x$$

$$y'(x \cos y - \cos x) = -y \sin x - \sin y$$

$$y' = \frac{y \sin x + \sin y}{\cos x - x \cos y}$$

107. $y = \sqrt{x}$

$$\frac{dy}{dt} = 2 \text{ units/sec}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{x}} \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2\sqrt{x} \frac{dy}{dt} = 4\sqrt{x}$$

(a) When $x = \frac{1}{2}$, $\frac{dx}{dt} = 2\sqrt{2}$ units/sec.

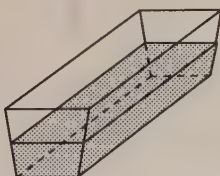
(b) When $x = 1$, $\frac{dx}{dt} = 4$ units/sec.

(c) When $x = 4$, $\frac{dx}{dt} = 8$ units/sec.

109. $\frac{s}{h} = \frac{1/2}{2}$

$$s = \frac{1}{4}h$$

$$\frac{dV}{dt} = 1$$

Width of water at depth h :

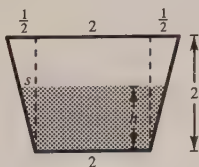
$$w = 2 + 2s = 2 + 2\left(\frac{1}{4}h\right) = \frac{4 + h}{2}$$

$$V = \frac{5}{2} \left(2 + \frac{4 + h}{2} \right) h = \frac{5}{4} (8 + h)h$$

$$\frac{dV}{dt} = \frac{5}{2} (4 + h) \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2(dV/dt)}{5(4 + h)}$$

When $h = 1$, $\frac{dh}{dt} = \frac{2}{25}$ m/min.



105. $x^2 + y^2 = 20$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

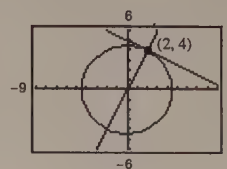
At $(2, 4)$: $y' = -\frac{1}{2}$

Tangent line: $y - 4 = -\frac{1}{2}(x - 2)$

$$x + 2y - 10 = 0$$

Normal line: $y - 4 = 2(x - 2)$

$$2x - y = 0$$



111. $s(t) = 60 - 4.9t^2$

$$s'(t) = -9.8t$$

$$s = 35 = 60 - 4.9t^2$$

$$4.9t^2 = 25$$

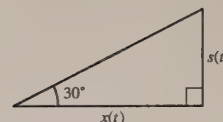
$$t = \frac{5}{\sqrt{4.9}}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{s(t)}{x(t)}$$

$$x(t) = \sqrt{3}s(t)$$

$$\frac{dx}{dt} = \sqrt{3} \frac{ds}{dt} = \sqrt{3}(-9.8) \frac{5}{\sqrt{4.9}}$$

$$\approx -38.34 \text{ m/sec}$$



Problem Solving for Chapter 2

1. (a)
- $x^2 + (y - r)^2 = r^2$
- Circle

$$x^2 = y \quad \text{Parabola}$$

Substituting,

$$(y - r)^2 = r^2 - y$$

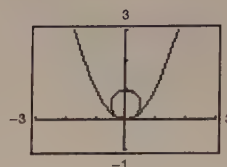
$$y^2 - 2ry + r^2 = r^2 - y$$

$$y^2 - 2ry + y = 0$$

$$y(y - 2r + 1) = 0$$

Since you want only one solution, let $1 - 2r = 0 \Rightarrow r = \frac{1}{2}$

$$\text{Graph } y = x^2 \text{ and } x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$



- (b) Let
- (x, y)
- be a point of tangency:
- $x^2 + (y - b)^2 = 1 \Rightarrow 2x + 2(y - b)y' = 0 \Rightarrow y' = \frac{x}{b - y}$
- (circle).

 $y = x^2 \Rightarrow y' = 2x$ (parabola). Equating,

$$2x = \frac{x}{b - y}$$

$$2(b - y) = 1$$

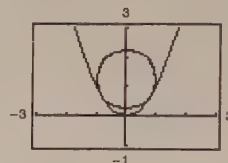
$$b - y = \frac{1}{2} \Rightarrow b = y + \frac{1}{2}$$

Also, $x^2 + (y - b)^2 = 1$ and $y = x^2$ imply

$$y + (y - b)^2 = 1 \Rightarrow y + \left[y - \left(y + \frac{1}{2}\right)\right]^2 = 1 \Rightarrow y + \frac{1}{4} = 1 \Rightarrow y = \frac{3}{4} \text{ and } b = \frac{5}{4}$$

$$\text{Center: } \left(0, \frac{5}{4}\right)$$

$$\text{Graph } y = x^2 \text{ and } x^2 + \left(y - \frac{5}{4}\right)^2 = 1$$



3. (a) $f(x) = \cos x$ $P_1(x) = a_0 + a_1x$
 $f(0) = 1$ $P_1(0) = a_0 \Rightarrow a_0 = 1$
 $f'(0) = 0$ $P'_1(0) = a_1 \Rightarrow a_1 = 0$
 $P_1(x) = 1$

- (b) $f(x) = \cos x$ $P_2(x) = a_0 + a_1x + a_2x^2$
 $f(0) = 1$ $P_2(0) = a_0 \Rightarrow a_0 = 1$
 $f'(0) = 0$ $P'_2(0) = a_1 \Rightarrow a_1 = 0$
 $f''(0) = -1$ $P''_2(0) = 2a_2 \Rightarrow a_2 = -\frac{1}{2}$
 $P_2(x) = 1 - \frac{1}{2}x^2$

(c)

x	-1.0	-0.1	-0.001	0	0.001	0.1	1.0
$\cos x$	0.5403	0.9950	≈ 1	1	≈ 1	0.9950	0.5403
$P_2(x)$	0.5	0.9950	≈ 1	1	≈ 1	0.9950	0.5

 $P_2(x)$ is a good approximation of $f(x) = \cos x$ when x is near 0.

- (d) $f(x) = \sin x$ $P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$
 $f(0) = 0$ $P_3(0) = a_0 \Rightarrow a_0 = 0$
 $f'(0) = 1$ $P'_3(0) = a_1 \Rightarrow a_1 = 1$
 $f''(0) = 0$ $P''_3(0) = 2a_2 \Rightarrow a_2 = 0$
 $f'''(0) = -1$ $P'''_3(0) = 6a_3 \Rightarrow a_3 = -\frac{1}{6}$
 $P_3(x) = x - \frac{1}{6}x^3$

5. Let $p(x) = Ax^3 + Bx^2 + Cx + D$

$$p'(x) = 3Ax^2 + 2Bx + C$$

At $(1, 1)$: $A + B + C + D = 1$ Equation 1

$$3A + 2B + C = 14$$
 Equation 2

At $(-1, -3)$: $-A + B - C + D = -3$ Equation 3

$$3A - 2B + C = -2$$
 Equation 4

Adding Equations 1 and 3: $2B + 2D = -2$

Subtracting Equations 1 and 3: $2A + 2C = 4$

Adding Equations 2 and 4: $6A + 2C = 12$

Subtracting Equations 2 and 4: $4B = 16$

Hence, $B = 4$ and $D = \frac{1}{2}(-2 - 2B) = -5$

Subtracting $2A + 2C = 4$ and $6A + 2C = 12$, you obtain $4A = 8 \Rightarrow A = 2$. Finally, $C = \frac{1}{2}(4 - 2A) = 0$

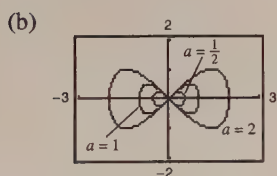
Thus, $p(x) = 2x^3 + 4x^2 - 5$.

7. (a) $x^4 = a^2x^2 - a^2y^2$

$$a^2y^2 = a^2x^2 - x^4$$

$$y = \frac{\pm \sqrt{a^2x^2 - x^4}}{a}$$

Graph: $y_1 = \frac{\sqrt{a^2x^2 - x^4}}{a}$ and $y_2 = -\frac{\sqrt{a^2x^2 - x^4}}{a}$



$(\pm a, 0)$ are the x -intercepts, along with $(0, 0)$.

(c) Differentiating implicitly,

$$4x^3 = 2a^2x - 2a^2yy'$$

$$y' = \frac{2a^2x - 4x^3}{2a^2y} = \frac{x(a^2 - 2x^2)}{a^2y} = 0 \Rightarrow 2x^2 = a^2 \Rightarrow x = \frac{\pm a}{\sqrt{2}}$$

$$\left(\frac{a^2}{2}\right)^2 = a^2\left(\frac{a^2}{2}\right) - a^2y^2$$

$$\frac{a^4}{4} = \frac{a^4}{2} - a^2y^2$$

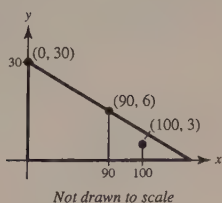
$$a^2y^2 = \frac{a^4}{4}$$

$$y^2 = \frac{a^2}{4}$$

$$y = \pm \frac{a}{2}$$

Four points: $\left(\frac{a}{\sqrt{2}}, \frac{a}{2}\right), \left(\frac{a}{\sqrt{2}}, -\frac{a}{2}\right), \left(-\frac{a}{\sqrt{2}}, \frac{a}{2}\right), \left(-\frac{a}{\sqrt{2}}, -\frac{a}{2}\right)$

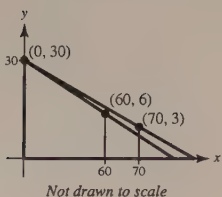
9. (a)

Line determined by $(0, 30)$ and $(90, 6)$:

$$y - 30 = \frac{30 - 6}{0 - 90}(x - 0) = -\frac{24}{90}x = -\frac{4}{15}x \Rightarrow y = -\frac{4}{15}x + 30$$

When $x = 100$, $y = -\frac{4}{15}(100) + 30 = \frac{10}{3} > 3 \Rightarrow$ Shadow determined by man.

(b)

Line determined by $(0, 30)$ and $(60, 6)$:

$$y - 30 = \frac{30 - 6}{0 - 60}(x - 0) = -\frac{2}{5}x \Rightarrow y = -\frac{2}{5}x + 30$$

When $x = 70$, $y = -\frac{2}{5}(70) + 30 = 2 < 3 \Rightarrow$ Shadow determined by child.(c) Need $(0, 30)$, $(d, 6)$, $(d + 10, 3)$ collinear.

$$\frac{30 - 6}{0 - d} = \frac{6 - 3}{d - (d + 10)} \Rightarrow \frac{24}{-d} = \frac{3}{-10} \Rightarrow d = 80 \text{ feet}$$

(d) Let y be the distance from the base of the street light to the tip of the shadow. We know that $\frac{dx}{dt} = -5$.For $x > 80$, the shadow is determined by the man.

$$\frac{y}{30} = \frac{y - x}{6} \Rightarrow y = \frac{5}{4}x \text{ and } \frac{dy}{dt} = \frac{5}{4} \frac{dx}{dt} = \frac{-25}{4}.$$

For $x < 80$, the shadow is determined by the child.

$$\frac{y}{30} = \frac{y - x - 10}{3} \Rightarrow y = \frac{10}{9}x + \frac{100}{9} \text{ and } \frac{dy}{dt} = \frac{10}{9} \frac{dx}{dt} = \frac{-50}{9}.$$

Therefore,

$$\frac{dy}{dt} = \begin{cases} -\frac{25}{4} & x > 80 \\ -\frac{50}{9} & 0 < x < 80 \end{cases}$$

 $\frac{dy}{dt}$ is not continuous at $x = 80$.

$$\begin{aligned} 11. L'(x) &= \lim_{\Delta x \rightarrow 0} \frac{L(x + \Delta x) - L(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{L(x) + L(\Delta x) - L(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{L(\Delta x)}{\Delta x} \end{aligned}$$

$$\text{Also, } L'(0) = \lim_{\Delta x \rightarrow 0} \frac{L(\Delta x) - L(0)}{\Delta x}$$

But, $L(0) = 0$ because $L(0) = L(0 + 0) = L(0) + L(0) \Rightarrow L(0) = 0$.Thus, $L'(x) = L'(0)$, for all x .The graph of L is a line through the origin of slope $L'(0)$.

13. (a)

z (degrees)	0.1	0.01	0.0001
$\frac{\sin z}{z}$	0.0174524	0.0174533	0.0174533

(b) $\lim_{z \rightarrow 0} \frac{\sin z}{z} \approx 0.0174533$

In fact, $\lim_{z \rightarrow 0} \frac{\sin z}{z} = \frac{\pi}{180}$.

(c) $\frac{d}{dz}(\sin z) = \lim_{\Delta z \rightarrow 0} \frac{\sin(z + \Delta z) - \sin z}{\Delta z}$

$$= \lim_{\Delta z \rightarrow 0} \frac{\sin z \cdot \cos \Delta z + \sin \Delta z \cdot \cos z - \sin z}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \left[\sin z \left(\frac{\cos \Delta z - 1}{\Delta z} \right) \right] + \lim_{\Delta z \rightarrow 0} \left[\cos z \left(\frac{\sin \Delta z}{\Delta z} \right) \right]$$

$$= (\sin z)(0) + (\cos z) \left(\frac{\pi}{180} \right) = \frac{\pi}{180} \cos z$$

(d) $S(90) = \sin\left(\frac{\pi}{180} 90\right) = \sin \frac{\pi}{2} = 1$; $C(180) = \cos\left(\frac{\pi}{180} 180\right) = -1$

$$\frac{d}{dz}S(z) = \frac{d}{dz} \sin(cz) = c \cdot \cos(cz) = \frac{\pi}{180} C(z)$$

(e) The formulas for the derivatives are more complicated in degrees.

15. $j(t) = a'(t)$

(a) $j(t)$ is the rate of change of the acceleration.

(b) From Exercise 102 in Section 2.3:

$$s(t) = -8.25t^2 + 66t$$

$$v(t) = -16.5t + 66$$

$$a(t) = -16.5$$

$$a'(t) = j(t) = 0$$

The acceleration is constant.

CHAPTER 3

Applications of Differentiation

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CHAPTER 3

Applications of Differentiation

Section 3.1 Extrema on an Interval

Solutions to Odd-Numbered Exercises

1. $f(x) = \frac{x^2}{x^2 + 4}$

$$f'(x) = \frac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \frac{8x}{(x^2 + 4)^2}$$

$$f'(0) = 0$$

5. $f(x) = (x + 2)^{2/3}$

$$f'(x) = \frac{2}{3}(x + 2)^{-1/3}$$

$f'(-2)$ is undefined.

9. Critical numbers: $x = 1, 2, 3$

$x = 1, 3$: absolute maximum

$x = 2$: absolute minimum

13. $g(t) = t\sqrt{4-t}, t < 3$

$$g'(t) = t\left[\frac{1}{2}(4-t)^{-1/2}(-1)\right] + (4-t)^{1/2}$$

$$= \frac{1}{2}(4-t)^{-1/2}[-t + 2(4-t)]$$

$$= \frac{8-3t}{2\sqrt{4-t}}$$

Critical number is $t = \frac{8}{3}$.

17. $f(x) = 2(3-x), [-1, 2]$

$$f'(x) = -2 \Rightarrow \text{No critical numbers}$$

Left endpoint: $(-1, 8)$ Maximum

Right endpoint: $(2, 2)$ Minimum

3. $f(x) = x + \frac{27}{2x^2} = x + \frac{27}{2}x^{-2}$

$$f'(x) = 1 - 27x^{-3} = 1 - \frac{27}{x^3}$$

$$f'(3) = 1 - \frac{27}{3^3} = 1 - 1 = 0$$

7. Critical numbers: $x = 2$

$x = 2$: absolute maximum

11. $f(x) = x^2(x-3) = x^3 - 3x^2$

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

Critical numbers: $x = 0, x = 2$

15. $h(x) = \sin^2 x + \cos x, 0 < x < 2\pi$

$$h'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1)$$

On $(0, 2\pi)$, critical numbers: $x = \frac{\pi}{3}, x = \pi, x = \frac{5\pi}{3}$

19. $f(x) = -x^2 + 3x, [0, 3]$

$$f'(x) = -2x + 3$$

Left endpoint: $(0, 0)$ Minimum

Critical number: $(\frac{3}{2}, \frac{9}{4})$ Maximum

Right endpoint: $(3, 0)$ Minimum

21. $f(x) = x^3 - \frac{3}{2}x^2, [-1, 2]$

$$f'(x) = 3x^2 - 3x = 3x(x - 1)$$

Left endpoint: $\left(-1, -\frac{5}{2}\right)$ Minimum

Right endpoint: $(2, 2)$ Maximum

Critical number: $(0, 0)$

Critical number: $\left(1, -\frac{1}{2}\right)$

25. $g(t) = \frac{t^2}{t^2 + 3}, [-1, 1]$

$$g'(t) = \frac{6t}{(t^2 + 3)^2}$$

Left endpoint: $\left(-1, \frac{1}{4}\right)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $\left(1, \frac{1}{4}\right)$ Maximum

29. $f(x) = \cos \pi x, \left[0, \frac{1}{6}\right]$

$$f'(x) = -\pi \sin \pi x$$

Left endpoint: $(0, 1)$ Maximum

Right endpoint: $\left(\frac{1}{6}, \frac{\sqrt{3}}{2}\right)$ Minimum

33. (a) Minimum: $(0, -3)$

Maximum: $(2, 1)$

(b) Minimum: $(0, -3)$

(c) Maximum: $(2, 1)$

(d) No extrema

23. $f(x) = 3x^{2/3} - 2x, [-1, 1]$

$$f'(x) = 2x^{-1/3} - 2 = \frac{2(1 - \sqrt[3]{x})}{\sqrt[3]{x}}$$

Left endpoint: $(-1, 5)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $(1, 1)$

27. $h(s) = \frac{1}{s - 2}, [0, 1]$

$$h'(s) = \frac{-1}{(s - 2)^2}$$

Left endpoint: $\left(0, -\frac{1}{2}\right)$ Maximum

Right endpoint: $(1, -1)$ Minimum

31. $y = \frac{4}{x} + \tan \frac{\pi x}{8}, [1, 2]$

$$y' = \frac{-4}{x^2} + \frac{\pi}{8} \sec^2 \frac{\pi x}{8} = 0$$

$$\frac{\pi}{8} \sec^2 \frac{\pi x}{8} = \frac{4}{x^2}$$

On the interval $[1, 2]$, this equation has no solutions. Thus, there are no critical numbers.

Left endpoint: $(1, \sqrt{2} + 3) \approx (1, 4.4142)$ Maximum

Right endpoint: $(2, 3)$ Minimum

35. $f(x) = x^2 - 2x$

(a) Minimum: $(1, -1)$

Maximum: $(-1, 3)$

(b) Maximum: $(3, 3)$

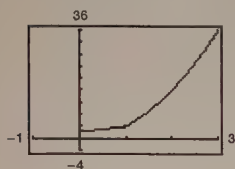
(c) Minimum: $(1, -1)$

(d) Minimum: $(1, -1)$

$$37. f(x) = \begin{cases} 2x + 2, & 0 \leq x \leq 1 \\ 4x^2, & 1 < x \leq 3 \end{cases}$$

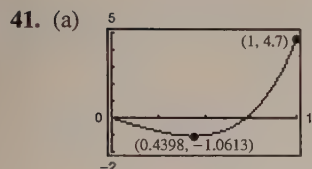
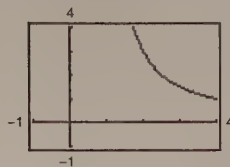
Left endpoint: (0, 2) Minimum

Right endpoint: (3, 36) Maximum



$$39. f(x) = \frac{3}{x-1}, (1, 4]$$

Right endpoint: (4, 1) Minimum



Maximum: (1, 4.7) (endpoint)

Minimum: (0.4398, -1.0613)

$$(b) \quad f(x) = 3.2x^5 + 5x^3 - 3.5x, [0, 1]$$

$$f'(x) = 16x^4 + 15x^2 - 3.5$$

$$16x^4 + 15x^2 - 3.5 = 0$$

$$x^2 = \frac{-15 \pm \sqrt{(15)^2 - 4(16)(-3.5)}}{2(16)}$$

$$= \frac{-15 \pm \sqrt{449}}{32}$$

$$x = \sqrt{\frac{-15 + \sqrt{449}}{32}} \approx 0.4398$$

$$f(0) = 0$$

$$f(1) = 4.7 \text{ Maximum (endpoint)}$$

$$f\left(\sqrt{\frac{-15 + \sqrt{449}}{32}}\right) \approx -1.0613$$

Minimum: (0.4398, -1.0613)

$$43. f(x) = (1 + x^3)^{1/2}, [0, 2]$$

$$f'(x) = \frac{3}{2}x^2(1 + x^3)^{-1/2}$$

$$f''(x) = \frac{3}{4}(x^4 + 4x)(1 + x^3)^{-3/2}$$

$$f'''(x) = -\frac{3}{8}(x^6 + 20x^3 - 8)(1 + x^3)^{-5/2}$$

Setting $f''' = 0$, we have $x^6 + 20x^3 - 8 = 0$.

$$x^3 = \frac{-20 \pm \sqrt{400 - 4(1)(-8)}}{2}$$

$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1$$

In the interval $[0, 2]$, choose

$$x = \sqrt[3]{-10 + \sqrt{108}} = \sqrt{3} - 1 \approx 0.732.$$

$$\left| f''\left(\sqrt[3]{-10 + \sqrt{108}}\right) \right| \approx 1.47 \text{ is the maximum value.}$$

$$45. f(x) = (x + 1)^{2/3}, [0, 2]$$

$$f'(x) = \frac{2}{3}(x + 1)^{-1/3}$$

$$f''(x) = -\frac{2}{9}(x + 1)^{-4/3}$$

$$f'''(x) = \frac{8}{27}(x + 1)^{-7/3}$$

$$f^{(4)}(x) = -\frac{56}{81}(x + 1)^{-10/3}$$

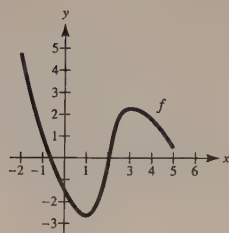
$$f^{(5)}(x) = \frac{560}{243}(x + 1)^{-13/3}$$

$$|f^{(4)}(0)| = \frac{56}{81} \text{ is the maximum value.}$$

47. $f(x) = \tan x$

f is continuous on $[0, \pi/4]$ but not on $[0, \pi]$. $\lim_{x \rightarrow \pi/2^-} \tan x = \infty$.

49.



51. (a) Yes

(b) No

53. (a) No

(b) Yes

55. $P = VI - RI^2 = 12I - 0.5I^2, 0 \leq I \leq 15$

$P = 0$ when $I = 0$.

$P = 67.5$ when $I = 15$.

$P' = 12 - I = 0$

Critical number: $I = 12$ ampsWhen $I = 12$ amps, $P = 72$, the maximum output.

No, a 20-amp fuse would not increase the power output.

 P is decreasing for $I > 12$.

57.

$$S = 6hs + \frac{3s^2}{2} \left(\frac{\sqrt{3} - \cos \theta}{\sin \theta} \right), \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$$

$$\frac{dS}{d\theta} = \frac{3s^2}{2} (-\sqrt{3} \csc \theta \cot \theta + \csc^2 \theta)$$

$$= \frac{3s^2}{2} \csc \theta (-\sqrt{3} \cot \theta + \csc \theta) = 0$$

$\csc \theta = \sqrt{3} \cot \theta$

$\sec \theta = \sqrt{3}$

$\theta = \operatorname{arcsec} \sqrt{3} \approx 0.9553$ radians

$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2} (\sqrt{3})$$

$$S\left(\frac{\pi}{2}\right) = 6hs + \frac{3s^2}{2} (\sqrt{3})$$

$$S(\operatorname{arcsec} \sqrt{3}) = 6hs + \frac{3s^2}{2} (\sqrt{2})$$

 S is minimum when $\theta = \operatorname{arcsec} \sqrt{3} \approx 0.9553$ radians.

59. (a) $y = ax^2 + bx + c$

$y' = 2ax + b$

The coordinates of B are $(500, 30)$, and those of A are $(-500, 45)$.From the slopes at A and B ,

$-1000a + b = -0.09$

$1000a + b = 0.06$

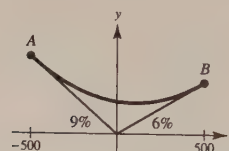
Solving these two equations, you obtain $a = 3/40000$ and $b = -3/200$. From the points $(500, 30)$ and $(-500, 45)$, you obtain

$$30 = \frac{3}{40000} 500^2 + 500 \left(\frac{-3}{200} \right) + c$$

$$45 = \frac{3}{40000} 500^2 - 500 \left(\frac{-3}{200} \right) + c.$$

In both cases, $c = 18.75 = \frac{75}{4}$. Thus,

$$y = \frac{3}{40000} x^2 - \frac{3}{200} x + \frac{75}{4}.$$



—CONTINUED—

59. —CONTINUED—

(b)

x	-500	-400	-300	-200	-100	0	100	200	300	400	500
d	0	.75	3	6.75	12	18.75	12	6.75	3	.75	0

For $-500 \leq x \leq 0$, $d = (ax^2 + bx + c) - (-0.09x)$.

For $0 \leq x \leq 500$, $d = (ax^2 + bx + c) - (0.06x)$.

(c) The lowest point on the highway is (100, 18), which is not directly over the point where the two hillsides come together.

61. True. See Exercise 25.

63. True.

Section 3.2 Rolle's Theorem and the Mean Value Theorem

1. Rolle's Theorem does not apply to $f(x) = 1 - |x - 1|$ over $[0, 2]$ since f is not differentiable at $x = 1$.

3. $f(x) = x^2 - x - 2 = (x - 2)(x + 1)$

x -intercepts: $(-1, 0)$, $(2, 0)$

$$f'(x) = 2x - 1 = 0 \text{ at } x = \frac{1}{2}.$$

5. $f(x) = x\sqrt{x+4}$

x -intercepts: $(-4, 0)$, $(0, 0)$

$$f'(x) = \frac{1}{2}(x+4)^{-1/2} + (x+4)^{1/2}$$

$$= (x+4)^{-1/2} \left(\frac{x}{2} + (x+4) \right)$$

$$f'(x) = \left(\frac{3}{2}x + 4 \right) (x+4)^{-1/2} = 0 \text{ at } x = -\frac{8}{3}$$

7. $f(x) = x^2 - 2x$, $[0, 2]$

$$f(0) = f(2) = 0$$

f is continuous on $[0, 2]$. f is differentiable on $(0, 2)$.
Rolle's Theorem applies.

$$f'(x) = 2x - 2$$

$$2x - 2 = 0 \Rightarrow x = 1$$

c value: 1

9. $f(x) = (x-1)(x-2)(x-3)$, $[1, 3]$

$$f(1) = f(3) = 0$$

f is continuous on $[1, 3]$. f is differentiable on $(1, 3)$.
Rolle's Theorem applies.

$$f(x) = x^3 - 6x^2 + 11x - 6$$

$$f'(x) = 3x^2 - 12x + 11$$

$$3x^2 - 12x + 11 = 0 \Rightarrow x = \frac{6 \pm \sqrt{3}}{3}$$

$$c = \frac{6 - \sqrt{3}}{3}, c = \frac{6 + \sqrt{3}}{3}$$

11. $f(x) = x^{2/3} - 1$, $[-8, 8]$

$$f(-8) = f(8) = 3$$

f is continuous on $[-8, 8]$. f is not differentiable on $(-8, 8)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.

13. $f(x) = \frac{x^2 - 2x - 3}{x + 2}, [-1, 3]$

$$f(-1) = f(3) = 0$$

f is continuous on $[-1, 3]$. (Note: The discontinuity, $x = -2$, is not in the interval.) f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = \frac{(x+2)(2x-2) - (x^2-2x-3)(1)}{(x+2)^2} = 0$$

$$\frac{x^2 + 4x - 1}{(x+2)^2} = 0$$

$$x = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

c value: $-2 + \sqrt{5}$

15. $f(x) = \sin x, [0, 2\pi]$

$$f(0) = f(2\pi) = 0$$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$$f'(x) = \cos x$$

c values: $\frac{\pi}{2}, \frac{3\pi}{2}$

17. $f(x) = \frac{6x}{\pi} - 4 \sin^2 x, \left[0, \frac{\pi}{6}\right]$

$$f(0) = f\left(\frac{\pi}{6}\right) = 0$$

f is continuous on $[0, \pi/6]$. f is differentiable on $(0, \pi/6)$. Rolle's Theorem applies.

$$f'(x) = \frac{6}{\pi} - 8 \sin x \cos x = 0$$

$$\frac{6}{\pi} = 8 \sin x \cos x$$

$$\frac{3}{4\pi} = \frac{1}{2} \sin 2x$$

$$\frac{3}{2\pi} = \sin 2x$$

$$\frac{1}{2} \arcsin\left(\frac{3}{2\pi}\right) = x$$

$$x \approx 0.2489$$

c value: 0.2489

19. $f(x) = \tan x, [0, \pi]$

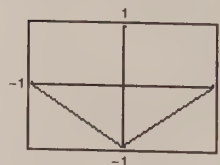
$$f(0) = f(\pi) = 0$$

f is not continuous on $[0, \pi]$ since $f(\pi/2)$ does not exist. Rolle's Theorem does not apply.

21. $f(x) = |x| - 1, [-1, 1]$

$$f(-1) = f(1) = 0$$

f is continuous on $[-1, 1]$. f is not differentiable on $(-1, 1)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.



$$23. f(x) = 4x - \tan \pi x, \left[-\frac{1}{4}, \frac{1}{4}\right]$$

$$f\left(-\frac{1}{4}\right) = f\left(\frac{1}{4}\right) = 0$$

f is continuous on $[-1/4, 1/4]$. f is differentiable on $(-1/4, 1/4)$. Rolle's Theorem applies.

$$f'(x) = 4 - \pi \sec^2 \pi x = 0$$

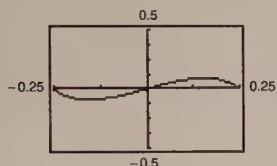
$$\sec^2 \pi x = \frac{4}{\pi}$$

$$\sec \pi x = \pm \frac{2}{\sqrt{\pi}}$$

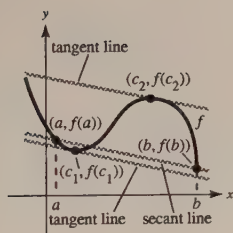
$$x = \pm \frac{1}{\pi} \operatorname{arcsec} \frac{2}{\sqrt{\pi}} = \pm \frac{1}{\pi} \arccos \frac{\sqrt{\pi}}{2}$$

$$\approx \pm 0.1533 \text{ radian}$$

c values: ± 0.1533 radian



27.



31. $f(x) = x^2$ is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$.

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 4}{3} = -1$$

$f'(x) = 2x = -1$ when $x = -\frac{1}{2}$. Therefore,

$$c = -\frac{1}{2}.$$

$$25. f(t) = -16t^2 + 48t + 32$$

$$(a) f(1) = f(2) = 64$$

(b) $v = f'(t)$ must be 0 at some time in $(1, 2)$.

$$f'(t) = -32t + 48 = 0$$

$$t = \frac{3}{2} \text{ seconds}$$

$$29. f(x) = \frac{1}{x-3}, [0, 6]$$

f has a discontinuity at $x = 3$.

33. $f(x) = x^{2/3}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = 1$$

$$f'(x) = \frac{2}{3}x^{-1/3} = 1$$

$$x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$c = \frac{8}{27}$$

35. $f(x) = \sqrt{2-x}$ is continuous on $[-7, 2]$ and differentiable on $(-7, 2)$.

$$\frac{f(2) - f(-7)}{2 - (-7)} = \frac{0 - 3}{9} = -\frac{1}{3}$$

$$f'(x) = \frac{-1}{2\sqrt{2-x}} = -\frac{1}{3}$$

$$2\sqrt{2-x} = 3$$

$$\sqrt{2-x} = \frac{3}{2}$$

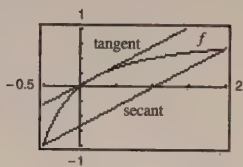
$$2-x = \frac{9}{4}$$

$$x = -\frac{1}{4}$$

$$c = -\frac{1}{4}$$

39. $f(x) = \frac{x}{x+1}$ on $[-\frac{1}{2}, 2]$.

(a)



(b) Secant line:

$$\text{slope} = \frac{f(2) - f(-1/2)}{2 - (-1/2)} = \frac{2/3 - (-1)}{5/2} = \frac{2}{3}$$

$$y - \frac{2}{3} = \frac{2}{3}(x - 2)$$

$$3y - 2 = 2x - 4$$

$$3y - 2x + 2 = 0$$

37. $f(x) = \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = \cos x = 0$$

$$c = \frac{\pi}{2}$$

$$(c) f'(x) = \frac{1}{(x+1)^2} = \frac{2}{3}$$

$$(x+1)^2 = \frac{3}{2}$$

$$x = -1 \pm \sqrt{\frac{3}{2}} = -1 \pm \frac{\sqrt{6}}{2}$$

In the interval $[-1/2, 2]$, $c = -1 + (\sqrt{6}/2)$.

$$f(c) = \frac{-1 + (\sqrt{6}/2)}{[-1 + (\sqrt{6}/2)] + 1} = \frac{-2 + \sqrt{6}}{\sqrt{6}} = \frac{-2}{\sqrt{6}} +$$

$$\text{Tangent line: } y - 1 + \frac{2}{\sqrt{6}} = \frac{2}{3}\left(x - \frac{\sqrt{6}}{2} + 1\right)$$

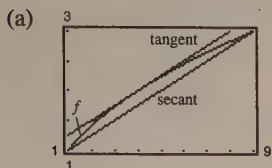
$$y - 1 + \frac{\sqrt{6}}{3} = \frac{2}{3}x - \frac{\sqrt{6}}{3} + \frac{2}{3}$$

$$3y - 2x - 5 + 2\sqrt{6} = 0$$

41. $f(x) = \sqrt{x}$, $[1, 9]$

$(1, 1), (9, 3)$

$$m = \frac{3-1}{9-1} = \frac{1}{4}$$



(b) Secant line: $y - 1 = \frac{1}{4}(x - 1)$

$$y = \frac{1}{4}x + \frac{3}{4}$$

$$0 = x - 4y + 3$$

(c) $f'(x) = \frac{1}{2\sqrt{x}}$

$$\frac{f(9) - f(1)}{9 - 1} = \frac{1}{4}$$

$$\frac{1}{2\sqrt{c}} = \frac{1}{4}$$

$$\sqrt{c} = 2$$

$$c = 4$$

$$(c, f(c)) = (4, 2)$$

$$m = f'(4) = \frac{1}{4}$$

Tangent line: $y - 2 = \frac{1}{4}(x - 4)$

$$y = \frac{1}{4}x + 1$$

$$0 = x - 4y + 4$$

43. $s(t) = -4.9t^2 + 500$

(a) $V_{\text{avg}} = \frac{s(3) - s(0)}{3 - 0} = \frac{455.9 - 500}{3} = -14.7 \text{ m/sec}$

(b) $s(t)$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$.
Therefore, the Mean Value Theorem applies.

$$v(t) = s'(t) = -9.8t = -14.7 \text{ m/sec}$$

$$t = \frac{-14.7}{-9.8} = 1.5 \text{ seconds}$$

45. No. Let $f(x) = x^2$ on $[-1, 2]$.

$$f'(x) = 2x$$

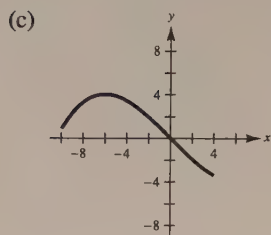
$f'(0) = 0$ and zero is in the interval $(-1, 2)$ but
 $f(-1) \neq f(2)$.

47. Let $S(t)$ be the position function of the plane. If $t = 0$ corresponds to 2 P.M., $S(0) = 0$, $S(5.5) = 2500$ and the Mean Value Theorem says that there exists a time t_0 , $0 < t_0 < 5.5$, such that

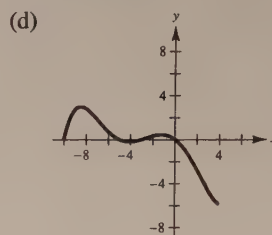
$$S'(t_0) = v(t_0) = \frac{2500 - 0}{5.5 - 0} \approx 454.54.$$

Applying the Intermediate Value Theorem to the velocity function on the intervals $[0, t_0]$ and $[t_0, 5.5]$, you see that there are at least two times during the flight when the speed was 400 miles per hour. ($0 < 400 < 454.54$)

49. (a) f is continuous on $[-10, 4]$ and changes sign, $(f(-8) > 0, f(3) < 0)$. By the Intermediate Value Theorem, there exists at least one value of x in $[-10, 4]$ satisfying $f(x) = 0$.

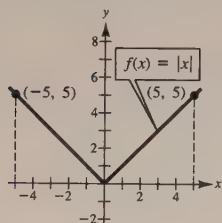


- (b) There exist real numbers a and b such that $-10 < a < b < 4$ and $f(a) = f(b) = 2$. Therefore, by Rolle's Theorem there exists at least one number c in $(-10, 4)$ such that $f'(c) = 0$. This is called a critical number.



- (e) No, f' did not have to be continuous on $[-10, 4]$.

51. f is continuous on $[-5, 5]$ and does not satisfy the conditions of the Mean Value Theorem.
 $\Rightarrow f$ is not differentiable on $(-5, 5)$.
 Example: $f(x) = |x|$



53. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

55. True. A polynomial is continuous and differentiable everywhere.

57. Suppose that $p(x) = x^{2n+1} + ax + b$ has two real roots x_1 and x_2 . Then by Rolle's Theorem, since $p(x_1) = p(x_2) = 0$, there exists c in (x_1, x_2) such that $p'(c) = 0$. But $p'(x) = (2n+1)x^{2n} + a \neq 0$, since $n > 0, a > 0$. Therefore, $p(x)$ cannot have two real roots.

59. If $p(x) = Ax^2 + Bx + C$, then

$$\begin{aligned} p'(x) &= 2Ax + B = \frac{f(b) - f(a)}{b - a} = \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a} \\ &= \frac{A(b^2 - a^2) + B(b - a)}{b - a} \\ &= \frac{(b - a)[A(b + a) + B]}{b - a} \\ &= A(b + a) + B. \end{aligned}$$

Thus, $2Ax = A(b + a)$ and $x = (b + a)/2$ which is the midpoint of $[a, b]$.

61. $f(x) = \frac{1}{2} \cos x$ differentiable on $(-\infty, \infty)$.

$$f'(x) = -\frac{1}{2} \sin x$$

$$-\frac{1}{2} \leq f'(x) \leq \frac{1}{2} \Rightarrow f'(x) < 1 \text{ for all real numbers.}$$

Thus, from Exercise 60, f has, at most, one fixed point. ($x \approx 0.4502$)

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

1. $f(x) = x^2 - 6x + 8$

Increasing on: $(3, \infty)$

Decreasing on: $(-\infty, 3)$

5. $f(x) = \frac{1}{x^2} = x^{-2}$

$$f'(x) = \frac{-2}{x^3}$$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on $(-\infty, 0)$

Decreasing on $(0, \infty)$

9. $y = x\sqrt{16 - x^2}$ Domain: $[-4, 4]$

$$y' = \frac{-2(x^2 - 8)}{\sqrt{16 - x^2}} = \frac{-2}{\sqrt{16 - x^2}}(x - 2\sqrt{2})(x + 2\sqrt{2})$$

Critical numbers: $x = \pm 2\sqrt{2}$

Test intervals:	$-4 < x < -2\sqrt{2}$	$-2\sqrt{2} < x < 2\sqrt{2}$	$2\sqrt{2} < x < 4$
Sign of y' :	$y' < 0$	$y' > 0$	$y' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on $(-2\sqrt{2}, 2\sqrt{2})$

Decreasing on $(-4, -2\sqrt{2}), (2\sqrt{2}, 4)$

11. $f(x) = x^2 - 6x$

$$f'(x) = 2x - 6 = 0$$

Critical number: $x = 3$

Test intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(3, \infty)$

Decreasing on: $(-\infty, 3)$

Relative minimum: $(3, -9)$

3. $y = \frac{x^3}{4} - 3x$

Increasing on: $(-\infty, -2), (2, \infty)$

Decreasing on: $(-2, 2)$

7. $g(x) = x^2 - 2x - 8$

$$g'(x) = 2x - 2$$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $g'(x)$:	$g' < 0$	$g' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(1, \infty)$

Decreasing on: $(-\infty, 1)$

13. $f(x) = -2x^2 + 4x + 3$

$$f'(x) = -4x + 4 = 0$$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 1)$

Decreasing on: $(1, \infty)$

Relative maximum: $(1, 5)$

15. $f(x) = 2x^3 + 3x^2 - 12x$

$$f'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1) = 0$$

Critical numbers: $x = -2, 1$

Test intervals:	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (1, \infty)$ Decreasing on: $(-2, 1)$ Relative maximum: $(-2, 20)$ Relative minimum: $(1, -7)$

17. $f(x) = x^2(3-x) = 3x^2 - x^3$

$$f'(x) = 6x - 3x^2 = 3x(2-x)$$

Critical numbers: $x = 0, 2$

Test intervals:	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(0, 2)$ Decreasing on: $(-\infty, 0), (2, \infty)$ Relative maximum: $(2, 4)$ Relative minimum: $(0, 0)$

19. $f(x) = \frac{x^5 - 5x}{5}$

$$f'(x) = x^4 - 1$$

Critical numbers: $x = -1, 1$

Test intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$ Decreasing on: $(-1, 1)$ Relative maximum: $(-1, \frac{4}{5})$ Relative minimum: $(1, -\frac{4}{5})$

21. $f(x) = x^{1/3} + 1$

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

 Critical number: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

 Increasing on: $(-\infty, \infty)$

No relative extrema

25. $f(x) = 5 - |x - 5|$

$$f'(x) = -\frac{x-5}{|x-5|} = \begin{cases} 1, & x < 5 \\ -1, & x > 5 \end{cases}$$

 Critical number: $x = 5$

Test intervals:	$-\infty < x < 5$	$5 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

 Increasing on: $(-\infty, 5)$

 Decreasing on: $(5, \infty)$

 Relative maximum: $(5, 5)$

27. $f(x) = x + \frac{1}{x}$

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

 Critical numbers: $x = -1, 1$

 Discontinuity: $x = 0$

Test intervals:	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

 Increasing on: $(-\infty, -1), (1, \infty)$

 Decreasing on: $(-1, 0), (0, 1)$

 Relative maximum: $(-1, -2)$

 Relative minimum: $(1, 2)$

23. $f(x) = (x - 1)^{2/3}$

$$f'(x) = \frac{2}{3(x - 1)^{1/3}}$$

 Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

 Increasing on: $(1, \infty)$

 Decreasing on: $(-\infty, 1)$

 Relative minimum: $(1, 0)$

29. $f(x) = \frac{x^2}{x^2 - 9}$

$$f'(x) = \frac{(x^2 - 9)(2x) - (x^2)(2x)}{(x^2 - 9)^2} = \frac{-18x}{(x^2 - 9)^2}$$

Critical number: $x = 0$

Discontinuities: $x = -3, 3$

Test intervals:	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$	$f' < 0$	$f' < 0$
Conclusion:	Increasing	Increasing	Decreasing	Decreasing

Increasing on: $(-\infty, -3), (-3, 0)$

Decreasing on: $(0, 3), (3, \infty)$

Relative maximum: $(0, 0)$

31. $f(x) = \frac{x^2 - 2x + 1}{x + 1}$

$$f'(x) = \frac{(x + 1)(2x - 2) - (x^2 - 2x + 1)(1)}{(x + 1)^2} = \frac{x^2 + 2x - 3}{(x + 1)^2} = \frac{(x + 3)(x - 1)}{(x + 1)^2}$$

Critical numbers: $x = -3, 1$

Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -3), (1, \infty)$

Decreasing on: $(-3, -1), (-1, 1)$

Relative maximum: $(-3, -8)$

Relative minimum: $(1, 0)$

33. $f(x) = \frac{x}{2} + \cos x, 0 < x < 2\pi$

$$f'(x) = \frac{1}{2} - \sin x = 0$$

Critical numbers: $x = \frac{\pi}{6}, \frac{5\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{6}$	$\frac{\pi}{6} < x < \frac{5\pi}{6}$	$\frac{5\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{6}\right), \left(\frac{5\pi}{6}, 2\pi\right)$

Relative maximum: $\left(\frac{\pi}{6}, \frac{\pi + 6\sqrt{3}}{12}\right)$

Decreasing on: $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

Relative minimum: $\left(\frac{5\pi}{6}, \frac{5\pi - 6\sqrt{3}}{12}\right)$

35. $f(x) = \sin^2 x + \sin x, 0 < x < 2\pi$

$$f'(x) = 2 \sin x \cos x + \cos x = \cos x(2 \sin x + 1) = 0$$

Critical numbers: $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{7\pi}{6}$	$\frac{7\pi}{6} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{11\pi}{6}$	$\frac{11\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{2}\right), \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{2}, \frac{7\pi}{6}\right), \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$

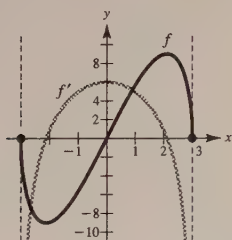
Relative minima: $\left(\frac{7\pi}{6}, -\frac{1}{4}\right), \left(\frac{11\pi}{6}, -\frac{1}{4}\right)$

Relative maxima: $\left(\frac{\pi}{2}, 2\right), \left(\frac{3\pi}{2}, 0\right)$

37. $f(x) = 2x\sqrt{9-x^2}, [-3, 3]$

(a) $f'(x) = \frac{2(9-2x^2)}{\sqrt{9-x^2}}$

(b)



(c) $\frac{2(9-2x^2)}{\sqrt{9-x^2}} = 0$

Critical numbers: $x = \pm \frac{3}{\sqrt{2}} = \pm \frac{3\sqrt{2}}{2}$

(d) Intervals:

$$\left(-3, -\frac{3\sqrt{2}}{2}\right) \quad \left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) \quad \left(\frac{3\sqrt{2}}{2}, 3\right)$$

$$f'(x) < 0$$

$$f'(x) > 0$$

$$f'(x) < 0$$

Decreasing

Increasing

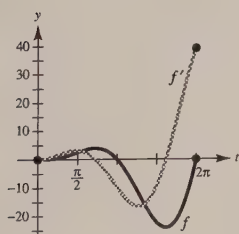
Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

39. $f(t) = t^2 \sin t, [0, 2\pi]$

(a) $f'(t) = t^2 \cos t + 2t \sin t$
 $= t(t \cos t + 2 \sin t)$

(b)



(c) $t(t \cos t + 2 \sin t) = 0$

$$t = 0 \text{ or } t = -2 \tan t$$

$$t \cot t = -2$$

$$t \approx 2.2889, 5.0870 \text{ (graphing utility)}$$

Critical numbers: $t = 2.2889, t = 5.0870$

(d) Intervals:

$$(0, 2.2889)$$

$$(2.2889, 5.0870)$$

$$(5.0870, 2\pi)$$

$$f'(t) > 0$$

$$f'(t) < 0$$

$$f'(t) > 0$$

Increasing

Decreasing

Increasing

f is increasing when f' is positive and decreasing when f' is negative.

$$41. f(x) = \frac{x^5 - 4x^3 + 3x}{x^2 - 1} = \frac{(x^2 - 1)(x^3 - 3x)}{x^2 - 1} = x^3 - 3x, x \neq \pm 1$$

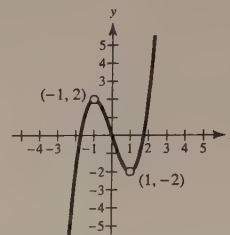
$$f(x) = g(x) = x^3 - 3x \text{ for all } x \neq \pm 1.$$

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1), x \neq \pm 1 \quad f'(x) \neq 0$$

f symmetric about origin

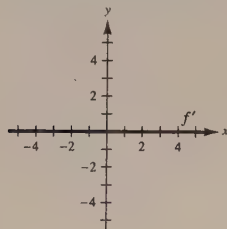
zeros of f : $(0, 0), (\pm\sqrt{3}, 0)$

No relative extrema

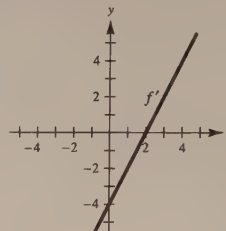


Holes at $(-1, 2)$ and $(1, -2)$

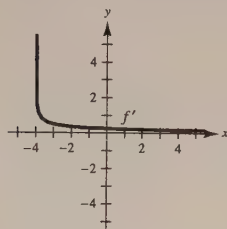
$$43. f(x) = c \text{ is constant} \Rightarrow f'(x) = 0$$



$$45. f \text{ is quadratic} \Rightarrow f' \text{ is a line.}$$



$$47. f \text{ has positive, but decreasing slope}$$



In Exercises 49–53, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.

$$49. g(x) = f(x) + 5$$

$$g'(x) = f'(x)$$

$$g'(0) = f'(0) < 0$$

$$51. g(x) = -f(x)$$

$$g'(x) = -f'(x)$$

$$g'(-6) = -f'(-6) < 0$$

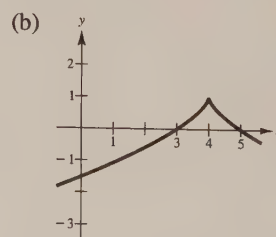
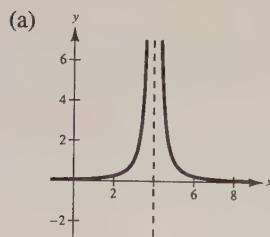
$$53. g(x) = f(x - 10)$$

$$g'(x) = f'(x - 10)$$

$$g'(0) = f'(-10) > 0$$

$$55. f'(x) = \begin{cases} > 0, & x < 4 \Rightarrow f \text{ is increasing on } (-\infty, 4). \\ \text{undefined}, & x = 4 \\ < 0, & x > 4 \Rightarrow f \text{ is decreasing on } (4, \infty). \end{cases}$$

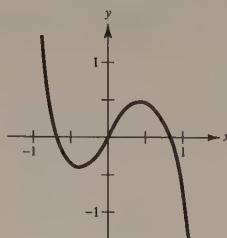
Two possibilities for $f(x)$ are given below.



57. The critical numbers are in intervals $(-0.50, -0.25)$ and $(0.25, 0.50)$ since the sign of f' changes in these intervals. f is decreasing on approximately $(-1, -0.40)$, $(0.48, 1)$, and increasing on $(-0.40, 0.48)$.

Relative minimum when $x \approx -0.40$.

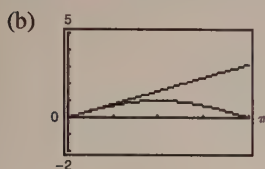
Relative maximum when $x \approx 0.48$.



59. $f(x) = x$, $g(x) = \sin x$, $0 < x < \pi$

x	0.5	1	1.5	2	2.5	3
$f(x)$	0.5	1	1.5	2	2.5	3
$g(x)$	0.479	0.841	0.997	0.909	0.598	0.141

$f(x)$ seems greater than $g(x)$ on $(0, \pi)$.



$x > \sin x$ on $(0, \pi)$

61. $v = k(R - r)r^2 = k(Rr^2 - r^3)$

$$v' = k(2Rr - 3r^2)$$

$$= kr(2R - 3r) = 0$$

$$r = 0 \text{ or } \frac{2}{3}R$$

Maximum when $r = \frac{2}{3}R$.

- (c) Let $h(x) = f(x) - g(x) = x - \sin x$

$$h'(x) = 1 - \cos x > 0 \text{ on } (0, \pi).$$

Therefore, $h(x)$ is increasing on $(0, \pi)$. Since $h(0) = 0$, $h(x) > 0$ on $(0, \pi)$. Thus,

$$x - \sin x > 0$$

$$x > \sin x$$

$$f(x) > g(x) \text{ on } (0, \pi).$$

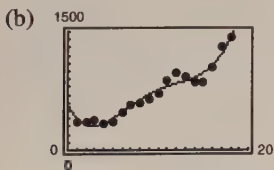
63. $P = \frac{vR_1R_2}{(R_1 + R_2)^2}$, v and R_1 are constant

$$\frac{dP}{dR_2} = \frac{(R_1 + R_2)^2(vR_1) - vR_1R_2[2(R_1 + R_2)(1)]}{(R_1 + R_2)^4}$$

$$= \frac{vR_1(R_1 - R_2)}{(R_1 + R_2)^3} = 0 \Rightarrow R_2 = R_1$$

Maximum when $R_1 = R_2$.

65. (a) $B = 0.1198t^4 - 4.4879t^3 + 56.9909t^2 - 223.0222t + 579.9541$



- (c) $B' = 0$ for $t \approx 2.78$, or 1983, (311.1 thousand bankruptcies)

Actual minimum: 1984 (344.3 thousand bankruptcies)

67. (a) Use a cubic polynomial

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0.$$

(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1.$

$$(0, 0): \quad 0 = a_0 \quad (f(0) = 0)$$

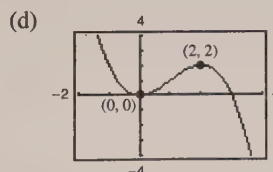
$$0 = a_1 \quad (f'(0) = 0)$$

$$(2, 2): \quad 2 = 8a_3 + 4a_2 \quad (f(2) = 2)$$

$$0 = 12a_3 + 4a_2 \quad (f'(2) = 0)$$

- (c) The solution is $a_0 = a_1 = 0$, $a_2 = \frac{3}{2}$, $a_3 = -\frac{1}{2}$:

$$f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2.$$



69. (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$(0, 0): 0 = a_0 \quad (f(0) = 0)$

$0 = a_1 \quad (f'(0) = 0)$

$(4, 0): 0 = 256a_4 + 64a_3 + 16a_2 \quad (f(4) = 0)$

$0 = 256a_4 + 48a_3 + 8a_2 \quad (f'(4) = 0)$

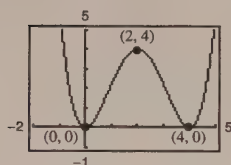
$(2, 4): 4 = 16a_4 + 8a_3 + 4a_2 \quad (f(2) = 4)$

$0 = 32a_4 + 12a_3 + 4a_2 \quad (f'(2) = 0)$

(c) The solution is $a_0 = a_1 = 0, a_2 = 4, a_3 = -2, a_4 = \frac{1}{4}$.

$$f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$$

(d)



71. True

Let $h(x) = f(x) + g(x)$ where f and g are increasing. Then $h'(x) = f'(x) + g'(x) > 0$ since $f'(x) > 0$ and $g'(x) > 0$.

73. False

Let $f(x) = x^3$, then $f'(x) = 3x^2$ and f only has one critical number. Or, let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3(x^2 + 1)$ has no critical numbers.

75. False. For example, $f(x) = x^3$ does not have a relative extrema at the critical number $x = 0$.

77. Assume that $f'(x) < 0$ for all x in the interval (a, b) and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, we know there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) < 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) < 0$, which implies that $f(x_2) < f(x_1)$. Thus, f is decreasing on the interval.

79. Let $f(x) = (1 + x)^n - nx - 1$. Then

$$f'(x) = n(1 + x)^{n-1} - n$$

$$= n[(1 + x)^{n-1} - 1] > 0 \text{ since } x > 0 \text{ and } n > 1.$$

Thus, $f(x)$ is increasing on $(0, \infty)$. Since $f(0) = 0 \Rightarrow f(x) > 0$ on $(0, \infty)$

$$(1 + x)^n - nx - 1 > 0 \Rightarrow (1 + x)^n > 1 + nx.$$

Section 3.4 , Concavity and the Second Derivative Test

1. $y = x^2 - x - 2, y'' = 2$

Concave upward: $(-\infty, \infty)$

5. $f(x) = \frac{x^2 + 1}{x^2 - 1}, y'' = \frac{4(3x^2 + 1)}{(x^2 - 1)^3}$

Concave upward: $(-\infty, -1), (1, \infty)$

Concave downward: $(-1, 1)$

9. $y = 2x - \tan x, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y' = 2 - \sec^2 x$

$y'' = -2 \sec^2 x \tan x$

Concave upward: $\left(-\frac{\pi}{2}, 0\right)$

Concave downward: $\left(0, \frac{\pi}{2}\right)$

13. $f(x) = \frac{1}{4}x^4 - 2x^2$

$f'(x) = x^3 - 4x$

$f''(x) = 3x^2 - 4$

$f''(x) = 3x^2 - 4 = 0$ when $x = \pm \frac{2}{\sqrt{3}}$.

3. $f(x) = \frac{24}{x^2 + 12}, y'' = \frac{-144(4 - x^2)}{(x^2 + 12)^3}$

Concave upward: $(-\infty, -2), (2, \infty)$

Concave downward: $(-2, 2)$

7. $f(x) = 3x^2 - x^3$

$f'(x) = 6x - 3x^2$

$f''(x) = 6 - 6x$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

11. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12$

$f''(x) = 6(x - 2) = 0$ when $x = 2$.

The concavity changes at $x = 2$. $(2, 8)$ is a point of inflection.

Concave upward: $(2, \infty)$

Concave downward: $(-\infty, 2)$

Test interval:	$-\infty < x < -\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}} < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection: $\left(\pm \frac{2}{\sqrt{3}}, -\frac{20}{9}\right)$

15. $f(x) = x(x-4)^3$

$$f'(x) = x[3(x-4)^2] + (x-4)^3$$

$$= (x-4)^2(4x-4)$$

$$f''(x) = 4(x-4)[2(x-4)] + 4(x-4)^2$$

$$= 4(x-4)[2(x-4) + (x-4)]$$

$$= 4(x-4)(3x-6) = 12(x-4)(x-2)$$

$$f''(x) = 12(x-4)(x-2) = 0 \text{ when } x = 2, 4.$$

Test interval:	$-\infty < x < 2$	$2 < x < 4$	$4 < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection: $(2, -16), (4, 0)$

17. $f(x) = x\sqrt{x+3}$, Domain: $[-3, \infty)$

$$f'(x) = x\left(\frac{1}{2}\right)(x+3)^{-1/2} + \sqrt{x+3} = \frac{3(x+2)}{2\sqrt{x+3}}$$

$$f''(x) = \frac{6\sqrt{x+3} - 3(x+2)(x+3)^{-1/2}}{4(x+3)} = \frac{3(x+4)}{4(x+3)^{3/2}}$$

 $f''(x) > 0$ on the entire domain of f (except for $x = -3$, for which $f''(x)$ is undefined). There are no points of inflection.Concave upward on $(-3, \infty)$

19. $f(x) = \frac{x}{x^2+1}$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}$$

Test intervals:	$-\infty < x < -\sqrt{3}$	$-\sqrt{3} < x < 0$	$0 < x < \sqrt{3}$	$\sqrt{3} < x < \infty$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection: $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right), (0, 0), \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$

21. $f(x) = \sin\left(\frac{x}{2}\right), 0 \leq x \leq 4\pi$

$$f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$f''(x) = -\frac{1}{4}\sin\left(\frac{x}{2}\right)$$

$$f''(x) = 0 \text{ when } x = 0, 2\pi, 4\pi.$$

Point of inflection: $(2\pi, 0)$

Test interval:	$0 < x < 2\pi$	$2\pi < x < 4\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward

23. $f(x) = \sec\left(x - \frac{\pi}{2}\right), 0 < x < 4\pi$

$$f'(x) = \sec\left(x - \frac{\pi}{2}\right) \tan\left(x - \frac{\pi}{2}\right)$$

$$f''(x) = \sec^3\left(x - \frac{\pi}{2}\right) + \sec\left(x - \frac{\pi}{2}\right) \tan^2\left(x - \frac{\pi}{2}\right) \neq 0 \text{ for any } x \text{ in the domain of } f.$$

Concave upward: $(0, \pi), (2\pi, 3\pi)$

Concave downward: $(\pi, 2\pi), (3\pi, 4\pi)$

No points of inflection

25. $f(x) = 2 \sin x + \sin 2x, 0 \leq x \leq 2\pi$

$$f'(x) = 2 \cos x + 2 \cos 2x$$

$$f''(x) = -2 \sin x - 4 \sin 2x = -2 \sin x(1 + 4 \cos x)$$

$$f''(x) = 0 \text{ when } x = 0, 1.823, \pi, 4.460.$$

Test interval:	$0 < x < 1.823$	$1.823 < x < \pi$	$\pi < x < 4.460$	$4.460 < x < 2\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection: $(1.823, 1.452), (\pi, 0), (4.46, -1.452)$

27. $f(x) = x^4 - 4x^3 + 2$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

Critical numbers: $x = 0, x = 3$

However, $f''(0) = 0$, so we must use the First Derivative Test. $f'(x) < 0$ on the intervals $(-\infty, 0)$ and $(0, 3)$; hence, $(0, 2)$ is not an extremum. $f''(3) > 0$ so $(3, -25)$ is a relative minimum.

31. $f(x) = x^3 - 3x^2 + 3$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

$$f''(x) = 6x - 6 = 6(x - 1)$$

Critical numbers: $x = 0, x = 2$

$$f''(0) = -6 < 0$$

Therefore, $(0, 3)$ is a relative maximum.

$$f''(2) = 6 > 0$$

Therefore, $(2, -1)$ is a relative minimum.

29. $f(x) = (x - 5)^2$

$$f'(x) = 2(x - 5)$$

$$f''(x) = 2$$

Critical number: $x = 5$

$$f''(5) > 0$$

Therefore, $(5, 0)$ is a relative minimum.

33. $g(x) = x^2(6 - x)^3$

$$g'(x) = x(x - 6)^2(12 - 5x)$$

$$g''(x) = 4(6 - x)(5x^2 - 24x + 18)$$

Critical numbers: $x = 0, \frac{12}{5}, 6$

$$g''(0) = 432 > 0$$

Therefore, $(0, 0)$ is a relative minimum.

$$g''\left(\frac{12}{5}\right) = -155.52 < 0$$

Therefore, $\left(\frac{12}{5}, 268.7\right)$ is a relative minimum.

$$g''(6) = 0$$

Test fails by the First Derivative Test, $(6, 0)$ is not an extremum.

35. $f(x) = x^{2/3} - 3$

$$f'(x) = \frac{2}{3x^{1/3}}$$

$$f''(x) = \frac{-2}{9x^{4/3}}$$

Critical number: $x = 0$

However, $f''(0)$ is undefined, so we must use the First Derivative Test. Since $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$, $(0, -3)$ is a relative minimum.

39. $f(x) = \cos x - x$, $0 \leq x \leq 4\pi$

$$f'(x) = -\sin x - 1 \leq 0$$

Therefore, f is non-increasing and there are no relative extrema.

41. $f(x) = 0.2x^2(x - 3)^3$, $[-1, 4]$

(a) $f'(x) = 0.2x(5x - 6)(x - 3)^2$

$$f''(x) = (x - 3)(4x^2 - 9.6x + 3.6)$$

$$= 0.4(x - 3)(10x^2 - 24x + 9)$$

(b) $f''(0) < 0 \Rightarrow (0, 0)$ is a relative maximum.

$$f''\left(\frac{6}{5}\right) > 0 \Rightarrow (1.2, -1.6796)$$
 is a relative minimum.

Points of inflection:

$$(3, 0), (0.4652, -0.7049), (1.9348, -0.9049)$$

43. $f(x) = \sin x - \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$, $[0, \pi]$

(a) $f'(x) = \cos x - \cos 3x + \cos 5x$

$$f'(x) = 0 \text{ when } x = \frac{\pi}{6}, x = \frac{\pi}{2}, x = \frac{5\pi}{6}$$

$$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$$

$$f''(x) = 0 \text{ when } x = \frac{\pi}{6}, x = \frac{5\pi}{6}, x \approx 1.1731, x \approx 1.9685$$

(b) $f''\left(\frac{\pi}{2}\right) < 0 \Rightarrow \left(\frac{\pi}{2}, 1.53333\right)$ is a relative maximum.

Points of inflection: $\left(\frac{\pi}{6}, 0.2667\right), (1.1731, 0.9638),$

$$(1.9685, 0.9637), \left(\frac{5\pi}{6}, 0.2667\right)$$

Note: $(0, 0)$ and $(\pi, 0)$ are not points of inflection since they are endpoints.

37. $f(x) = x + \frac{4}{x}$

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}$$

$$f''(x) = \frac{8}{x^3}$$

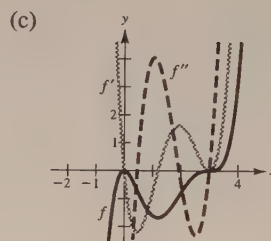
Critical numbers: $x = \pm 2$

$$f''(-2) < 0$$

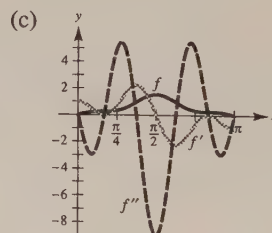
Therefore, $(-2, -4)$ is a relative maximum.

$$f''(2) > 0$$

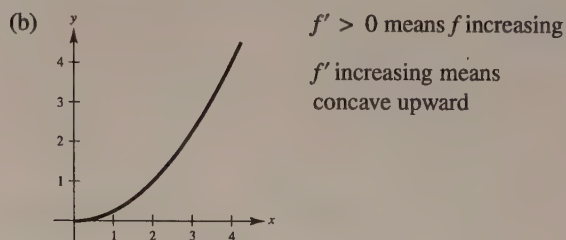
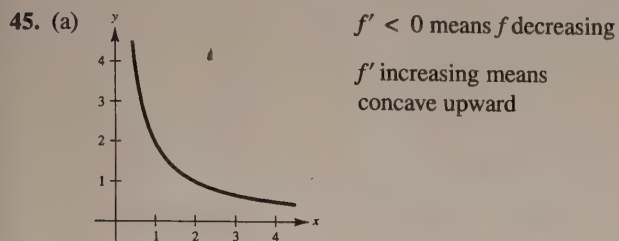
Therefore, $(2, 4)$ is a relative minimum.



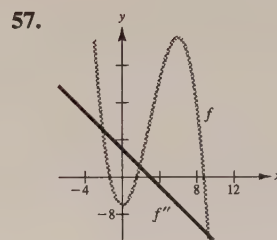
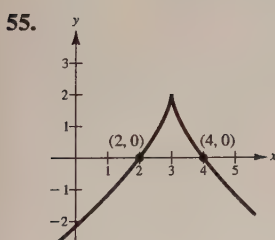
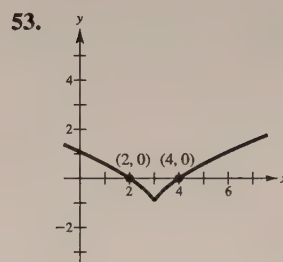
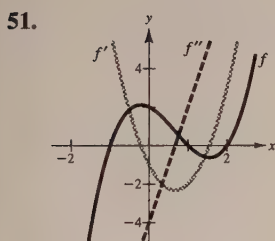
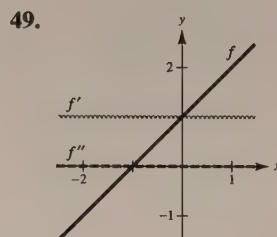
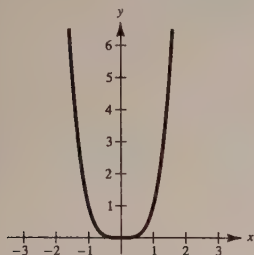
f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.



47. Let $f(x) = x^4$.
 $f''(x) = 12x^2$
 $f''(0) = 0$, but $(0, 0)$ is not a point of inflection.



f'' is linear.
 f' is quadratic.
 f is cubic.
 f concave upwards on $(-\infty, 3)$, downward on $(3, \infty)$.

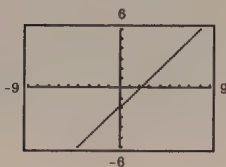
59. (a) $n = 1$:

$$f(x) = x - 2$$

$$f'(x) = 1$$

$$f''(x) = 0$$

No inflection points

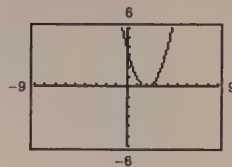
 $n = 2$:

$$f(x) = (x - 2)^2$$

$$f'(x) = 2(x - 2)$$

$$f''(x) = 2$$

No inflection points

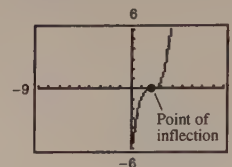
Relative minimum:
(2, 0) $n = 3$:

$$f(x) = (x - 2)^3$$

$$f'(x) = 3(x - 2)^2$$

$$f''(x) = 6(x - 2)$$

Inflection point: (2, 0)

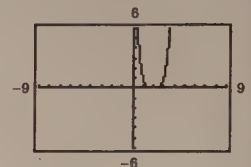
 $n = 4$:

$$f(x) = (x - 2)^4$$

$$f'(x) = 4(x - 2)^3$$

$$f''(x) = 12(x - 2)^2$$

No inflection points:

Relative minimum:
(2, 0)

Conclusion: If $n \geq 3$ and n is odd, then $(2, 0)$ is an inflection point. If $n \geq 2$ and n is even, then $(2, 0)$ is a relative minimum.

(b) Let $f(x) = (x - 2)^n$, $f'(x) = n(x - 2)^{n-1}$, $f''(x) = n(n - 1)(x - 2)^{n-2}$.

For $n \geq 3$ and odd, $n - 2$ is also odd and the concavity changes at $x = 2$.

For $n \geq 4$ and even, $n - 2$ is also even and the concavity does not change at $x = 2$.

Thus, $x = 2$ is an inflection point if and only if $n \geq 3$ is odd.

61. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: (3, 3)

Relative minimum: (5, 1)

Point of inflection: (4, 2)

$$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$$

$$\left. \begin{aligned} f(3) &= 27a + 9b + 3c + d = 3 \\ f(5) &= 125a + 25b + 5c + d = 1 \end{aligned} \right\} 98a + 16b + 2c = -2 \Rightarrow 49a + 8b + c = -1$$

$$f'(3) = 27a + 6b + c = 0, f''(4) = 24a + 2b = 0$$

$$49a + 8b + c = -1 \quad 24a + 2b = 0$$

$$\underline{27a + 6b + c = 0} \quad \underline{22a + 2b = -1}$$

$$22a + 2b = -1 \quad 2a = 1$$

$$a = \frac{1}{2}, b = -6, c = \frac{45}{2}, d = -24$$

$$f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$$

63. $f(x) = ax^3 + bx^2 + cx + d$

Maximum: $(-4, 1)$

Minimum: $(0, 0)$

(a) $f'(x) = 3ax^2 + 2bx + c$, $f''(x) = 6ax + 2b$

$$f(0) = 0 \Rightarrow d = 0$$

$$f(-4) = 1 \Rightarrow -64a + 16b - 4c = 1$$

$$f'(-4) = 0 \Rightarrow 48a - 8b + c = 0$$

$$f'(0) = 0 \Rightarrow c = 0$$

Solving this system yields $a = \frac{1}{32}$ and $b = 6a = \frac{3}{16}$.

$$f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$$

(b) The plane would be descending at the greatest rate at the point of inflection.

$$f''(x) = 6ax + 2b = \frac{3}{16}x + \frac{3}{8} = 0 \Rightarrow x = -2.$$

Two miles from touchdown.

65. $D = 2x^4 - 5Lx^3 + 3L^2x^2$

$$D' = 8x^3 - 15Lx^2 + 6L^2x = x(8x^2 - 15Lx + 6L^2) = 0$$

$$x = 0 \text{ or } x = \frac{15L \pm \sqrt{33}L}{16} = \left(\frac{15 \pm \sqrt{33}}{16}\right)L$$

By the Second Derivative Test, the deflection is maximum when

$$x = \left(\frac{15 - \sqrt{33}}{16}\right)L \approx 0.578L.$$

69. $S = \frac{5000t^2}{8 + t^2}$

$$S'(t) = \frac{80,000t}{(8 + t^2)^2}$$

$$S''(t) = \frac{80,000(8 - 3t^2)}{(8 + t^2)^3}$$

$$S''(t) = 0 \text{ for } t = \sqrt{8/3} \approx 1.633.$$

Sales are increasing at the greatest rate at $t = 1.633$ years.

71. $f(x) = 2(\sin x + \cos x)$, $f\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

$$f'(x) = 2(\cos x - \sin x), \quad f'\left(\frac{\pi}{4}\right) = 0$$

$$f''(x) = 2(-\sin x - \cos x), \quad f''\left(\frac{\pi}{4}\right) = -2\sqrt{2}$$

$$P_1(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) = 2\sqrt{2}$$

$$P_1'(x) = 0$$

$$P_2(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) + \frac{1}{2}(-2\sqrt{2})\left(x - \frac{\pi}{4}\right)^2 = 2\sqrt{2} - \sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2'(x) = -2\sqrt{2}\left(x - \frac{\pi}{4}\right)$$

$$P_2''(x) = -2\sqrt{2}$$

The values of f , P_1 , P_2 , and their first derivatives are equal at $x = \pi/4$. The values of the second derivatives of f and P_2 are equal at $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.

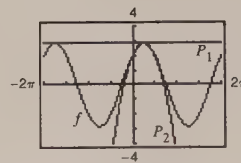
67. $C = 0.5x^2 + 15x + 5000$

$$\bar{C} = \frac{C}{x} = 0.5x + 15 + \frac{5000}{x}$$

$$\bar{C} = \text{average cost per unit}$$

$$\frac{d\bar{C}}{dx} = 0.5 - \frac{5000}{x^2} = 0 \text{ when } x = 100$$

By the First Derivative Test, \bar{C} is minimized when $x = 100$ units.



$$73. f(x) = \sqrt{1-x}, \quad f(0) = 1$$

$$f'(x) = -\frac{1}{2\sqrt{1-x}}, \quad f'(0) = -\frac{1}{2}$$

$$f''(x) = -\frac{1}{4(1-x)^{3/2}}, \quad f''(0) = -\frac{1}{4}$$

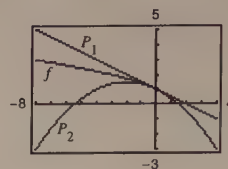
$$P_1(x) = 1 + \left(-\frac{1}{2}\right)(x-0) = 1 - \frac{x}{2}$$

$$P_1'(x) = -\frac{1}{2}$$

$$P_2(x) = 1 + \left(-\frac{1}{2}\right)(x-0) + \frac{1}{2}\left(-\frac{1}{4}\right)(x-0)^2 = 1 - \frac{x}{2} - \frac{x^2}{8}$$

$$P_2'(x) = -\frac{1}{2} - \frac{x}{4}$$

$$P_2''(x) = -\frac{1}{4}$$



The values of f , P_1 , P_2 , and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

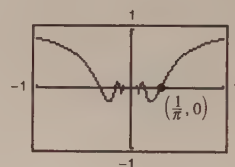
$$75. f(x) = x \sin\left(\frac{1}{x}\right)$$

$$f'(x) = x \left[-\frac{1}{x^2} \cos\left(\frac{1}{x}\right) \right] + \sin\left(\frac{1}{x}\right) = -\frac{1}{x} \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

$$f''(x) = -\frac{1}{x} \left[\frac{1}{x^2} \sin\left(\frac{1}{x}\right) \right] + \frac{1}{x^2} \cos\left(\frac{1}{x}\right) - \frac{1}{x^2} \cos\left(\frac{1}{x}\right) = -\frac{1}{x^3} \sin\left(\frac{1}{x}\right) = 0$$

$$x = \frac{1}{\pi}$$

$$\text{Point of inflection: } \left(\frac{1}{\pi}, 0\right)$$



When $x > 1/\pi$, $f'' < 0$, so the graph is concave downward.

77. Assume the zeros of f are all real. Then express the function as $f(x) = a(x - r_1)(x - r_2)(x - r_3)$ where r_1 , r_2 , and r_3 are the distinct zeros of f . From the Product Rule for a function involving three factors, we have

$$f'(x) = a[(x - r_1)(x - r_2) + (x - r_1)(x - r_3) + (x - r_2)(x - r_3)]$$

$$\begin{aligned} f''(x) &= a[(x - r_1) + (x - r_2) + (x - r_1) + (x - r_3) + (x - r_2) + (x - r_3)] \\ &= a[6x - 2(r_1 + r_2 + r_3)]. \end{aligned}$$

Consequently, $f''(x) = 0$ if

$$x = \frac{2(r_1 + r_2 + r_3)}{6} = \frac{r_1 + r_2 + r_3}{3} = (\text{Average of } r_1, r_2, \text{ and } r_3).$$

79. True. Let $y = ax^3 + bx^2 + cx + d$, $a \neq 0$. Then $y'' = 6ax + 2b = 0$ when $x = -(b/3a)$, and the concavity changes at this point.

81. False.

$$f(x) = 3 \sin x + 2 \cos x$$

$$f'(x) = 3 \cos x - 2 \sin x$$

$$3 \cos x - 2 \sin x = 0$$

$$3 \cos x = 2 \sin x$$

$$\frac{3}{2} = \tan x$$

$$\text{Critical number: } x = \tan^{-1}\left(\frac{3}{2}\right)$$

$$f\left(\tan^{-1}\left(\frac{3}{2}\right)\right) \approx 3.60555 \text{ is the maximum value of } y.$$

 83. False. Concavity is determined by f'' .

 For example, let $f(x) = x$ and $C = 2$. Then

$$f'(C) = f'(2) > 0 \text{ but } f \text{ is not concave upward at } x = 2.$$

Section 3.5 Limits at Infinity

1. $f(x) = \frac{3x^2}{x^2 + 2}$

No vertical asymptotes

 Horizontal asymptote: $y = 3$

Matches (f)

3. $f(x) = \frac{x}{x^2 + 2}$

No vertical asymptotes

 Horizontal asymptote: $y = 0$

$$f(1) < 1$$

Matches (d)

5. $f(x) = \frac{4 \sin x}{x^2 + 1}$

No vertical asymptotes

 Horizontal asymptote: $y = 0$

$$f(1) > 1$$

Matches (b)

7. $f(x) = \frac{4x + 3}{2x - 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	7	2.26	2.025	2.0025	2.0003	2	2

$$\lim_{x \rightarrow \infty} f(x) = 2$$

9. $f(x) = \frac{-6x}{\sqrt{4x^2 + 5}}$

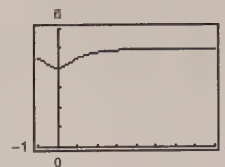
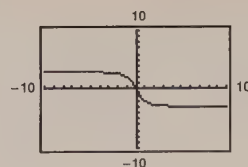
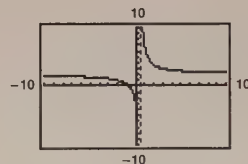
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	-2	-2.98	-2.9998	-3	-3	-3	-3

$$\lim_{x \rightarrow \infty} f(x) = -3$$

11. $f(x) = 5 - \frac{1}{x^2 + 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	4.5	4.99	4.9999	4.999999	5	5	5

$$\lim_{x \rightarrow \infty} f(x) = 5$$



$$13. (a) h(x) = \frac{f(x)}{x^2} = \frac{5x^3 - 3x^2 + 10}{x^2} = 5x - 3 + \frac{10}{x^2}$$

$$\lim_{x \rightarrow \infty} h(x) = \infty \quad (\text{Limit does not exist})$$

$$(b) h(x) = \frac{f(x)}{x^3} = \frac{5x^3 - 3x^2 + 10}{x^3} = 5 - \frac{3}{x} + \frac{10}{x^3}$$

$$\lim_{x \rightarrow \infty} h(x) = 5$$

$$(c) h(x) = \frac{f(x)}{x^4} = \frac{5x^3 - 3x^2 + 10}{x^4} = \frac{5}{x} - \frac{3}{x^2} + \frac{10}{x^4}$$

$$\lim_{x \rightarrow \infty} h(x) = 0$$

$$17. (a) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x^2 - 4} = 0$$

$$(b) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x^{3/2} - 4} = -\frac{2}{3}$$

$$(c) \lim_{x \rightarrow \infty} \frac{5 - 2x^{3/2}}{3x - 4} = -\infty \quad (\text{Limit does not exist})$$

$$21. \lim_{x \rightarrow \infty} \frac{x}{x^2 - 1} = \lim_{x \rightarrow \infty} \frac{1/x}{1 - (1/x^2)} = \frac{0}{1} = 0$$

$$15. (a) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x^3 - 1} = 0$$

$$(b) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x^2 - 1} = 1$$

$$(c) \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x - 1} = \infty \quad (\text{Limit does not exist})$$

$$19. \lim_{x \rightarrow \infty} \frac{2x - 1}{3x + 2} = \lim_{x \rightarrow \infty} \frac{2 - (1/x)}{3 + (2/x)} = \frac{2 - 0}{3 + 0} = \frac{2}{3}$$

$$23. \lim_{x \rightarrow -\infty} \frac{5x^2}{x + 3} = \lim_{x \rightarrow -\infty} \frac{5x}{1 + (3/x)} = -\infty$$

Limit does not exist.

$$25. \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2 - x}} = \lim_{x \rightarrow -\infty} \frac{1}{\frac{\sqrt{x^2 - x}}{-\sqrt{x^2}}}, \quad (\text{for } x < 0 \text{ we have } x = -\sqrt{x^2})$$

$$= \lim_{x \rightarrow -\infty} \frac{-1}{\sqrt{1 - (1/x)}} = -1$$

$$27. \lim_{x \rightarrow -\infty} \frac{2x + 1}{\sqrt{x^2 - x}} = \lim_{x \rightarrow -\infty} \frac{2 + \frac{1}{x}}{\left(\frac{\sqrt{x^2 - x}}{-\sqrt{x^2}} \right)} \quad (\text{for } x < 0, x = -\sqrt{x^2})$$

$$= \lim_{x \rightarrow -\infty} \frac{-2 - (1/x)}{\sqrt{1 - (1/x)}} = -2$$

29. Since $(-1/x) \leq (\sin(2x))/x \leq (1/x)$ for all $x \neq 0$, we have by the Squeeze Theorem,

$$\lim_{x \rightarrow \infty} -\frac{1}{x} \leq \lim_{x \rightarrow \infty} \frac{\sin(2x)}{x} \leq \lim_{x \rightarrow \infty} \frac{1}{x}$$

$$0 \leq \lim_{x \rightarrow \infty} \frac{\sin(2x)}{x} \leq 0.$$

$$\text{Therefore, } \lim_{x \rightarrow \infty} \frac{\sin(2x)}{x} = 0.$$

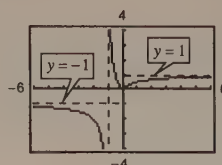
$$31. \lim_{x \rightarrow \infty} \frac{1}{2x + \sin x} = 0$$

$$33. (a) f(x) = \frac{|x|}{x+1}$$

$$\lim_{x \rightarrow \infty} \frac{|x|}{x+1} = 1$$

$$\lim_{x \rightarrow -\infty} \frac{|x|}{x+1} = -1$$

Therefore, $y = 1$ and $y = -1$ are both horizontal asymptotes.



$$35. \lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = 1$$

(Let $x = 1/t$.)

$$37. \lim_{x \rightarrow -\infty} (x + \sqrt{x^2 + 3}) = \lim_{x \rightarrow -\infty} \left[(x + \sqrt{x^2 + 3}) \cdot \frac{x - \sqrt{x^2 + 3}}{x - \sqrt{x^2 + 3}} \right] = \lim_{x \rightarrow -\infty} \frac{-3}{x - \sqrt{x^2 + 3}} = 0$$

$$39. \lim_{x \rightarrow \infty} (x - \sqrt{x^2 + x}) = \lim_{x \rightarrow \infty} \left[(x - \sqrt{x^2 + x}) \cdot \frac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}} \right]$$

$$= \lim_{x \rightarrow \infty} \frac{-x}{x + \sqrt{x^2 + x}} = \lim_{x \rightarrow \infty} \frac{-1}{1 + \sqrt{1 + (1/x)}} = -\frac{1}{2}$$

41.

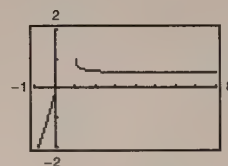
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	0.513	0.501	0.500	0.500	0.500	0.500

$$\lim_{x \rightarrow \infty} (x - \sqrt{x(x-1)}) = \lim_{x \rightarrow \infty} \frac{x - \sqrt{x^2 - x}}{1} \cdot \frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}}$$

$$= \lim_{x \rightarrow \infty} \frac{x}{x + \sqrt{x^2 - x}}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{1 + \sqrt{1 - (1/x)}}$$

$$= \frac{1}{2}$$

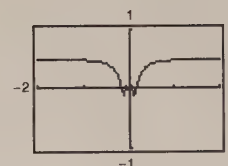


43.

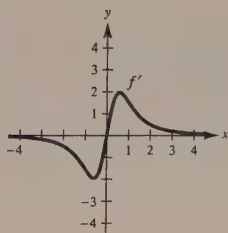
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.479	0.500	0.500	0.500	0.500	0.500	0.500

Let $x = 1/t$.

$$\lim_{x \rightarrow \infty} x \sin \left(\frac{1}{2x} \right) = \lim_{t \rightarrow 0^+} \frac{\sin(t/2)}{t} = \lim_{t \rightarrow 0^+} \frac{1}{2} \frac{\sin(t/2)}{t/2} = \frac{1}{2}$$



45. (a)



$$(b) \lim_{x \rightarrow \infty} f(x) = 3 \quad \lim_{x \rightarrow \infty} f'(x) = 0$$

(c) Since $\lim_{x \rightarrow \infty} f(x) = 3$, the graph approaches that of a horizontal line, $\lim_{x \rightarrow \infty} f'(x) = 0$.

$$49. y = \frac{2+x}{1-x}$$

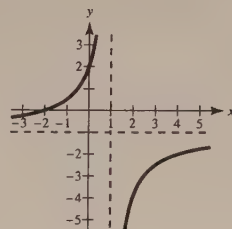
Intercepts: $(-2, 0), (0, 2)$

Symmetry: none

Horizontal asymptote: $y = -1$ since

$$\lim_{x \rightarrow -\infty} \frac{2+x}{1-x} = -1 = \lim_{x \rightarrow \infty} \frac{2+x}{1-x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)



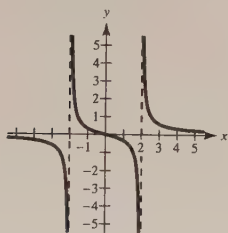
$$51. y = \frac{x}{x^2 - 4}$$

Intercept: $(0, 0)$

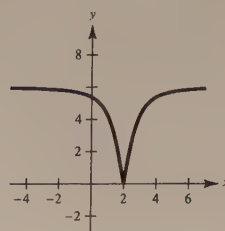
Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 2$



$$47. \text{ Yes. For example, let } f(x) = \frac{6|x-2|}{\sqrt{(x-2)^2 + 1}}.$$



$$53. y = \frac{x^2}{x^2 + 9}$$

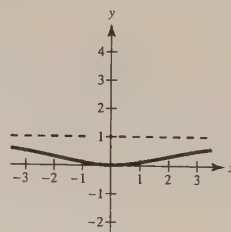
Intercept: $(0, 0)$

Symmetry: y -axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x \rightarrow -\infty} \frac{x^2}{x^2 + 9} = 1 = \lim_{x \rightarrow \infty} \frac{x^2}{x^2 + 9}.$$

Relative minimum: $(0, 0)$



55. $y = \frac{2x^2}{x^2 - 4}$

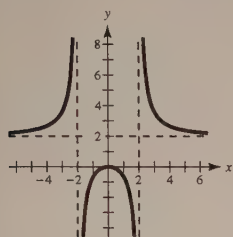
Intercept: $(0, 0)$

Symmetry: y -axis

Horizontal asymptote: $y = 2$

Vertical asymptotes: $x = \pm 2$

Relative maximum: $(0, 0)$



59. $y = \frac{2x}{1 - x}$

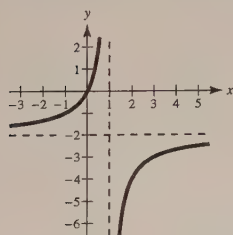
Intercept: $(0, 0)$

Symmetry: none

Horizontal asymptote: $y = -2$ since

$$\lim_{x \rightarrow -\infty} \frac{2x}{1 - x} = -2 = \lim_{x \rightarrow \infty} \frac{2x}{1 - x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)



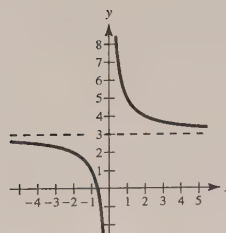
63. $y = 3 + \frac{2}{x}$

Intercept: $y = 0 = 3 + \frac{2}{x} \Rightarrow \frac{2}{x} = -3 \Rightarrow x = -\frac{2}{3}; \left(-\frac{2}{3}, 0\right)$

Symmetry: none

Horizontal asymptote: $y = 3$

Vertical asymptote: $x = 0$



57. $xy^2 = 4$

Domain: $x > 0$

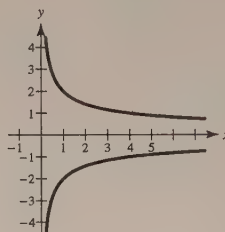
Intercepts: none

Symmetry: x -axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0 = \lim_{x \rightarrow \infty} -\frac{2}{\sqrt{x}}.$$

Discontinuity: $x = 0$ (Vertical asymptote)



61. $y = 2 - \frac{3}{x^2}$

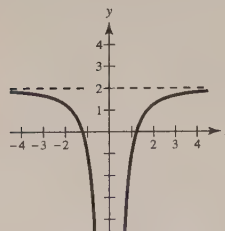
Intercepts: $(\pm\sqrt{3/2}, 0)$

Symmetry: y -axis

Horizontal asymptote: $y = 2$ since

$$\lim_{x \rightarrow -\infty} \left(2 - \frac{3}{x^2}\right) = 2 = \lim_{x \rightarrow \infty} \left(2 - \frac{3}{x^2}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)



$$65. y = \frac{x^3}{\sqrt{x^2 - 4}}$$

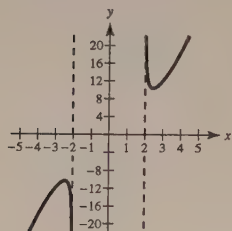
Domain: $(-\infty, -2), (2, \infty)$

Intercepts: none

Symmetry: origin

Horizontal asymptote: none

Vertical asymptotes: $x = \pm 2$ (discontinuities)



$$67. f(x) = 5 - \frac{1}{x^2} = \frac{5x^2 - 1}{x^2}$$

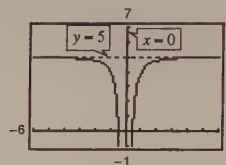
Domain: $(-\infty, 0), (0, \infty)$

$$f'(x) = \frac{2}{x^3} \Rightarrow \text{No relative extrema}$$

$$f''(x) = -\frac{6}{x^4} \Rightarrow \text{No points of inflection}$$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 5$



$$69. f(x) = \frac{x}{x^2 - 4}$$

$$f'(x) = \frac{(x^2 - 4) - x(2x)}{(x^2 - 4)^2}$$

$$= \frac{-(x^2 + 4)}{(x^2 - 4)^2} \neq 0 \text{ for any } x \text{ in the domain of } f.$$

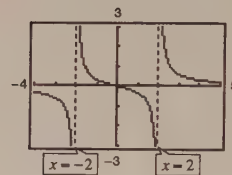
$$f''(x) = \frac{(x^2 - 4)^2(-2x) + (x^2 + 4)(2)(x^2 - 4)(2x)}{(x^2 - 4)^4}$$

$$= \frac{2x(x^2 + 12)}{(x^2 - 4)^3} = 0 \text{ when } x = 0.$$

Since $f''(x) > 0$ on $(-2, 0)$ and $f''(x) < 0$ on $(0, 2)$, then $(0, 0)$ is a point of inflection.

Vertical asymptotes: $x = \pm 2$

Horizontal asymptote: $y = 0$



$$71. f(x) = \frac{x - 2}{x^2 - 4x + 3} = \frac{x - 2}{(x - 1)(x - 3)}$$

$$f'(x) = \frac{(x^2 - 4x + 3) - (x - 2)(2x - 4)}{(x^2 - 4x + 3)^2} = \frac{-x^2 + 4x - 5}{(x^2 - 4x + 3)^2} \neq 0$$

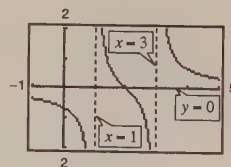
$$f''(x) = \frac{(x^2 - 4x + 3)^2(-2x + 4) - (-x^2 + 4x - 5)(2)(x^2 - 4x + 3)(2x - 4)}{(x^2 - 4x + 3)^4}$$

$$= \frac{2(x^3 - 6x^2 + 15x - 14)}{(x^2 - 4x + 3)^3} = \frac{2(x - 2)(x^2 - 4x + 7)}{(x^2 - 4x + 3)^3} = 0 \text{ when } x = 2.$$

Since $f''(x) > 0$ on $(1, 2)$ and $f''(x) < 0$ on $(2, 3)$, then $(2, 0)$ is a point of inflection.

Vertical asymptotes: $x = 1, x = 3$

Horizontal asymptote: $y = 0$



$$73. f(x) = \frac{3x}{\sqrt{4x^2 + 1}}$$

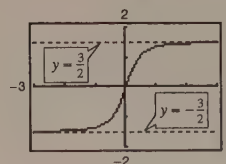
$$f'(x) = \frac{3}{(4x^2 + 1)^{3/2}} \Rightarrow \text{No relative extrema}$$

$$f''(x) = \frac{-36x}{(4x^2 + 1)^{5/2}} = 0 \text{ when } x = 0.$$

Point of inflection: (0, 0)

Horizontal asymptotes: $y = \pm \frac{3}{2}$

No vertical asymptotes



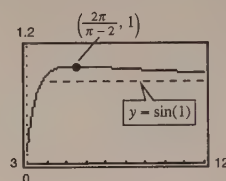
$$75. g(x) = \sin\left(\frac{x}{x-2}\right), 3 < x < \infty$$

$$g'(x) = \frac{-2 \cos\left(\frac{x}{x-2}\right)}{(x-2)^2}$$

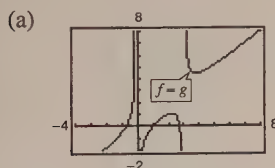
Horizontal asymptote: $y = \sin(1)$

$$\text{Relative maximum: } \frac{x}{x-2} = \frac{\pi}{2} \Rightarrow x = \frac{2\pi}{\pi-2} \approx 5.5039$$

No vertical asymptotes

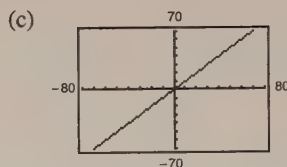


$$77. f(x) = \frac{x^3 - 3x^2 + 2}{x(x-3)}, g(x) = x + \frac{2}{x(x-3)}$$



(b)

$$\begin{aligned} f(x) &= \frac{x^3 - 3x^2 + 2}{x(x-3)} \\ &= \frac{x^2(x-3)}{x(x-3)} + \frac{2}{x(x-3)} \\ &= x + \frac{2}{x(x-3)} = g(x) \end{aligned}$$



The graph appears as the slant asymptote $y = x$.

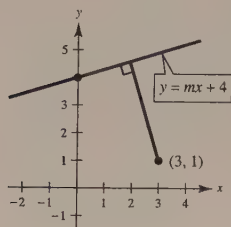
$$79. C = 0.5x + 500$$

$$\bar{C} = \frac{C}{x}$$

$$\bar{C} = 0.5 + \frac{500}{x}$$

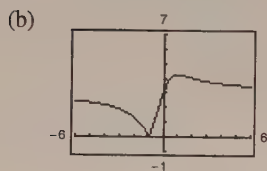
$$\lim_{x \rightarrow \infty} \left(0.5 + \frac{500}{x}\right) = 0.5$$

81. line: $mx - y + 4 = 0$



$$(a) d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}}$$

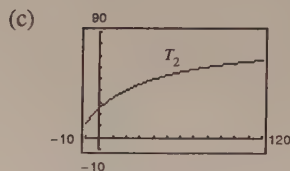
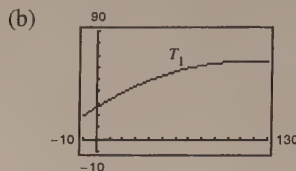
$$= \frac{|3m + 3|}{\sqrt{m^2 + 1}}$$



$$(c) \lim_{m \rightarrow \infty} d(m) = 3 = \lim_{m \rightarrow -\infty} d(m)$$

The line approaches the vertical line $x = 0$. Hence, the distance approaches 3.

83. (a) $T_1(t) = -0.003t^2 + 0.677t + 26.564$



$$T_2 = \frac{1451 + 86t}{58 + t}$$

$$(d) T_1(0) \approx 26.6$$

$$T_2(0) \approx 25.0$$

$$(e) \lim_{t \rightarrow \infty} T_2 = \frac{86}{1} = 86$$

(f) The limiting temperature is 86.
 T_1 has no horizontal asymptote.

85. Answers will vary. See page 195.

87. False. Let $f(x) = \frac{2x}{\sqrt{x^2 + 2}}$. (See Exercise 2.)

Section 3.6 A Summary of Curve Sketching

1. f has constant negative slope. Matches (D)

5. (a) $f'(x) = 0$ for $x = -2$ and $x = 2$

f' is negative for $-2 < x < 2$ (decreasing function).

f' is positive for $x > 2$ and $x < -2$
(increasing function).

(b) $f''(x) = 0$ at $x = 0$ (Inflection point).

f'' is positive for $x > 0$ (Concave upwards).

f'' is negative for $x < 0$ (Concave downward).

3. The slope is periodic, and zero at $x = 0$. Matches (A)

(c) f' is increasing on $(0, \infty)$. ($f'' > 0$)

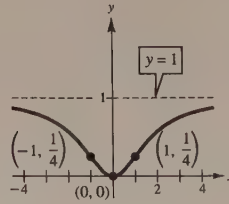
(d) $f'(x)$ is minimum at $x = 0$. The rate of change of f at $x = 0$ is less than the rate of change of f for all other values of x .

$$7. y = \frac{x^2}{x^2 + 3}$$

$$y' = \frac{6x}{(x^2 + 3)^2} = 0 \text{ when } x = 0.$$

$$y'' = \frac{18(1 - x^2)}{(x^2 + 3)^3} = 0 \text{ when } x = \pm 1.$$

Horizontal asymptote: $y = 1$



	y	y'	y''	Conclusion
$-\infty < x < -1$		-	-	Decreasing, concave down
$x = -1$	$\frac{1}{4}$	-	0	Point of inflection
$-1 < x < 0$		-	+	Decreasing, concave up
$x = 0$	0	0	+	Relative minimum
$0 < x < 1$		+	+	Increasing, concave up
$x = 1$	$\frac{1}{4}$	+	0	Point of inflection
$1 < x < \infty$		+	-	Increasing, concave down

$$9. y = \frac{1}{x-2} - 3$$

$$y' = -\frac{1}{(x-2)^2} < 0 \text{ when } x \neq 2.$$

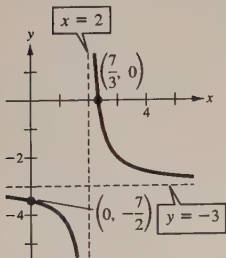
$$y'' = \frac{2}{(x-2)^3}$$

No relative extrema, no points of inflection

Intercepts: $(\frac{7}{3}, 0), (0, -\frac{7}{2})$

Vertical asymptote: $x = 2$

Horizontal asymptote: $y = -3$



$$11. y = \frac{2x}{x^2 - 1}$$

$$y' = \frac{-2(x^2 + 1)}{(x^2 - 1)^2} < 0 \text{ if } x \neq \pm 1.$$

$$y'' = \frac{4x(x^2 + 3)}{(x^2 - 1)^3} = 0 \text{ if } x = 0.$$

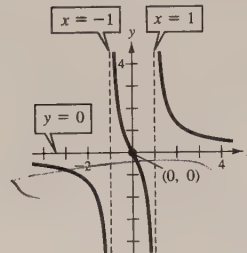
Inflection point: $(0, 0)$

Intercept: $(0, 0)$

Vertical asymptote: $x = \pm 1$

Horizontal asymptote: $y = 0$

Symmetry with respect to the origin



13. $g(x) = x + \frac{4}{x^2 + 1}$

$$g'(x) = 1 - \frac{8x}{(x^2 + 1)^2} = \frac{x^4 + 2x^2 - 8x + 1}{(x^2 + 1)^2} = 0 \text{ when } x \approx 0.1292, 1.6085$$

$$g''(x) = \frac{8(3x^2 - 1)}{(x^2 + 1)^3} = 0 \text{ when } x = \pm \frac{\sqrt{3}}{3}$$

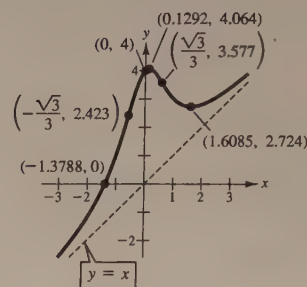
$g''(0.1292) < 0$, therefore, $(0.1292, 4.064)$ is relative maximum.

$g''(1.6085) > 0$, therefore, $(1.6085, 2.724)$ is a relative minimum.

Points of inflection: $\left(-\frac{\sqrt{3}}{3}, 2.423\right), \left(\frac{\sqrt{3}}{3}, 3.577\right)$

Intercepts: $(0, 4), (-1.3788, 0)$

Slant asymptote: $y = x$



15. $f(x) = \frac{x^2 + 1}{x} = x + \frac{1}{x}$

$$f'(x) = 1 - \frac{1}{x^2} = 0 \text{ when } x = \pm 1.$$

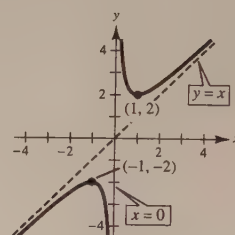
$$f''(x) = \frac{2}{x^3} \neq 0$$

Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$



17. $y = \frac{x^2 - 6x + 12}{x - 4} = x - 2 + \frac{4}{x - 4}$

$$y' = 1 - \frac{4}{(x - 4)^2}$$

$$= \frac{(x - 2)(x - 6)}{(x - 4)^2} = 0 \text{ when } x = 2, 6.$$

$$y'' = \frac{8}{(x - 4)^3}$$

$y'' < 0$ when $x = 2$.

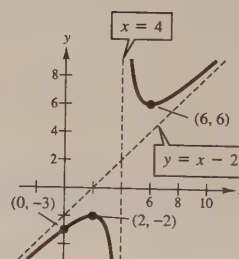
Therefore, $(2, -2)$ is a relative maximum.

$y'' > 0$ when $x = 6$.

Therefore, $(6, 6)$ is a relative minimum.

Vertical asymptote: $x = 4$

Slant asymptote: $y = x - 2$



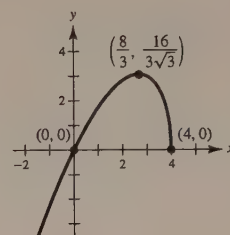
19. $y = x\sqrt{4-x}$

Domain: $(-\infty, 4]$

$$y' = \frac{8-3x}{2\sqrt{4-x}} = 0 \text{ when } x = \frac{8}{3} \text{ and undefined when } x = 4.$$

$$y'' = \frac{3x-16}{4(4-x)^{3/2}} = 0 \text{ when } x = \frac{16}{3} \text{ and undefined when } x = 4.$$

Note: $x = \frac{16}{3}$ is not in the domain.



	y	y'	y''	Conclusion
$-\infty < x < \frac{8}{3}$		+	-	Increasing, concave down
$x = \frac{8}{3}$	$\frac{16}{3\sqrt{3}}$	0	-	Relative maximum
$\frac{8}{3} < x < 4$		-	-	Decreasing, concave down
$x = 4$	0	Undefined	Undefined	Endpoint

21. $h(x) = x\sqrt{9-x^2}$ Domain: $-3 \leq x \leq 3$

$$h'(x) = \frac{9-2x^2}{\sqrt{9-x^2}} = 0 \text{ when } x = \pm \frac{3}{\sqrt{2}} = \pm \frac{3\sqrt{2}}{2}$$

$$h''(x) = \frac{x(2x^2-27)}{(9-x^2)^{3/2}} = 0 \text{ when } x = 0$$

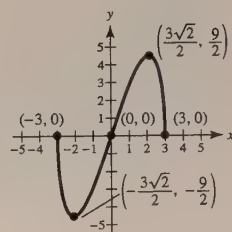
Relative maximum: $\left(\frac{3\sqrt{2}}{2}, \frac{9}{2}\right)$

Relative minimum: $\left(-\frac{3\sqrt{2}}{2}, -\frac{9}{2}\right)$

Intercepts: $(0, 0), (\pm 3, 0)$

Symmetric with respect to the origin

Point of inflection: $(0, 0)$

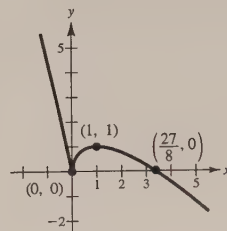


23. $y = 3x^{2/3} - 2x$

$$y' = 2x^{-1/3} - 2 = \frac{2(1-x^{1/3})}{x^{1/3}}$$

$$= 0 \text{ when } x = 1 \text{ and undefined when } x = 0.$$

$$y'' = \frac{-2}{3x^{4/3}} < 0 \text{ when } x \neq 0.$$



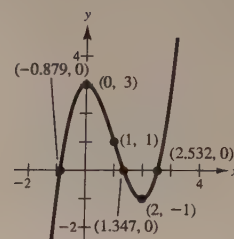
	y	y'	y''	Conclusion
$-\infty < x < 0$		-	-	Decreasing, concave down
$x = 0$	0	Undefined	Undefined	Relative minimum
$0 < x < 1$		+	-	Increasing, concave down
$x = 1$	1	0	-	Relative maximum
$1 < x < \infty$		-	-	Decreasing, concave down

25. $y = x^3 - 3x^2 + 3$

$y' = 3x^2 - 6x = 3x(x - 2) = 0$ when $x = 0, x = 2$

$y'' = 6x - 6 = 6(x - 1) = 0$ when $x = 1$

	y	y'	y''	Conclusion
$-\infty < x < 0$		+	-	Increasing, concave down
$x = 0$	3	0	-	Relative maximum
$0 < x < 1$		-	-	Decreasing, concave down
$x = 1$	1	-	0	Point of inflection
$1 < x < 2$		-	+	Decreasing, concave up
$x = 2$	-1	0	+	Relative minimum
$2 < x < \infty$		+	+	Increasing, concave up



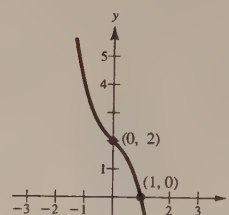
27. $y = 2 - x - x^3$

$y' = -1 - 3x^2$

No critical numbers

$y'' = -6x = 0$ when $x = 0$.

	y	y'	y''	Conclusion
$-\infty < x < 0$		-	+	Decreasing, concave up
$x = 0$	2	-	0	Point of inflection
$0 < x < \infty$		-	-	Decreasing, concave down

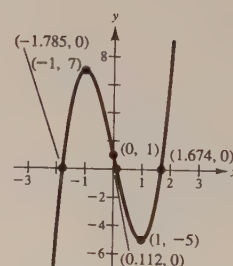


29. $f(x) = 3x^3 - 9x + 1$

$f'(x) = 9x^2 - 9 = 9(x^2 - 1) = 0$ when $x = \pm 1$

$f''(x) = 18x = 0$ when $x = 0$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < -1$		+	-	Increasing, concave down
$x = -1$	7	0	-	Relative maximum
$-1 < x < 0$		-	-	Decreasing, concave down
$x = 0$	1	-	0	Point of inflection
$0 < x < 1$		-	+	Decreasing, concave up
$x = 1$	-5	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

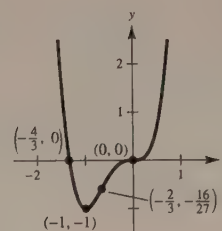


31. $y = 3x^4 + 4x^3$

$$y' = 12x^3 + 12x^2 = 12x^2(x + 1) = 0 \text{ when } x = 0, x = -1.$$

$$y'' = 36x^2 + 24x = 12x(3x + 2) = 0 \text{ when } x = 0, x = -\frac{2}{3}.$$

	y	y'	y''	Conclusion
$-\infty < x < -1$		-	+	Decreasing, concave up
$x = -1$	-1	0	+	Relative minimum
$-1 < x < -\frac{2}{3}$		+	+	Increasing, concave up
$x = -\frac{2}{3}$	$-\frac{16}{27}$	+	0	Point of inflection
$-\frac{2}{3} < x < 0$		+	-	Increasing, concave down
$x = 0$	0	0	0	Point of inflection
$0 < x < \infty$		+	+	Increasing, concave up

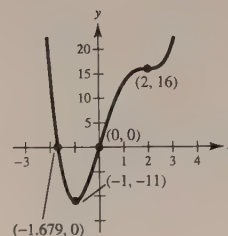


33. $f(x) = x^4 - 4x^3 + 16x$

$$f'(x) = 4x^3 - 12x^2 + 16 = 4(x + 1)(x - 2)^2 = 0 \text{ when } x = -1, x = 2.$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2) = 0 \text{ when } x = 0, x = 2.$$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < -1$		-	+	Decreasing, concave up
$x = -1$	-11	0	+	Relative minimum
$-1 < x < 0$		+	+	Increasing, concave up
$x = 0$	0	+	0	Point of inflection
$0 < x < 2$		+	-	Increasing, concave down
$x = 2$	16	0	0	Point of inflection
$2 < x < \infty$		+	+	Increasing, concave up

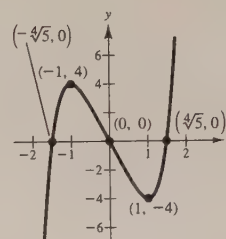


35. $y = x^5 - 5x$

$$y' = 5x^4 - 5 = 5(x^4 - 1) = 0 \text{ when } x = \pm 1.$$

$$y'' = 20x^3 = 0 \text{ when } x = 0.$$

	y	y'	y''	Conclusion
$-\infty < x < -1$		+	-	Increasing, concave down
$x = -1$	4	0	-	Relative maximum
$-1 < x < 0$		-	-	Decreasing, concave down
$x = 0$	0	-	0	Point of inflection
$0 < x < 1$		-	+	Decreasing, concave up
$x = 1$	-4	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

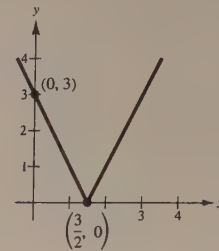


37. $y = |2x - 3|$

$$y' = \frac{2(2x - 3)}{|2x - 3|} \text{ undefined at } x = \frac{3}{2}$$

$$y'' = 0$$

	y	y'	Conclusion
$-\infty < x < \frac{3}{2}$		-	Decreasing
$x = \frac{3}{2}$	0	Undefined	Relative minimum
$\frac{3}{2} < x < \infty$		+	Increasing



39. $y = \sin x - \frac{1}{18} \sin 3x, 0 \leq x \leq 2\pi$

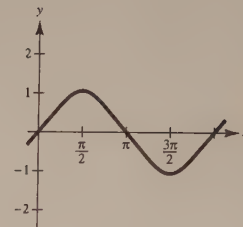
$$y' = \cos x - \frac{1}{6} \cos 3x = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$y'' = -\sin x + \frac{1}{2} \sin 3x = 0 \text{ when } x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$$

Relative maximum: $\left(\frac{\pi}{2}, \frac{19}{18}\right)$

Relative minimum: $\left(\frac{3\pi}{2}, -\frac{19}{18}\right)$

Inflection points: $\left(\frac{\pi}{6}, \frac{4}{9}\right), \left(\frac{5\pi}{6}, \frac{4}{9}\right), (\pi, 0), \left(\frac{7\pi}{6}, -\frac{4}{9}\right), \left(\frac{11\pi}{6}, -\frac{4}{9}\right)$



41. $y = 2x - \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$y' = 2 - \sec^2 x = 0 \text{ when } x = \pm \frac{\pi}{4}$$

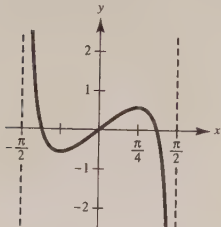
$$y'' = -2\sec^2 x \tan x = 0 \text{ when } x = 0$$

Relative maximum: $\left(\frac{\pi}{4}, \frac{\pi}{2} - 1\right)$

Relative minimum: $\left(-\frac{\pi}{4}, 1 - \frac{\pi}{2}\right)$

Inflection point: $(0, 0)$

Vertical asymptotes: $x = \pm \frac{\pi}{2}$

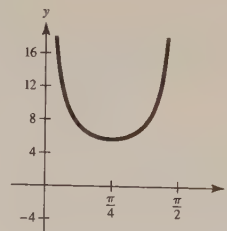


43. $y = 2(\csc x + \sec x), 0 < x < \frac{\pi}{2}$

$$y' = 2(\sec x \tan x - \csc x \cot x) = 0 \implies x = \frac{\pi}{4}$$

Relative minimum: $\left(\frac{\pi}{4}, 4\sqrt{2}\right)$

Vertical asymptotes: $x = 0, x = \frac{\pi}{2}$



$$45. g(x) = x \tan x, \quad -\frac{3\pi}{2} < x < \frac{3\pi}{2}$$

$$g'(x) = \frac{x + \sin x \cos x}{\cos^2 x} = 0 \text{ when } x = 0$$

$$g''(x) = \frac{2(\cos x + x \sin x)}{\cos^3 x}$$

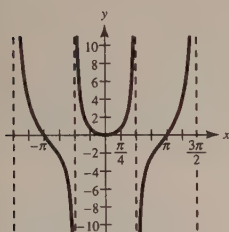
$$\text{Vertical asymptotes: } x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\text{Intercepts: } (-\pi, 0), (0, 0), (\pi, 0)$$

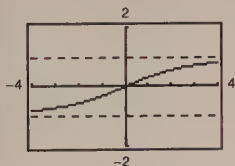
Symmetric with respect to y-axis.

$$\text{Increasing on } \left(0, \frac{\pi}{2}\right) \text{ and } \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

$$\text{Points of inflection: } (\pm 2.80, 0)$$



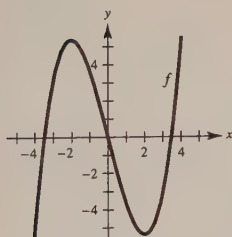
$$49. y = \frac{x}{\sqrt{x^2 + 7}}$$



(0, 0) point of inflection

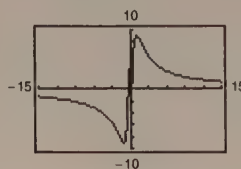
$y = \pm 1$ horizontal asymptotes

53.



(any vertical translate of f will do)

$$47. f(x) = \frac{20x}{x^2 + 1} - \frac{1}{x} = \frac{19x^2 - 1}{x(x^2 + 1)}$$



$x = 0$ vertical asymptote

$y = 0$ horizontal asymptote

Minimum: $(-1.10, -9.05)$

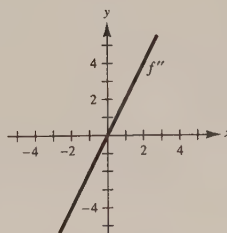
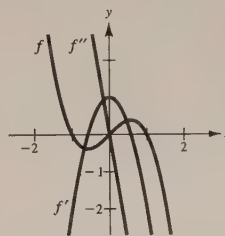
Maximum: $(1.10, 9.05)$

Points of inflection: $(-1.84, -7.86), (1.84, 7.86)$

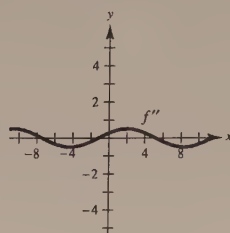
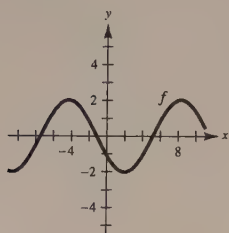
51. f is cubic.

f' is quadratic.

f'' is linear.



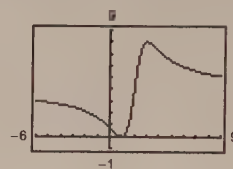
55.

(any vertical translate of f will do)

57. Since the slope is negative, the function is decreasing on $(2, 8)$, and hence $f(3) > f(5)$.

$$59. f(x) = \frac{4(x-1)^2}{x^2 - 4x + 5}$$

Vertical asymptote: none

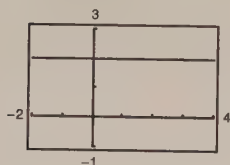
Horizontal asymptote: $y = 4$ 

The graph crosses the horizontal asymptote $y = 4$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

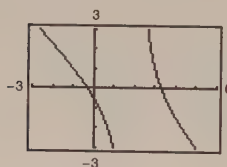
$$61. h(x) = \frac{6-2x}{3-x}$$

$$= \frac{2(3-x)}{3-x} = \begin{cases} 2, & \text{if } x \neq 3 \\ \text{Undefined,} & \text{if } x = 3 \end{cases}$$

The rational function is not reduced to lowest terms.

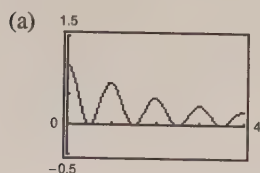
hole at $(3, 2)$

$$63. f(x) = \frac{-x^2 - 3x - 1}{x - 2} = -x + 1 + \frac{3}{x - 2}$$



The graph appears to approach the slant asymptote $y = -x + 1$.

$$65. f(x) = \frac{\cos^2 \pi x}{\sqrt{x^2 + 1}}, (0, 4)$$

On $(0, 4)$ there seem to be 7 critical numbers:

0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5

$$(b) f'(x) = \frac{-\cos \pi x (x \cos \pi x + 2\pi(x^2 + 1)\sin \pi x)}{(x^2 + 1)^{3/2}} = 0$$

$$\text{Critical numbers} \approx \frac{1}{2}, 0.97, \frac{3}{2}, 1.98, \frac{5}{2}, 2.98, \frac{7}{2}.$$

The critical numbers where maxima occur appear to be integers in part (a), but approximating them using f' shows that they are not integers.

67. Vertical asymptote: $x = 5$

Horizontal asymptote: $y = 0$

$$y = \frac{1}{x - 5}$$

71. $f(x) = \frac{ax}{(x - b)^2}$

- (a) The graph has a vertical asymptote at $x = b$. If $a > 0$, the graph approaches ∞ as $x \rightarrow b$. If $a < 0$, the graph approaches $-\infty$ as $x \rightarrow b$. The graph approaches its vertical asymptote faster as $|a| \rightarrow 0$.

73. $f(x) = \frac{3x^n}{x^4 + 1}$

- (a) For n even, f is symmetric about the y -axis. For n odd, f is symmetric about the origin.
 (b) The x -axis will be the horizontal asymptote if the degree of the numerator is less than 4. That is, $n = 0, 1, 2, 3$.
 (c) $n = 4$ gives $y = 3$ as the horizontal asymptote.

69. Vertical asymptote: $x = 5$

Slant asymptote: $y = 3x + 2$

$$y = 3x + 2 + \frac{1}{x - 5} = \frac{3x^2 - 13x - 9}{x - 5}$$

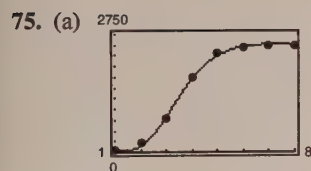
- (b) As b varies, the position of the vertical asymptote changes: $x = b$. Also, the coordinates of the minimum ($a > 0$) or maximum ($a < 0$) are changed.

- (d) There is a slant asymptote $y = 3x$ if $n = 5$:

$$\frac{3x^5}{x^4 + 1} = 3x - \frac{3x}{x^4 + 1}$$

(e)

n	0	1	2	3	4	5
M	1	2	3	2	1	0
N	2	3	4	5	2	3



- (b) When $t = 10$, $N(10) \approx 2434$ bacteria.
 (c) N is a maximum when $t \approx 7.2$ (seventh day).
 (d) $N'(t) = 0$ for $t \approx 3.2$
 (e) $\lim_{t \rightarrow \infty} N(t) = \frac{13,250}{7} \approx 1892.86$

Section 3.7 Optimization Problems

1. (a)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$

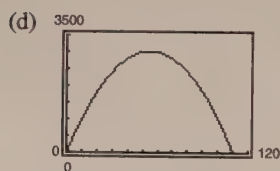
1. —CONTINUED—

(b)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$
70	$110 - 70$	$70(110 - 70) = 2800$
80	$110 - 80$	$80(110 - 80) = 2400$
90	$110 - 90$	$90(110 - 90) = 1800$
100	$110 - 100$	$100(110 - 100) = 1000$

The maximum is attained near $x = 50$ and 60 .

(c) $P = x(110 - x) = 110x - x^2$



The solution appears to be $x = 55$.

(e) $\frac{dP}{dx} = 110 - 2x = 0$ when $x = 55$.

$$\frac{d^2P}{dx^2} = -2 < 0$$

P is a maximum when $x = 110 - x = 55$.

The two numbers are 55 and 55.

3. Let x and y be two positive numbers such that $xy = 192$.

$$S = x + y = x + \frac{192}{x}$$

$$\frac{dS}{dx} = 1 - \frac{192}{x^2} = 0 \text{ when } x = \sqrt{192}.$$

$$\frac{d^2S}{dx^2} = \frac{384}{x^3} > 0 \text{ when } x = \sqrt{192}.$$

S is a minimum when $x = y = \sqrt{192}$.

5. Let x be a positive number.

$$S = x + \frac{1}{x}$$

$$\frac{dS}{dx} = 1 - \frac{1}{x^2} = 0 \text{ when } x = 1.$$

$$\frac{d^2S}{dx^2} = \frac{2}{x^3} > 0 \text{ when } x = 1.$$

The sum is a minimum when $x = 1$ and $1/x = 1$.

7. Let x be the length and y the width of the rectangle.

$$2x + 2y = 100$$

$$y = 50 - x$$

$$A = xy = x(50 - x)$$

$$\frac{dA}{dx} = 50 - 2x = 0 \text{ when } x = 25.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = 25.$$

A is maximum when $x = y = 25$ meters.

9. Let x be the length and y the width of the rectangle.

$$xy = 64$$

$$y = \frac{64}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{64}{x}\right) = 2x + \frac{128}{x}$$

$$\frac{dP}{dx} = 2 - \frac{128}{x^2} = 0 \text{ when } x = 8.$$

$$\frac{d^2P}{dx^2} = \frac{256}{x^3} > 0 \text{ when } x = 8.$$

P is minimum when $x = y = 8$ feet.

$$11. d = \sqrt{(x-4)^2 + (\sqrt{x}-0)^2}$$

$$= \sqrt{x^2 - 7x + 16}$$

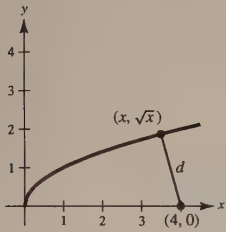
Since d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^2 - 7x + 16.$$

$$f'(x) = 2x - 7 = 0$$

$$x = \frac{7}{2}$$

By the First Derivative Test, the point nearest to $(4, 0)$ is $(7/2, \sqrt{7/2})$.



$$15. \frac{dQ}{dx} = kx(Q_0 - x) = kQ_0x - kx^2$$

$$\frac{d^2Q}{dx^2} = kQ_0 - 2kx$$

$$= k(Q_0 - 2x) = 0 \text{ when } x = \frac{Q_0}{2}.$$

$$\frac{d^3Q}{dx^3} = -2k < 0 \text{ when } x = \frac{Q_0}{2}.$$

dQ/dx is maximum when $x = Q_0/2$.

$$13. d = \sqrt{(x-2)^2 + [x^2 - (1/2)]^2}$$

$$= \sqrt{x^4 - 4x + (17/4)}$$

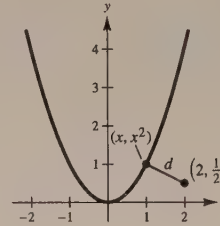
Since d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^4 - 4x + \frac{17}{4}.$$

$$f'(x) = 4x^3 - 4 = 0$$

$$x = 1$$

By the First Derivative Test, the point nearest to $(2, \frac{1}{2})$ is $(1, 1)$.



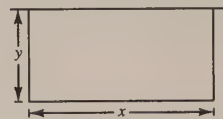
$$17. xy = 180,000 \text{ (see figure)}$$

$S = x + 2y = \left(x + \frac{360,000}{x}\right)$ where S is the length of fence needed.

$$\frac{dS}{dx} = 1 - \frac{360,000}{x^2} = 0 \text{ when } x = 600.$$

$$\frac{d^2S}{dx^2} = \frac{720,000}{x^3} > 0 \text{ when } x = 600.$$

S is a minimum when $x = 600$ meters and $y = 300$ meters.



$$19. (a) A = 4(\text{area of side}) + 2(\text{area of Top})$$

$$(a) A = 4(3)(11) + 2(3)(3) = 150 \text{ square inches}$$

$$(b) A = 4(5)(5) + 2(5)(5) = 150 \text{ square inches}$$

$$(c) A = 4(3.25)(6) + 2(6)(6) = 150 \text{ square inches}$$

$$(c) S = 4xy + 2x^2 = 150 \Rightarrow y = \frac{150 - 2x^2}{4x}$$

$$V = x^2y = x^2\left(\frac{150 - 2x^2}{4x}\right) = \frac{75}{2}x - \frac{1}{2}x^3$$

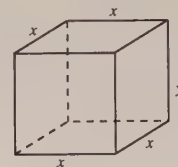
$$V' = \frac{75}{2} - \frac{3}{2}x^2 = 0 \Rightarrow x = \pm 5$$

$$(b) V = (\text{length})(\text{width})(\text{height})$$

$$(a) V = (3)(3)(11) = 99 \text{ cubic inches}$$

$$(b) V = (5)(5)(5) = 125 \text{ cubic inches}$$

$$(c) V = (6)(6)(3.25) = 117 \text{ cubic inches}$$



By the First Derivative Test, $x = 5$ yields the maximum volume. Dimensions: $5 \times 5 \times 5$. (A cube!)

21. (a) $V = x(s - 2x)^2, 0 < x < \frac{s}{2}$

$$\frac{dV}{dx} = 2x(s - 2x)(-2) + (s - 2x)^2$$

$$= (s - 2x)(s - 6x) = 0 \text{ when } x = \frac{s}{2}, \frac{s}{6} \text{ (} s/2 \text{ is not in the domain).}$$

$$\frac{d^2V}{dx^2} = 24x - 8s$$

$$\frac{d^2V}{dx^2} < 0 \text{ when } x = \frac{s}{6}.$$

$$V = \frac{2s^3}{27} \text{ is maximum when } x = \frac{s}{6}.$$

(b) If the length is doubled, $V = \frac{2}{27}(2s)^3 = 8(\frac{2}{27}s^3)$. Volume is increased by a factor of 8.

23. $16 = 2y + x + \pi\left(\frac{x}{2}\right)$

$$32 = 4y + 2x + \pi x$$

$$y = \frac{32 - 2x - \pi x}{4}$$

$$A = xy + \frac{\pi\left(\frac{x}{2}\right)^2} = \left(\frac{32 - 2x - \pi x}{4}\right)x + \frac{\pi x^2}{8}$$

$$= 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

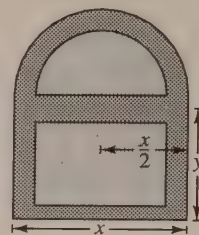
$$\frac{dA}{dx} = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x = 8 - x\left(1 + \frac{\pi}{4}\right)$$

$$= 0 \text{ when } x = \frac{8}{1 + (\pi/4)} = \frac{32}{4 + \pi}.$$

$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0 \text{ when } x = \frac{32}{4 + \pi}$$

$$y = \frac{32 - 2[32/(4 + \pi)] - \pi[32/(4 + \pi)]}{4} = \frac{16}{4 + \pi}$$

The area is maximum when $y = \frac{16}{4 + \pi}$ feet and $x = \frac{32}{4 + \pi}$ feet.

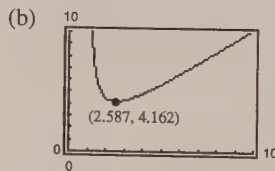


25. (a) $\frac{y-2}{0-1} = \frac{0-2}{x-1}$

$$y = 2 + \frac{2}{x-1}$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2 + \frac{2}{x-1}\right)^2}$$

$$= \sqrt{x^2 + 4 + \frac{8}{x-1} + \frac{4}{(x-1)^2}}, \quad x > 1$$



L is minimum when $x \approx 2.587$ and $L \approx 4.162$.

—CONTINUED—

25. —CONTINUED—

$$(c) \text{ Area} = A(x) = \frac{1}{2}xy = \frac{1}{2}x\left(2 + \frac{2}{x-1}\right) = x + \frac{x}{x-1}$$

$$A'(x) = 1 + \frac{(x-1) - x}{(x-1)^2} = 1 - \frac{1}{(x-1)^2} = 0$$

$$(x-1)^2 = 1$$

$$x-1 = \pm 1$$

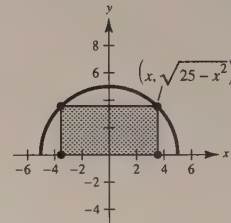
$$x = 0, 2 \text{ (select } x = 2\text{)}$$

Then $y = 4$ and $A = 4$.

Vertices: $(0, 0)$, $(2, 0)$, $(0, 4)$

$$27. A = 2xy = 2x\sqrt{25-x^2} \text{ (see figure)}$$

$$\begin{aligned} \frac{dA}{dx} &= 2x\left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{25-x^2}}\right) + 2\sqrt{25-x^2} \\ &= 2\left(\frac{25-2x^2}{\sqrt{25-x^2}}\right) = 0 \text{ when } x = y = \frac{5\sqrt{2}}{2} \approx 3.54. \end{aligned}$$



By the First Derivative Test, the inscribed rectangle of maximum area has vertices

$$\left(\pm \frac{5\sqrt{2}}{2}, 0\right), \left(\pm \frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right).$$

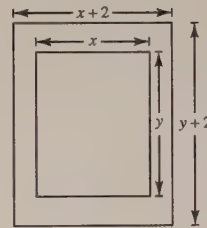
$$\text{Width: } \frac{5\sqrt{2}}{2}; \text{ Length: } 5\sqrt{2}$$

$$29. xy = 30 \Rightarrow y = \frac{30}{x}$$

$$A = (x+2)\left(\frac{30}{x} + 2\right) \text{ (see figure)}$$

$$\frac{dA}{dx} = (x+2)\left(\frac{-30}{x^2}\right) + \left(\frac{30}{x} + 2\right) = \frac{2(x^2-30)}{x^2} = 0 \text{ when } x = \sqrt{30}.$$

$$y = \frac{30}{\sqrt{30}} = \sqrt{30}$$



By the First Derivative Test, the dimensions $(x+2)$ by $(y+2)$ are $(2 + \sqrt{30})$ by $(2 + \sqrt{30})$ (approximately 7.477 by 7.477). These dimensions yield a minimum area.

$$31. V = \pi r^2 h = 22 \text{ cubic inches or } h = \frac{22}{\pi r^2}$$

(a)	Radius, r	Height	Surface Area
	0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
	0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
	0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
	0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$

31. —CONTINUED—

(b)

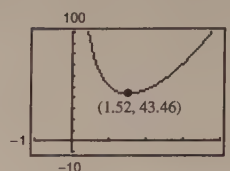
Radius, r	Height	Surface Area
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$
1.0	$\frac{22}{\pi(1.0)^2}$	$2\pi(1.0)\left[1.0 + \frac{22}{\pi(1.0)^2}\right] \approx 50.3$
1.2	$\frac{22}{\pi(1.2)^2}$	$2\pi(1.2)\left[1.2 + \frac{22}{\pi(1.2)^2}\right] \approx 45.7$
1.4	$\frac{22}{\pi(1.4)^2}$	$2\pi(1.4)\left[1.4 + \frac{22}{\pi(1.4)^2}\right] \approx 43.7$
1.6	$\frac{22}{\pi(1.6)^2}$	$2\pi(1.6)\left[1.6 + \frac{22}{\pi(1.6)^2}\right] \approx 43.6$
1.8	$\frac{22}{\pi(1.8)^2}$	$2\pi(1.8)\left[1.8 + \frac{22}{\pi(1.8)^2}\right] \approx 44.8$
2.0	$\frac{22}{\pi(2.0)^2}$	$2\pi(2.0)\left[2.0 + \frac{22}{\pi(2.0)^2}\right] \approx 47.1$

The minimum seems to be about 43.6 for $r = 1.6$.

(c) $S = 2\pi r^2 + 2\pi rh$

$$= 2\pi r(r + h) = 2\pi r\left[r + \frac{22}{\pi r^2}\right] = 2\pi r^2 + \frac{44}{r}$$

(d)



The minimum seems to be 43.46 for $r \approx 1.52$.

(e) $\frac{dS}{dr} = 4\pi r - \frac{44}{r^2} = 0$ when $r = \sqrt[3]{11/\pi} \approx 1.52$ in.

$$h = \frac{22}{\pi r^2} \approx 3.04 \text{ in.}$$

Note: Notice that

$$h = \frac{22}{\pi r^2} = \frac{22}{\pi(11/\pi)^{2/3}} = 2\left(\frac{11^{1/3}}{\pi^{1/3}}\right) = 2r.$$

33. Let x be the sides of the square ends and y the length of the package.

$$P = 4x + y = 108 \Rightarrow y = 108 - 4x$$

$$V = x^2y = x^2(108 - 4x) = 108x^2 - 4x^3$$

$$\frac{dV}{dx} = 216x - 12x^2$$

$$= 12x(18 - x) = 0 \text{ when } x = 18.$$

$$\frac{d^2V}{dx^2} = 216 - 24x = -216 < 0 \text{ when } x = 18.$$

The volume is maximum when $x = 18$ inches and $y = 108 - 4(18) = 36$ inches.

35. $V = \frac{1}{3}\pi x^2 h = \frac{1}{3}\pi x^2(r + \sqrt{r^2 - x^2})$ (see figure)

$$\frac{dV}{dx} = \frac{1}{3}\pi \left[\frac{-x^3}{\sqrt{r^2 - x^2}} + 2x(r + \sqrt{r^2 - x^2}) \right] = \frac{\pi x}{3\sqrt{r^2 - x^2}} (2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2) = 0$$

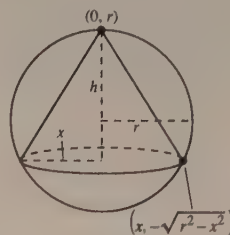
$$2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2 = 0$$

$$2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$$

$$4r^2(r^2 - x^2) = 9x^4 - 12x^2r^2 + 4r^4$$

$$0 = 9x^4 - 8x^2r^2 = x^2(9x^2 - 8r^2)$$

$$x = 0, \frac{2\sqrt{2}r}{3}$$



By the First Derivative Test, the volume is a maximum when

$$x = \frac{2\sqrt{2}r}{3} \text{ and } h = r + \sqrt{r^2 - x^2} = \frac{4r}{3}.$$

Thus, the maximum volume is

$$V = \frac{1}{3}\pi \left(\frac{8r^2}{9} \right) \left(\frac{4r}{3} \right) = \frac{32\pi r^3}{81} \text{ cubic units.}$$

37. No, there is no minimum area. If the sides are x and y , then $2x + 2y = 20 \Rightarrow y = 10 - x$.

The area is $A(x) = x(10 - x) = 10x - x^2$. This can be made arbitrarily small by selecting $x \approx 0$.

39. $V = 12 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$$h = \frac{12 - (4/3)\pi r^3}{\pi r^2} = \frac{12}{\pi r^2} - \frac{4}{3}r$$

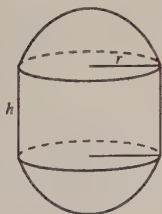
$$S = 4\pi r^2 + 2\pi r h = 4\pi r^2 + 2\pi r \left(\frac{12}{\pi r^2} - \frac{4}{3}r \right)$$

$$= 4\pi r^2 + \frac{24}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{24}{r}$$

$$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{24}{r^2} = 0 \text{ when } r = \sqrt[3]{9/\pi} \approx 1.42 \text{ cm.}$$

$$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{48}{r^3} > 0 \text{ when } r = \sqrt[3]{9/\pi} \text{ cm.}$$

The surface area is minimum when $r = \sqrt[3]{9/\pi}$ cm and $h = 0$. The resulting solid is a sphere of radius $r \approx 1.42$ cm.



41. Let x be the length of a side of the square and y the length of a side of the triangle.

$$4x + 3y = 10$$

$$A = x^2 + \frac{1}{2}y \left(\frac{\sqrt{3}}{2}y \right)$$

$$= \frac{(10 - 3y)^2}{16} + \frac{\sqrt{3}}{4}y^2$$

$$\frac{dA}{dy} = \frac{1}{8}(10 - 3y)(-3) + \frac{\sqrt{3}}{2}y = 0$$

$$-30 + 9y + 4\sqrt{3}y = 0$$

$$y = \frac{30}{9 + 4\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9 + 4\sqrt{3}}{8} > 0$$

A is minimum when

$$y = \frac{30}{9 + 4\sqrt{3}} \text{ and } x = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}.$$

43. Let
- S
- be the strength and
- k
- the constant of proportionality.

$$\text{Given } h^2 + w^2 = 24^2, h^2 = 24^2 - w^2,$$

$$S = kwh^2$$

$$S' = kw(576 - w^2) = k(576w - w^3)$$

$$\frac{dS}{dw} = k(576 - 3w^2) = 0 \text{ when } w = 8\sqrt{3}, h = 8\sqrt{6}.$$

$$\frac{d^2S}{dw^2} = -6kw < 0 \text{ when } w = 8\sqrt{3}.$$

These values yield a maximum.

$$47. \sin \alpha = \frac{h}{s} \Rightarrow s = \frac{h}{\sin \alpha}, \quad 0 < \alpha < \frac{\pi}{2}$$

$$\tan \alpha = \frac{h}{2} \Rightarrow h = 2 \tan \alpha \Rightarrow s = \frac{2 \tan \alpha}{\sin \alpha} = 2 \sec \alpha$$

$$I = \frac{k \sin \alpha}{s^2} = \frac{k \sin \alpha}{4 \sec^2 \alpha} = \frac{k}{4} \sin \alpha \cos^2 \alpha$$

$$\frac{dI}{d\alpha} = \frac{k}{4} [\sin \alpha (-2 \sin \alpha \cos \alpha) + \cos^2 \alpha (\cos \alpha)']$$

$$= \frac{k}{4} \cos \alpha [\cos^2 \alpha - 2 \sin^2 \alpha]$$

$$= \frac{k}{4} \cos \alpha [1 - 3 \sin^2 \alpha]$$

$$= 0 \text{ when } \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ or when } \sin \alpha = \pm \frac{1}{\sqrt{3}}.$$

Since α is acute, we have

$$\sin \alpha = \frac{1}{\sqrt{3}} \Rightarrow h = 2 \tan \alpha = 2 \left(\frac{1}{\sqrt{2}} \right) = \sqrt{2} \text{ feet.}$$

Since $(d^2I)/(d\alpha^2) = (k/4) \sin \alpha (9 \sin^2 \alpha - 7) < 0$ when $\sin \alpha = 1/\sqrt{3}$, this yields a maximum.

49.

$$S = \sqrt{x^2 + 4}, L = \sqrt{1 + (3-x)^2}$$

$$\text{Time} = T = \frac{\sqrt{x^2 + 4}}{2} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} + \frac{x-3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$\frac{x^2}{x^2 + 4} = \frac{9 - 6x + x^2}{4(x^2 - 6x + 10)}$$

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$

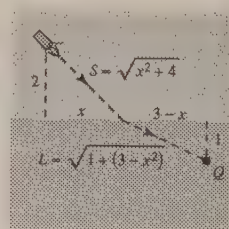
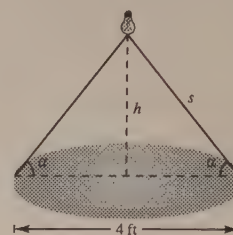
You need to find the roots of this equation in the interval $[0, 3]$. By using a computer or graphics calculator, you can determine that this equation has only one root in this interval ($x = 1$). Testing at this value and at the endpoints, you see that $x = 1$ yields the minimum time. Thus, the man should row to a point 1 mile from the nearest point on the coast.

$$45. \quad R = \frac{v_0^2}{g} \sin 2\theta$$

$$\frac{dR}{d\theta} = \frac{2v_0^2}{g} \cos 2\theta = 0 \text{ when } \theta = \frac{\pi}{4}, \frac{3\pi}{4}.$$

$$\frac{d^2R}{d\theta^2} = -\frac{4v_0^2}{g} \sin 2\theta < 0 \text{ when } \theta = \frac{\pi}{4}.$$

By the Second Derivative Test, R is maximum when $\theta = \pi/4$.



$$51. T = \frac{\sqrt{x^2 + 4}}{v_1} + \frac{\sqrt{x^2 - 6x + 10}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + 4}} + \frac{x - 3}{v_2\sqrt{x^2 - 6x + 10}} = 0$$

Since

$$\frac{x}{\sqrt{x^2 + 4}} = \sin \theta_1 \text{ and } \frac{x - 3}{\sqrt{x^2 - 6x + 10}} = -\sin \theta_2$$

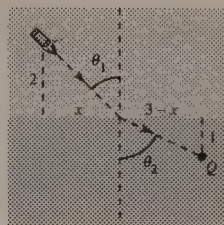
we have

$$\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0 \Rightarrow \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

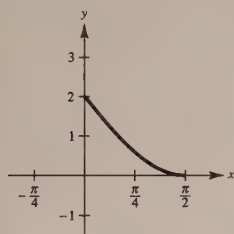
Since

$$\frac{d^2T}{dx^2} = \frac{4}{v_1(x^2 + 4)^{3/2}} + \frac{1}{v_2(x^2 - 6x + 10)^{3/2}} > 0$$

this condition yields a minimum time.



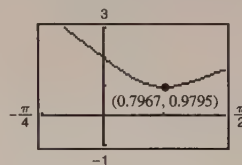
$$53. f(x) = 2 - 2 \sin x$$



(a) Distance from origin to y-intercept is 2.

Distance from origin to x-intercept is $\pi/2 \approx 1.57$.

$$(b) d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (2 - 2 \sin x)^2}$$



Minimum distance = 0.9795 at $x = 0.7967$.

$$(c) \text{ Let } f(x) = d^2(x) = x^2 + (2 - 2 \sin x)^2.$$

$$f'(x) = 2x + 2(2 - 2 \sin x)(-2 \cos x)$$

Setting $f'(x) = 0$, you obtain $x \approx 0.7967$, which corresponds to $d = 0.9795$.

$$55. F \cos \theta = k(W - F \sin \theta)$$

$$F = \frac{kW}{\cos \theta + k \sin \theta}$$

$$\frac{dF}{d\theta} = \frac{-kW(k \cos \theta - \sin \theta)}{(\cos \theta + k \sin \theta)^2} = 0$$

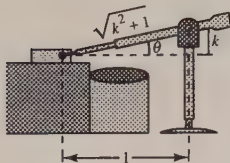
$$k \cos \theta = \sin \theta \Rightarrow k = \tan \theta \Rightarrow \theta = \arctan k$$

Since

$$\cos \theta + k \sin \theta = \frac{1}{\sqrt{k^2 + 1}} + \frac{k^2}{\sqrt{k^2 + 1}} = \sqrt{k^2 + 1},$$

the minimum force is

$$F = \frac{kW}{\cos \theta + k \sin \theta} = \frac{kW}{\sqrt{k^2 + 1}}.$$



57. (a)

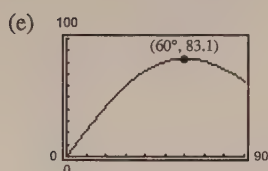
Base 1	Base 2	Altitude	Area
8	$8 + 16 \cos 10^\circ$	$8 \sin 10^\circ$	≈ 22.1
8	$8 + 16 \cos 20^\circ$	$8 \sin 20^\circ$	≈ 42.5
8	$8 + 16 \cos 30^\circ$	$8 \sin 30^\circ$	≈ 59.7
8	$8 + 16 \cos 40^\circ$	$8 \sin 40^\circ$	≈ 72.7
8	$8 + 16 \cos 50^\circ$	$8 \sin 50^\circ$	≈ 80.5
8	$8 + 16 \cos 60^\circ$	$8 \sin 60^\circ$	≈ 83.1

(b)

Base 1	Base 2	Altitude	Area
8	$8 + 16 \cos 10^\circ$	$8 \sin 10^\circ$	≈ 22.1
8	$8 + 16 \cos 20^\circ$	$8 \sin 20^\circ$	≈ 42.5
8	$8 + 16 \cos 30^\circ$	$8 \sin 30^\circ$	≈ 59.7
8	$8 + 16 \cos 40^\circ$	$8 \sin 40^\circ$	≈ 72.7
8	$8 + 16 \cos 50^\circ$	$8 \sin 50^\circ$	≈ 80.5
8	$8 + 16 \cos 60^\circ$	$8 \sin 60^\circ$	≈ 83.1
8	$8 + 16 \cos 70^\circ$	$8 \sin 70^\circ$	≈ 80.7
8	$8 + 16 \cos 80^\circ$	$8 \sin 80^\circ$	≈ 74.0
8	$8 + 16 \cos 90^\circ$	$8 \sin 90^\circ$	$= 64.0$

The maximum cross-sectional area is approximately 83.1 square feet.

$$\begin{aligned}
 \text{(c)} \quad A &= (a + b) \frac{h}{2} \\
 &= [8 + (8 + 16 \cos \theta)] \frac{8 \sin \theta}{2} \\
 &= 64(1 + \cos \theta) \sin \theta, 0^\circ < \theta < 90^\circ
 \end{aligned}$$



$$\begin{aligned}
 \text{(d)} \quad \frac{dA}{d\theta} &= 64(1 + \cos \theta) \cos \theta + (-64 \sin \theta) \sin \theta \\
 &= 64(\cos \theta + \cos^2 \theta - \sin^2 \theta) \\
 &= 64(2 \cos^2 \theta + \cos \theta - 1) \\
 &= 64(2 \cos \theta - 1)(\cos \theta + 1) \\
 &= 0 \text{ when } \theta = 60^\circ, 180^\circ, 300^\circ.
 \end{aligned}$$

The maximum occurs when $\theta = 60^\circ$.

$$59. \quad C = 100 \left(\frac{200}{x^2} + \frac{x}{x+30} \right), 1 \leq x$$

$$C' = 100 \left(-\frac{400}{x^3} + \frac{30}{(x+30)^2} \right)$$

Approximation: $x \approx 40.45$ units, or 4045 units

$$61. \quad S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2$$

$$\frac{dS_1}{dm} = 2(4m - 1)(4) + 2(5m - 6)(5) + 2(10m - 3)(10) = 282m - 128 = 0 \text{ when } m = \frac{64}{141}.$$

$$\text{Line: } y = \frac{64}{141}x$$

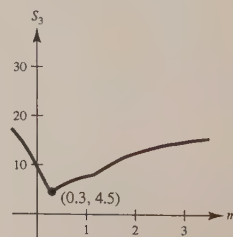
$$\begin{aligned}
 S &= \left| 4 \left(\frac{64}{141} \right) - 1 \right| + \left| 5 \left(\frac{64}{141} \right) - 6 \right| + \left| 10 \left(\frac{64}{141} \right) - 3 \right| \\
 &= \left| \frac{256}{141} - 1 \right| + \left| \frac{320}{141} - 6 \right| + \left| \frac{640}{141} - 3 \right| = \frac{858}{141} \approx 6.1 \text{ mi}
 \end{aligned}$$

$$63. \quad S_3 = \frac{|4m - 1|}{\sqrt{m^2 + 1}} + \frac{|5m - 6|}{\sqrt{m^2 + 1}} + \frac{|10m - 3|}{\sqrt{m^2 + 1}}$$

Using a graphing utility, you can see that the minimum occurs when $x \approx 0.3$.

Line: $y \approx 0.3x$

$$S_3 = \frac{|4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3|}{\sqrt{(0.3)^2 + 1}} \approx 4.5 \text{ mi.}$$



Section 3.8 , Newton's Method

1. $f(x) = x^2 - 3$

$f'(x) = 2x$

$x_1 = 1.7$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	-0.1100	3.4000	-0.0324	1.7324
2	1.7324	0.0012	3.4648	0.0003	1.7321

3. $f(x) = \sin x$

$f'(x) = \cos x$

$x_1 = 3$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	0.1411	-0.9900	-0.1425	3.1425
2	3.1425	-0.0009	-1.0000	0.0009	3.1416

5. $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

Approximation of the zero of f is 0.682.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	-0.3750	1.7500	-0.2143	0.7143
2	0.7143	0.0788	2.5307	0.0311	0.6832
3	0.6832	0.0021	2.4003	0.0009	0.6823

7. $f(x) = 3\sqrt{x-1} - x$

$f'(x) = \frac{3}{2\sqrt{x-1}} - 1$

Approximation of the zero of f is 1.146.

Similarly, the other zero is approximately 7.854.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.2000	0.1416	2.3541	0.0602	1.1398
2	1.1398	-0.0181	3.0118	-0.0060	1.1458
3	1.1458	-0.0003	2.9284	-0.0001	1.1459

9. $f(x) = x^3 + 3$

$f'(x) = 3x^2$

Approximation of the zero of f is -1.442.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.5000	-0.3750	6.7500	-0.0556	-1.4444
2	-1.4444	-0.0134	6.2589	-0.0021	-1.4423
3	-1.4423	-0.0003	6.2407	-0.0001	-1.4422

11. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

$f'(x) = 3x^2 - 7.8x + 4.79$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	-0.3360	1.6400	-0.2049	0.7049
2	0.7049	-0.0921	0.7824	-0.1177	0.8226
3	0.8226	-0.0231	0.4037	-0.0573	0.8799
4	0.8799	-0.0045	0.2495	-0.0181	0.8980
5	0.8980	-0.0004	0.2048	-0.0020	0.9000
6	0.9000	0.0000	0.2000	0.0000	0.9000

Approximation of the zero of f is 0.900.

11. —CONTINUED—

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.1	0.0000	-0.1600	-0.0000	1.1000

Approximation of the zero of f is 1.100.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.9	0.0000	0.8000	0.0000	1.9000

Approximation of the zero of f is 1.900.

13. $f(x) = x + \sin(x + 1)$

$f'(x) = 1 + \cos(x + 1)$

Approximation of the zero of f is -0.489.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	-0.0206	1.8776	-0.0110	-0.4890
2	-0.4890	0.0000	1.8723	0.0000	-0.4890

15. $h(x) = f(x) - g(x) = 2x + 1 - \sqrt{x + 4}$

$h'(x) = 2 - \frac{1}{2\sqrt{x + 4}}$

Point of intersection of the graphs of f and g occurs when $x \approx 0.569$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.6000	0.0552	1.7669	0.0313	0.5687
2	0.5687	-0.0001	1.7661	0.0000	0.5687

17. $h(x) = f(x) - g(x) = x - \tan x$

$h'(x) = 1 - \sec^2 x$

Point of intersection of the graphs of f and g occurs when $x \approx 4.493$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	4.5000	-0.1373	-21.5048	0.0064	4.4936
2	4.4936	-0.0039	-20.2271	0.0002	4.4934

19. $f(x) = x^2 - a = 0$

$f'(x) = 2x$

$$x_{i+1} = x_i - \frac{x_i^2 - a}{2x_i} = \frac{2x_i^2 - x_i^2 + a}{2x_i} = \frac{x_i^2 + a}{2x_i} = \frac{x_i}{2} + \frac{a}{2x_i}$$

21. $x_{i+1} = \frac{x_i^2 + 7}{2x_i}$

i	1	2	3	4	5
x_i	2.0000	2.7500	2.6477	2.6458	2.6458

$\sqrt{7} \approx 2.646$

23. $x_{i+1} = \frac{3x_i^4 + 6}{4x_i^3}$

i	1	2	3	4
x_i	1.5000	1.5694	1.5651	1.5651

$\sqrt[4]{6} \approx 1.565$

25. $f(x) = 1 + \cos x$

$f'(x) = -\sin x$

Approximation of the zero: 3.141

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	0.0100	-0.1411	-0.0709	3.0709
2	3.0709	0.0025	-0.0706	-0.0354	3.1063
3	3.1063	0.0006	-0.0353	-0.0176	3.1239
4	3.1239	0.0002	-0.0177	-0.0088	3.1327
5	3.1327	0.0000	-0.0089	-0.0044	3.1371
6	3.1371	0.0000	-0.0045	-0.0022	3.1393
7	3.1393	0.0000	-0.0023	-0.0011	3.1404
8	3.1404	0.0000	-0.0012	-0.0006	3.1410

27. $y = 2x^3 - 6x^2 + 6x - 1 = f(x)$

$y' = 6x^2 - 12x + 6 = f'(x)$

$x_1 = 1$

$f'(x) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$
1	1	1	0

29. $y = -x^3 + 6x^2 - 10x + 6 = f(x)$

$y' = -3x^2 + 12x - 10 = f'(x)$

$x_1 = 2$

$x_2 = 1$

$x_3 = 2$

$x_4 = 1$ and so on.

Fails to converge

31. Answers will vary. See page 222.

Newton's Method uses tangent lines to approximate c such that $f(c) = 0$.

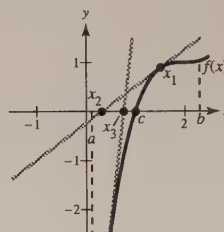
First, estimate an initial x_1 close to c (see graph).

Then determine x_2 by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

Calculate a third estimate by $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

Continue this process until $|x_n - x_{n+1}|$ is within the desired accuracy.

Let x_{n+1} be the final approximation of c .



33. Let $g(x) = f(x) - x = \cos x - x$

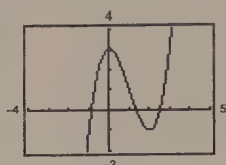
$g'(x) = -\sin x - 1$.

The fixed point is approximately 0.74.

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	-0.4597	-1.8415	0.2496	0.7504
2	0.7504	-0.0190	-1.6819	0.0113	0.7391
3	0.7391	0.0000	-1.6736	0.0000	0.7391

35. $f(x) = x^3 - 3x^2 + 3$, $f'(x) = 3x^2 - 6x$

(a)



(c) $x_1 = \frac{1}{4}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 2.405$$

Continuing, the zero is 2.532.

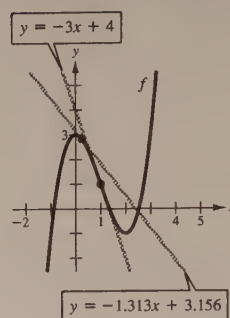
(e) If the initial guess x_1 is not "close to" the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

(b) $x_1 = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.333$$

Continuing, the zero is 1.347.

(d)



The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

37. $f(x) = \frac{1}{x} - a = 0$

$$f'(x) = -\frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{(1/x_n) - a}{-1/x_n^2} = x_n + x_n^2 \left(\frac{1}{x_n} - a \right) = x_n + x_n - x_n^2 a = 2x_n - x_n^2 a = x_n(2 - ax_n)$$

39. $f(x) = x \cos x$, $(0, \pi)$

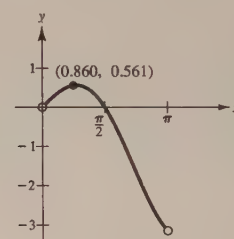
$$f'(x) = -x \sin x + \cos x = 0$$

Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

$$[F'(x) = -2 \sin x + x \cos x]$$

n	x_n	$F(x_n)$	$F'(x_n)$	$\frac{F(x_n)}{F'(x_n)}$	$x_n - \frac{F(x_n)}{F'(x_n)}$
1	0.9000	-0.0834	-2.1261	0.0392	0.8608
2	0.8608	-0.0010	-2.0778	0.0005	0.8603

Approximation to the critical number: 0.860



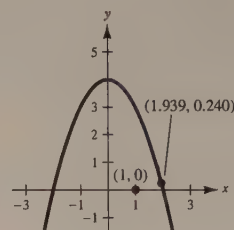
41. $y = f(x) = 4 - x^2, (1, 0)$

$$d = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + (4-x^2)^2} = \sqrt{x^4 - 7x^2 - 2x + 17}$$

d is minimized when $D = x^4 - 7x^2 - 2x + 17$ is a minimum.

$$g(x) = D' = 4x^3 - 14x - 2$$

$$g'(x) = 12x^2 - 14$$



n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	2.0000	2.0000	34.0000	0.0588	1.9412
2	1.9412	0.0830	31.2191	0.0027	1.9385
3	1.9385	-0.0012	31.0934	0.0000	1.9385

$$x \approx 1.939$$

Point closest to $(1, 0)$ is $\approx (1.939, 0.240)$.

43. Minimize: $T = \frac{\text{Distance rowed}}{\text{Rate rowed}} + \frac{\text{Distance walked}}{\text{Rate walked}}$

$$T = \frac{\sqrt{x^2 + 4}}{3} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$T' = \frac{x}{3\sqrt{x^2 + 4}} + \frac{x-3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$4x\sqrt{x^2 - 6x + 10} = -3(x-3)\sqrt{x^2 + 4}$$

$$16x^2(x^2 - 6x + 10) = 9(x-3)^2(x^2 + 4)$$

$$7x^4 - 42x^3 + 43x^2 + 216x - 324 = 0$$

Let $f(x) = 7x^4 - 42x^3 + 43x^2 + 216x - 324$ and $f'(x) = 28x^3 - 126x^2 + 86x + 216$. Since $f(1) = -100$ and $f(2) = 56$, the solution is in the interval $(1, 2)$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	19.5887	135.6240	0.1444	1.5556
2	1.5556	-1.0480	150.2780	-0.0070	1.5626
3	1.5626	0.0014	49.5591	0.0000	1.5626

Approximation: $x \approx 1.563$ miles

45. $2,500,000 = -76x^3 + 4830x^2 - 320,000$

$$76x^3 - 4830x^2 + 2,820,000 = 0$$

Let $f(x) = 76x^3 - 4830x^2 + 2,820,000$

$$f'(x) = 228x^2 - 9660x$$

From the graph, choose $x_1 = 40$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	40.0000	-44000.0000	-21600.0000	2.0370	37.9630
2	37.9630	17157.6209	-38131.4039	-0.4500	38.4130
3	38.4130	780.0914	-34642.2263	-0.0225	38.4355
4	38.4355	2.6308	-34465.3435	-0.0001	38.4356

The zero occurs when $x \approx 38.4356$ which corresponds to \$384,356.

47. False. Let $f(x) = (x^2 - 1)/(x - 1)$. $x = 1$ is a discontinuity. It is not a zero of $f(x)$. This statement would be true if $f(x) = p(x)/q(x)$ is given in **reduced** form.

49. True

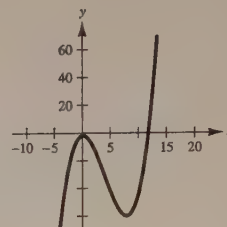
51. $f(x) = \frac{1}{4}x^3 - 3x^2 + \frac{3}{4}x - 2$

$$f'(x) = \frac{3}{4}x^2 - 6x + \frac{3}{4}$$

Let $x_1 = 12$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	12.0000	7.0000	36.7500	0.1905	11.8095
2	11.8095	0.2151	34.4912	0.0062	11.8033
3	11.8033	0.0015	34.4186	0.0000	11.8033

Approximation: $x \approx 11.803$



Section 3.9 Differentials

1. $f(x) = x^2$

$$f'(x) = 2x$$

Tangent line at (2, 4): $y - f(2) = f'(2)(x - 2)$

$$y - 4 = 4(x - 2)$$

$$y = 4x - 4$$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^2$	3.6100	3.9601	4	4.0401	4.4100
$T(x) = 4x - 4$	3.6000	3.9600	4	4.0400	4.4000

3. $f(x) = x^5$

$$f'(x) = 5x^4$$

Tangent line at (2, 32): $y - f(2) = f'(2)(x - 2)$

$$y - 32 = 80(x - 2)$$

$$y = 80x - 128$$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^5$	24.7610	31.2080	32	32.8080	40.8410
$T(x) = 80x - 128$	24.0000	31.2000	32	32.8000	40.0000

5. $f(x) = \sin x$

$$f'(x) = \cos x$$

Tangent line at (2, sin 2):

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sin 2 = (\cos 2)(x - 2)$$

$$y = (\cos 2)(x - 2) + \sin 2$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sin x$	0.9463	0.9134	0.9093	0.9051	0.8632
$T(x) = (\cos 2)(x - 2) + \sin 2$	0.9509	0.9135	0.9093	0.9051	0.8677

7. $y = f(x) = \frac{1}{2}x^3, f'(x) = \frac{3}{2}x^2, x = 2, \Delta x = dx = 0.1$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= f(2.1) - f(2)$$

$$= 0.6305$$

$$dy = f'(x)dx$$

$$= f'(2)(0.1)$$

$$= 6(0.1) = 0.6$$

9. $y = f(x) = x^4 + 1, f'(x) = 4x^3, x = -1, \Delta x = dx = 0.01$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= f(-0.99) - f(-1)$$

$$= [(-0.99)^4 + 1] - [(-1)^4 + 1] \approx -0.0394$$

$$dy = f'(x) dx$$

$$= f'(-1)(0.01)$$

$$= (-4)(0.01) = -0.04$$

11. $y = 3x^2 - 4$

$$dy = 6x dx$$

13. $y = \frac{x+1}{2x-1}$

$$dy = \frac{-3}{(2x-1)^2} dx$$

15. $y = x\sqrt{1-x^2}$

$$dy = \left(x \frac{-x}{\sqrt{1-x^2}} + \sqrt{1-x^2} \right) dx = \frac{1-2x^2}{\sqrt{1-x^2}} dx$$

17. $y = 2x - \cot^2 x$

$$dy = (2 + 2 \cot x \csc^2 x) dx$$

$$= (2 + 2 \cot x + 2 \cot^3 x) dx$$

19. $y = \frac{1}{3} \cos\left(\frac{6\pi x - 1}{2}\right)$

$$dy = -\pi \sin\left(\frac{6\pi x - 1}{2}\right) dx$$

21. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$$\approx 1 + (1)(-0.1) = 0.9$$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$$\approx 1 + (1)(0.04) = 1.04$$

23. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$$\approx 1 + \left(-\frac{1}{2}\right)(-0.1) = 1.05$$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$$\approx 1 + \left(-\frac{1}{2}\right)(0.04) = 0.98$$

25. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + \left(-\frac{1}{2}\right)(-0.07) = 8.035$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + \left(-\frac{1}{2}\right)(0.1) = 7.95$$

27. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + 0(-0.07) = 8$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + 0(0.1) = 8$$

29. $A = x^2$

$$x = 12$$

$$\Delta x = dx = \pm \frac{1}{64}$$

$$dA = 2x dx$$

$$\Delta A \approx dA = 2(12)\left(\pm \frac{1}{64}\right)$$

$$= \pm \frac{3}{8} \text{ square inches}$$

31. $A = \pi r^2$

$$r = 14$$

$$\Delta r = dr = \pm \frac{1}{4}$$

$$\Delta A \approx dA = 2\pi r dr = \pi(28)\left(\pm \frac{1}{4}\right)$$

$$= \pm 7\pi \text{ square inches}$$

33. (a) $x = 15$ centimeter

$\Delta x = dx = \pm 0.05$ centimeters

$A = x^2$

$dA = 2x dx = 2(15)(\pm 0.05)$

$= \pm 1.5$ square centimeters

Percentage error:

$$\frac{dA}{A} = \frac{1.5}{(15)^2} = 0.00666\ldots = \frac{2}{3}\%$$

(b) $\frac{dA}{A} = \frac{2x dx}{x^2} = \frac{2 dx}{x} \leq 0.025$

$$\frac{dx}{x} \leq \frac{0.025}{2} = 0.0125 = 1.25\%$$

37. $V = \pi r^2 h = 40\pi r^2$, $r = 5$ cm, $h = 40$ cm, $dr = 0.2$ cm

$\Delta V \approx dV = 80\pi r dr = 80\pi(5)(0.2) = 80\pi \text{ cm}^3$

39. (a) $T = 2\pi\sqrt{L/g}$

$$dT = \frac{\pi}{g\sqrt{L/g}} dL$$

Relative error:

$$\frac{dT}{T} = \frac{(\pi dL)/(g\sqrt{L/g})}{2\pi\sqrt{L/g}}$$

$$= \frac{dL}{2L}$$

$$= \frac{1}{2}(\text{relative error in } L)$$

$$= \frac{1}{2}(0.005) = 0.0025$$

Percentage error: $\frac{dT}{T}(100) = 0.25\% = \frac{1}{4}\%$

41. $\theta = 26^\circ 45' = 26.75^\circ$

$d\theta = \pm 15' = \pm 0.25^\circ$

(a) $h = 9.5 \csc \theta$

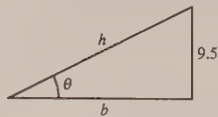
$dh = -9.5 \csc \theta \cot \theta d\theta$

$$\frac{dh}{h} = -\cot \theta d\theta$$

$$\left| \frac{dh}{h} \right| = (\cot 26.75^\circ)(0.25^\circ)$$

Converting to radians, $(\cot 0.4669)(0.0044)$

$\approx 0.0087 = 0.87\%$ (in radians).



35. $r = 6$ inches

$\Delta r = dr = \pm 0.02$ inches

(a) $V = \frac{4}{3}\pi r^3$

$dV = 4\pi r^2 dr = 4\pi(6)^2(\pm 0.02) = \pm 2.88\pi$ cubic inches

(b) $S = 4\pi r^2$

$dS = 8\pi r dr = 8\pi(6)(\pm 0.02) = \pm 0.96\pi$ square inches

(c) Relative error: $\frac{dV}{V} = \frac{4\pi r^2 dr}{(4/3)\pi r^3} = \frac{3dr}{r}$

$$= \frac{3}{6}(0.02) = 0.01 = 1\%$$

Relative error: $\frac{dS}{S} = \frac{8\pi r dr}{4\pi r^2} = \frac{2dr}{r}$

$$= \frac{2(0.02)}{6} = 0.00666\ldots = \frac{2}{3}\%$$

(b) $(0.0025)(3600)(24) = 216$ seconds

$= 3.6$ minutes

(b) $\left| \frac{dh}{h} \right| = \cot \theta d\theta \leq 0.02$

$$\frac{d\theta}{\theta} \leq \frac{0.02}{\theta(\cot \theta)} = \frac{0.02 \tan \theta}{\theta}$$

$$\frac{d\theta}{\theta} \leq \frac{0.02 \tan 26.75^\circ}{26.75^\circ} \approx \frac{0.02 \tan 0.4669}{0.4669}$$

$\approx 0.0216 = 2.16\%$ (in radians)

43. $r = \frac{v_0^2}{32} (\sin 2\theta)_t$

$v_0 = 2200$ ft/sec

θ changes from 10° to 11°

$$dr = \frac{(2200)^2}{16} (\cos 2\theta) d\theta$$

$$\theta = 10 \left(\frac{\pi}{180} \right)$$

$$d\theta = (11 - 10) \frac{\pi}{180}$$

$$\Delta r \approx dr$$

$$= \frac{(2200)^2}{16} \cos \left(\frac{20\pi}{180} \right) \left(\frac{\pi}{180} \right) \approx 4961 \text{ feet}$$

$$\approx 4961 \text{ feet}$$

47. Let $f(x) = \sqrt[4]{x}$, $x = 625$, $dx = -1$.

$$f(x + \Delta x) \approx f(x) + f'(x) dx = \sqrt[4]{x} + \frac{1}{4\sqrt[3]{x^3}} dx$$

$$f(x + \Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4(\sqrt[3]{625})^3} (-1)$$

$$= 5 - \frac{1}{500} = 4.998$$

Using a calculator, $\sqrt[4]{624} \approx 4.9980$.

45. Let $f(x) = \sqrt{x}$, $x = 100$, $dx = -0.6$.

$$f(x + \Delta x) \approx f(x) + f'(x) dx$$

$$= \sqrt{x} + \frac{1}{2\sqrt{x}} dx$$

$$f(x + \Delta x) = \sqrt{99.4}$$

$$\approx \sqrt{100} + \frac{1}{2\sqrt{100}} (-0.6) = 9.97$$

Using a calculator: $\sqrt{99.4} \approx 9.96995$

49. Let $f(x) = \sqrt{x}$, $x = 4$, $dx = 0.02$, $f'(x) = 1/(2\sqrt{x})$.

Then

$$f(4.02) \approx f(4) + f'(4) dx$$

$$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}} (0.02) = 2 + \frac{1}{4} (0.02).$$

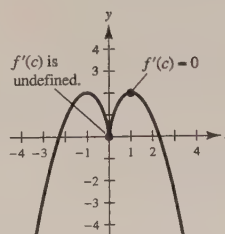
51. In general, when $\Delta x \rightarrow 0$, dy approaches Δy .

53. True

55. True

Review Exercises for Chapter 3

1. A number c in the domain of f is a critical number if $f'(c) = 0$ or f' is undefined at c .



3. $g(x) = 2x + 5 \cos x$, $[0, 2\pi]$

$$g'(x) = 2 - 5 \sin x$$

$$= 0 \text{ when } \sin x = \frac{2}{5}.$$

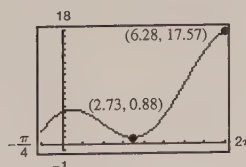
Critical numbers: $x \approx 0.41$, $x \approx 2.73$

Left endpoint: $(0, 5)$

Critical number: $(0.41, 5.41)$

Critical number: $(2.73, 0.88)$ Minimum

Right endpoint: $(2\pi, 17.57)$ Maximum

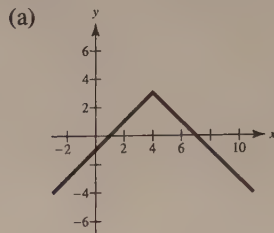


5. Yes. $f(-3) = f(2) = 0$. f is continuous on $[-3, 2]$, differentiable on $(-3, 2)$.

$$f'(x) = (x+3)(3x-1) = 0 \text{ for } x = \frac{1}{3}.$$

$$c = \frac{1}{3} \text{ satisfies } f'(c) = 0.$$

7. $f(x) = 3 - |x - 4|$



$$f(1) = f(7) = 0$$

(b) f is not differentiable at $x = 4$.

9. $f(x) = x^{2/3}, 1 \leq x \leq 8$

$$f'(x) = \frac{2}{3}x^{-1/3}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{8 - 1} = \frac{3}{7}$$

$$f'(c) = \frac{2}{3}c^{-1/3} = \frac{3}{7}$$

$$c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729} \approx 3.764$$

11. $f(x) = x - \cos x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$$f'(x) = 1 + \sin x$$

$$\frac{f(b) - f(a)}{b - a} = \frac{(\pi/2) - (-\pi/2)}{(\pi/2) - (-\pi/2)} = 1$$

$$f'(c) = 1 + \sin c = 1$$

$$c = 0$$

13. $f(x) = Ax^2 + Bx + C$

$$f'(x) = 2Ax + B$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{A(x_2^2 - x_1^2) + B(x_2 - x_1)}{x_2 - x_1}$$

$$= A(x_1 + x_2) + B$$

$$f'(c) = 2Ac + B = A(x_1 + x_2) + B$$

$$2Ac = A(x_1 + x_2)$$

$$c = \frac{x_1 + x_2}{2} = \text{Midpoint of } [x_1, x_2]$$

15. $f(x) = (x-1)^2(x-3)$

$$f'(x) = (x-1)^2(1) + (x-3)(2)(x-1)$$

$$= (x-1)(3x-7)$$

Critical numbers: $x = 1$ and $x = \frac{7}{3}$

Interval:	$-\infty < x < 1$	$1 < x < \frac{7}{3}$	$\frac{7}{3} < x < \infty$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

17. $h(x) = \sqrt{x}(x-3) = x^{3/2} - 3x^{1/2}$

Domain: $(0, \infty)$

$$h'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}$$

$$= \frac{3}{2}x^{-1/2}(x-1) = \frac{3(x-1)}{2\sqrt{x}}$$

Critical number: $x = 1$

Interval:	$0 < x < 1$	$1 < x < \infty$
Sign of $h'(x)$:	$h'(x) < 0$	$h'(x) > 0$
Conclusion:	Decreasing	Increasing

19. $h(t) = \frac{1}{4}t^4 - 8t$

$h'(t) = t^3 - 8 = 0$ when $t = 2$.

Relative minimum: $(2, -12)$

Test Interval:	$-\infty < t < 2$	$2 < t < \infty$
Sign of $h'(t)$:	$h'(t) < 0$	$h'(t) > 0$
Conclusion:	Decreasing	Increasing

21. $y = \frac{1}{3} \cos(12t) - \frac{1}{4} \sin(12t)$

$v = y' = -4 \sin(12t) - 3 \cos(12t)$

(a) When $t = \frac{\pi}{8}$, $y = \frac{1}{4}$ inch and $v = y' = 4$ inches/second.

(b) $y' = -4 \sin(12t) - 3 \cos(12t) = 0$ when $\frac{\sin(12t)}{\cos(12t)} = -\frac{3}{4} \Rightarrow \tan(12t) = -\frac{3}{4}$.

Therefore, $\sin(12t) = -\frac{3}{5}$ and $\cos(12t) = \frac{4}{5}$. The maximum displacement is

$$y = \left(\frac{1}{3}\right)\left(\frac{4}{5}\right) - \frac{1}{4}\left(-\frac{3}{5}\right) = \frac{5}{12} \text{ inch.}$$

(c) Period: $\frac{2\pi}{12} = \frac{\pi}{6}$

Frequency: $\frac{1}{\pi/6} = \frac{6}{\pi}$

23. $f(x) = x + \cos x$, $0 \leq x \leq 2\pi$

$f'(x) = 1 - \sin x$

$f''(x) = -\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

Test Interval:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

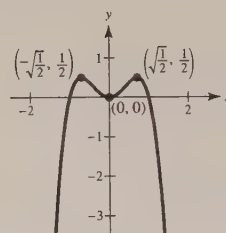
25. $g(x) = 2x^2(1 - x^2)$

$g'(x) = -4x(2x^2 - 1)$ Critical numbers: $x = 0, \pm \frac{1}{\sqrt{2}}$

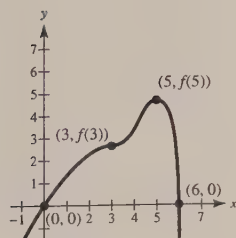
$g''(x) = 4 - 24x^2$

$g''(0) = 4 > 0$ Relative minimum at $(0, 0)$

$g''\left(\pm \frac{1}{\sqrt{2}}\right) = -8 < 0$ Relative maximums at $\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

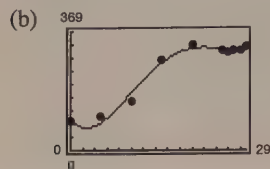


27.



29. The first derivative is positive and the second derivative is negative. The graph is increasing and is concave down.

31. (a) $D = 0.0034t^4 - 0.2352t^3 + 4.9423t^2 - 20.8641t + 94.4025$



(c) Maximum at (21.9, 319.5) (≈ 1992)

Minimum at (2.6, 69.6) (≈ 1972)

(d) Outlays increasing at greatest rate at the point of inflection (9.8, 173.7) (≈ 1979)

33. $\lim_{x \rightarrow \infty} \frac{2x^2}{3x^2 + 5} = \lim_{x \rightarrow \infty} \frac{2}{3 + 5/x^2} = \frac{2}{3}$

35. $\lim_{x \rightarrow \infty} \frac{5 \cos x}{x} = 0$, since $|5 \cos x| \leq 5$.

37. $h(x) = \frac{2x + 3}{x - 4}$

Discontinuity: $x = 4$

$$\lim_{x \rightarrow \infty} \frac{2x + 3}{x - 4} = \lim_{x \rightarrow \infty} \frac{2 + (3/x)}{1 - (4/x)} = 2$$

Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$

39. $f(x) = \frac{3}{x} - 2$

Discontinuity: $x = 0$

$$\lim_{x \rightarrow \infty} \left(\frac{3}{x} - 2 \right) = -2$$

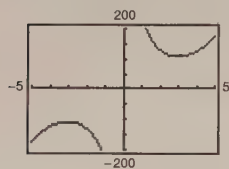
Vertical asymptote: $x = 0$

Horizontal asymptote: $y = -2$

41. $f(x) = x^3 + \frac{243}{x}$

Relative minimum: (3, 108)

Relative maximum: $(-3, -108)$

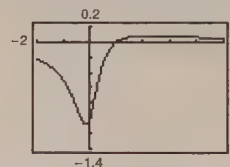


Vertical asymptote: $x = 0$

43. $f(x) = \frac{x - 1}{1 + 3x^2}$

Relative minimum: $(-0.155, -1.077)$

Relative maximum: $(2.155, 0.077)$



Horizontal asymptote: $y = 0$

45. $f(x) = 4x - x^2 = x(4 - x)$

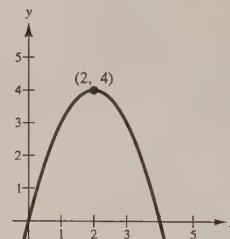
Domain: $(-\infty, \infty)$; Range: $(-\infty, 4]$

$$f'(x) = 4 - 2x = 0 \text{ when } x = 2.$$

$$f''(x) = -2$$

Therefore, (2, 4) is a relative maximum.

Intercepts: (0, 0), (4, 0)



47. $f(x) = x\sqrt{16 - x^2}$, Domain: $[-4, 4]$, Range: $[-8, 8]$

Domain: $[-4, 4]$; Range: $[-8, 8]$

$$f'(x) = \frac{16 - 2x^2}{\sqrt{16 - x^2}} = 0 \text{ when } x = \pm 2\sqrt{2} \text{ and undefined when } x = \pm 4.$$

$$f''(x) = \frac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$$

$$f''(-2\sqrt{2}) > 0$$

Therefore, $(-2\sqrt{2}, -8)$ is a relative minimum.

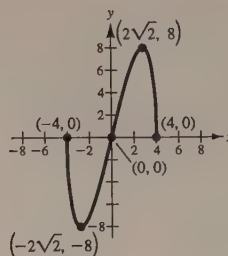
$$f''(2\sqrt{2}) < 0$$

Therefore, $(2\sqrt{2}, 8)$ is a relative maximum.

Point of inflection: $(0, 0)$

Intercepts: $(-4, 0)$, $(0, 0)$, $(4, 0)$

Symmetry with respect to origin



49. $f(x) = (x - 1)^3(x - 3)^2$

Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = (x - 1)^2(x - 3)(5x - 11) = 0 \text{ when } x = 1, \frac{11}{5}, 3.$$

$$f''(x) = 4(x - 1)(5x^2 - 22x + 23) = 0 \text{ when } x = 1, \frac{11 \pm \sqrt{6}}{5}.$$

$$f''(3) > 0$$

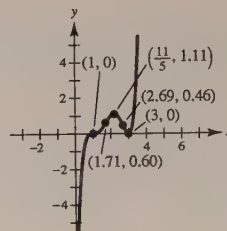
Therefore, $(3, 0)$ is a relative minimum.

$$f''\left(\frac{11}{5}\right) < 0$$

Therefore, $\left(\frac{11}{5}, \frac{3456}{3125}\right)$ is a relative maximum.

Points of inflection: $(1, 0)$, $\left(\frac{11 - \sqrt{6}}{5}, 0.60\right)$, $\left(\frac{11 + \sqrt{6}}{5}, 0.46\right)$

Intercepts: $(0, -9)$, $(1, 0)$, $(3, 0)$



51. $f(x) = x^{1/3}(x + 3)^{2/3}$

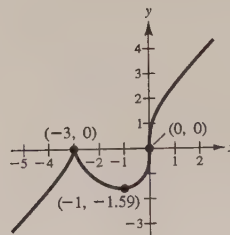
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = \frac{x + 1}{(x + 3)^{1/3}x^{2/3}} = 0 \text{ when } x = -1 \text{ and undefined when } x = -3, 0.$$

$$f''(x) = \frac{-2}{x^{5/3}(x + 3)^{4/3}} \text{ is undefined when } x = 0, -3.$$

By the First Derivative Test $(-3, 0)$ is a relative maximum and $(-1, -\sqrt[3]{4})$ is a relative minimum. $(0, 0)$ is a point of inflection.

Intercepts: $(-3, 0)$, $(0, 0)$



53. $f(x) = \frac{x+1}{x-1}$

Domain: $(-\infty, 1), (1, \infty)$; Range: $(-\infty, 1), (1, \infty)$

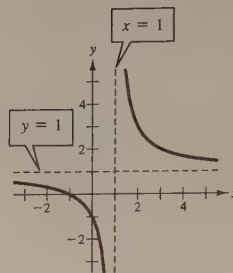
$$f'(x) = \frac{-2}{(x-1)^2} < 0 \text{ if } x \neq 1.$$

$$f''(x) = \frac{4}{(x-1)^3}$$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 1$

Intercepts: $(-1, 0), (0, -1)$



55. $f(x) = \frac{4}{1+x^2}$

Domain: $(-\infty, \infty)$; Range: $(0, 4]$

$$f'(x) = \frac{-8x}{(1+x^2)^2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{-8(1-3x^2)}{(1+x^2)^3} = 0 \text{ when } x = \pm \frac{\sqrt{3}}{3}.$$

$$f''(0) < 0$$

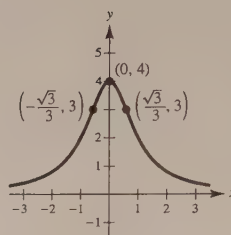
Therefore, $(0, 4)$ is a relative maximum.

Points of inflection: $(\pm\sqrt{3}/3, 3)$

Intercept: $(0, 4)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$



57. $f(x) = x^3 + x + \frac{4}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, -6], [6, \infty)$

$$f'(x) = 3x^2 + 1 - \frac{4}{x^2} = \frac{3x^4 + x^2 - 4}{x^2} = \frac{(3x^2 + 4)(x^2 - 1)}{x^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = 6x + \frac{8}{x^3} = \frac{6x^4 + 8}{x^3} \neq 0$$

$$f''(-1) < 0$$

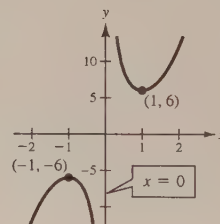
Therefore, $(-1, -6)$ is a relative maximum.

$$f''(1) > 0$$

Therefore, $(1, 6)$ is a relative minimum.

Vertical asymptote: $x = 0$

Symmetric with respect to origin



59. $f(x) = |x^2 - 9|$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$$f'(x) = \frac{2x(x^2 - 9)}{|x^2 - 9|} = 0 \text{ when } x = 0 \text{ and is undefined when } x = \pm 3.$$

$$f''(x) = \frac{2(x^2 - 9)}{|x^2 - 9|} \text{ is undefined at } x = \pm 3.$$

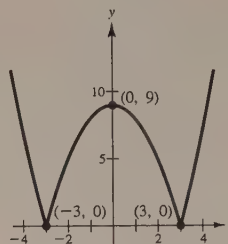
$$f''(0) < 0$$

Therefore, $(0, 9)$ is a relative maximum.

Relative minima: $(\pm 3, 0)$

Intercepts: $(\pm 3, 0)$, $(0, 9)$

Symmetric to the y-axis



61. $f(x) = x + \cos x$

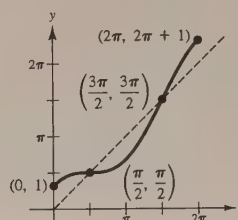
Domain: $[0, 2\pi]$; Range: $[1, 1 + 2\pi]$

$$f'(x) = 1 - \sin x \geq 0, f \text{ is increasing.}$$

$$f''(x) = -\cos x = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}.$$

$$\text{Points of inflection: } \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$$

Intercept: $(0, 1)$



63. $x^2 + 4y^2 - 2x - 16y + 13 = 0$

$$(a) (x^2 - 2x + 1) + 4(y^2 - 4y + 4) = -13 + 1 + 16$$

$$(x - 1)^2 + 4(y - 2)^2 = 4$$

$$\frac{(x - 1)^2}{4} + \frac{(y - 2)^2}{1} = 1$$

The graph is an ellipse:

Maximum: $(1, 3)$

Minimum: $(1, 1)$

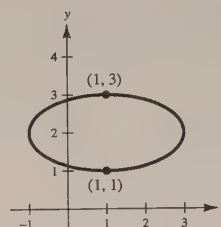
$$(b) x^2 + 4y^2 - 2x - 16y + 13 = 0$$

$$2x + 8y \frac{dy}{dx} - 2 - 16 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(8y - 16) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{8y - 16} = \frac{1 - x}{4y - 8}$$

The critical numbers are $x = 1$ and $y = 2$. These correspond to the points $(1, 1)$, $(1, 3)$, $(2, -1)$, and $(2, 3)$. Hence, the maximum is $(1, 3)$ and the minimum is $(1, 1)$.



65. Let
- $t = 0$
- at noon.

$$L = d^2 = (100 - 12t)^2 + (-10t)^2 = 10,000 - 2400t + 244t^2$$

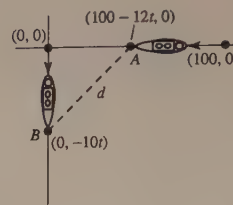
$$\frac{dL}{dt} = -2400 + 488t = 0 \text{ when } t = \frac{300}{61} \approx 4.92 \text{ hr.}$$

Ship A at $(40.98, 0)$; Ship B at $(0, -49.18)$

$$d^2 = 10,000 - 2400t + 244t^2$$

$$\approx 4098.36 \text{ when } t \approx 4.92 \approx 4:55 \text{ P.M..}$$

$$d \approx 64 \text{ km}$$



67. We have points
- $(0, y)$
- ,
- $(x, 0)$
- , and
- $(1, 8)$
- . Thus,

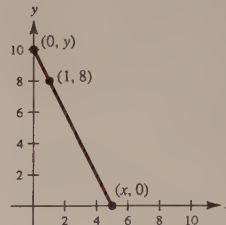
$$m = \frac{y-8}{0-1} = \frac{0-8}{x-1} \text{ or } y = \frac{8x}{x-1}.$$

$$\text{Let } f(x) = L^2 = x^2 + \left(\frac{8x}{x-1}\right)^2.$$

$$f'(x) = 2x + 128\left(\frac{x}{x-1}\right)\left[\frac{(x-1)-x}{(x-1)^2}\right] = 0$$

$$x - \frac{64x}{(x-1)^3} = 0$$

$$x[(x-1)^3 - 64] = 0 \text{ when } x = 0, 5 \text{ (minimum).}$$

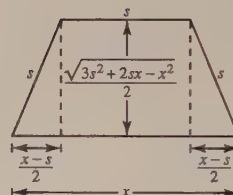
Vertices of triangle: $(0, 0)$, $(5, 0)$, $(0, 10)$ 

- 69.
- $A = (\text{Average of bases})(\text{Height})$

$$= \left(\frac{x+s}{2}\right) \frac{\sqrt{3s^2 + 2sx - x^2}}{2} \text{ (see figure)}$$

$$\frac{dA}{dx} = \frac{1}{4} \left[\frac{(s-x)(s+x)}{\sqrt{3s^2 + 2sx - x^2}} + \sqrt{3s^2 + 2sx - x^2} \right]$$

$$= \frac{2(2s-x)(s+x)}{4\sqrt{3s^2 + 2sx - x^2}} = 0 \text{ when } x = 2s.$$

 A is a maximum when $x = 2s$.

71. You can form a right triangle with vertices
- $(0, 0)$
- ,
- $(x, 0)$
- and
- $(0, y)$
- .

Assume that the hypotenuse of length L passes through $(4, 6)$.

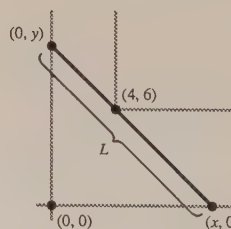
$$m = \frac{y-6}{0-4} = \frac{6-0}{4-x} \text{ or } y = \frac{6x}{x-4}$$

$$\text{Let } f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{6x}{x-4}\right)^2.$$

$$f'(x) = 2x + 72\left(\frac{x}{x-4}\right)\left[\frac{-4}{(x-4)^2}\right] = 0$$

$$x[(x-4)^3 - 144] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{144}.$$

$$L \approx 14.05 \text{ feet}$$



73. $\csc \theta = \frac{L_1}{6}$ or $L_1 = 6 \csc \theta$ (see figure)

$$L = L_1 + L_2 = \sec \theta \frac{L_2}{9} \text{ or } L_2 = 9 \sec \theta = 6 \csc \theta + 9 \sec \theta$$

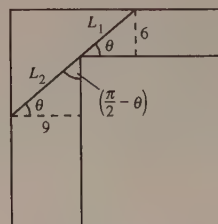
$$\frac{dL}{d\theta} = -6 \csc \theta \cot \theta + 9 \sec \theta \tan \theta = 0$$

$$\tan^3 \theta = \frac{2}{3} \Rightarrow \tan \theta = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$$

$$\sec \theta = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + \left(\frac{2}{3}\right)^{2/3}} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{3^{1/3}}$$

$$\csc \theta = \frac{\sec \theta}{\tan \theta} = \frac{\sqrt{3^{2/3} + 2^{2/3}}}{2^{1/3}}$$

$$L = 6 \frac{(3^{2/3} + 2^{2/3})^{1/2}}{2^{1/3}} + 9 \frac{(3^{2/3} + 2^{2/3})^{1/2}}{3^{1/3}} = 3(3^{2/3} + 2^{2/3})^{3/2} \text{ ft} \approx 21.07 \text{ ft (Compare to Exercise 72 using } a = 9 \text{ and } b = 6.)$$



75. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{600} + 5 \right) \left(\frac{110}{v} \right) = \frac{11v}{60} + \frac{550}{v}$$

$$\frac{dT}{dv} = \frac{11}{60} - \frac{550}{v^2} = \frac{11v^2 - 33,000}{60v^2}$$

$$= 0 \text{ when } v = \sqrt{3000} = 10\sqrt{30} \approx 54.8 \text{ mph.}$$

$$\frac{d^2T}{dv^2} = \frac{1100}{v^3} > 0 \text{ when } v = 10\sqrt{30} \text{ so this value yields a minimum.}$$

77. $f(x) = x^3 - 3x - 1$

From the graph you can see that $f(x)$ has three real zeros.

$$f'(x) = 3x^2 - 3$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.5000	0.1250	3.7500	0.0333	-1.5333
2	-1.5333	-0.0049	4.0530	-0.0012	-1.5321

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	0.3750	-2.2500	-0.1667	-0.3333
2	-0.3333	-0.0371	-2.6667	0.0139	-0.3472
3	-0.3472	-0.0003	-2.6384	0.0001	-0.3473

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.9000	0.1590	7.8300	0.0203	1.8797
2	1.8797	0.0024	7.5998	0.0003	1.8794

The three real zeros of $f(x)$ are $x \approx -1.532$, $x \approx -0.347$, and $x \approx 1.879$.

79. Find the zeros of
- $f(x) = x^4 - x - 3$
- .

$$f'(x) = 4x^3 - 1$$

From the graph you can see that $f(x)$ has two real zeros.

f changes sign in $[-2, -1]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.2000	0.2736	-7.9120	-0.0346	-1.1654
2	-1.1654	0.0100	-7.3312	-0.0014	-1.1640

On the interval $[-2, -1]$: $x \approx -1.164$.

f changes sign in $[1, 2]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.5000	0.5625	12.5000	0.0450	1.4550
2	1.4550	0.0268	11.3211	0.0024	1.4526
3	1.4526	-0.0003	11.2602	0.0000	1.4526

On the interval $[1, 2]$: $x \approx 1.453$.

- 81.
- $y = x(1 - \cos x) = x - x \cos x$

$$\frac{dy}{dx} = 1 + x \sin x - \cos x$$

$$dy = (1 + x \sin x - \cos x) dx$$

- 83.
- $S = 4\pi r^2$
- ,
- $dr = \Delta r = \pm 0.025$

$$dS = 8\pi r dr = 8\pi(9)(\pm 0.025)$$

$$= \pm 1.8\pi \text{ square cm}$$

$$\begin{aligned} \frac{dS}{S}(100) &= \frac{8\pi r dr}{4\pi r^2}(100) = \frac{2}{r} dr(100) \\ &= \frac{2(\pm 0.025)}{9}(100) \approx \pm 0.56\% \end{aligned}$$

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2 dr = 4\pi(9)^2(\pm 0.025)$$

$$= \pm 8.1\pi \text{ cubic cm}$$

$$\begin{aligned} \frac{dV}{V}(100) &= \frac{4\pi r^2 dr}{(4/3)\pi r^3}(100) = \frac{3}{r} dr(100) \\ &= \frac{3(\pm 0.025)}{9}(100) \approx \pm 0.83\% \end{aligned}$$

Problem Solving for Chapter 3

1. Assume $y_1 < d < y_2$. Let $g(x) = f(x) - d(x - a)$. g is continuous on $[a, b]$ and therefore has a minimum $(c, g(c))$ on $[a, b]$. The point c cannot be an endpoint of $[a, b]$ because

$$g'(a) = f'(a) - d = y_1 - d < 0$$

$$g'(b) = f'(b) - d = y_2 - d > 0$$

Hence, $a < c < b$ and $g'(c) = 0 \Rightarrow f'(c) = d$.

3. (a) For $a = -3, -2, -1, 0$, p has a relative maximum at $(0, 0)$.

For $a = 1, 2, 3$, p has a relative maximum at $(0, 0)$ and 2 relative minima.

$$(b) p'(x) = 4ax^3 - 12x = 4x(ax^2 - 3) = 0 \Rightarrow x = 0, \pm\sqrt{\frac{3}{a}}$$

$$p''(x) = 12ax^2 - 12 = 12(ax^2 - 1)$$

For $x = 0$, $p''(0) = -12 < 0 \Rightarrow p$ has a relative maximum at $(0, 0)$.

- (c) If $a > 0$, $x = \pm\sqrt{\frac{3}{a}}$ are the remaining critical numbers.

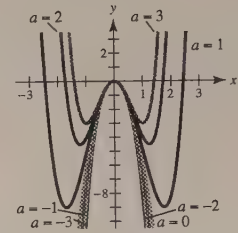
$$p''\left(\pm\sqrt{\frac{3}{a}}\right) = 12a\left(\frac{3}{a}\right) - 12 = 24 > 0 \Rightarrow p \text{ has relative minima for } a > 0.$$

- (d) $(0, 0)$ lies on $y = -3x^2$.

Let $x = \pm\sqrt{\frac{3}{a}}$. Then

$$p(x) = a\left(\frac{3}{a}\right)^2 - 6\left(\frac{3}{a}\right) = \frac{9}{a} - \frac{18}{a} = -\frac{9}{a}.$$

Thus, $y = -\frac{9}{a} = -3\left(\pm\sqrt{\frac{3}{a}}\right)^2 = -3x^2$ is satisfied by all the relative extrema of p .



5. $p(x) = x^4 + ax^2 + 1$

(a) $p'(x) = 4x^3 + 2ax = 2x(2x^2 + a)$

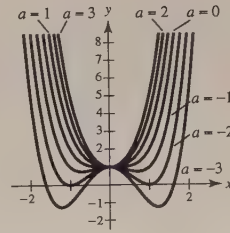
$$p''(x) = 12x^2 + 2a$$

For $a \geq 0$, there is one relative minimum at $(0, 1)$.

- (b) For $a < 0$, there is a relative maximum at $(0, 1)$.

(c) For $a < 0$, there are two relative minima at $x = \pm\sqrt{-\frac{a}{2}}$.

- (d) There are either 1 or 3 critical points. The above analysis shows that there cannot be exactly two relative extrema.



7. $f(x) = \frac{c}{x} + x^2$

$$f'(x) = -\frac{c}{x^2} + 2x = 0 \Rightarrow \frac{c}{x^2} = 2x \Rightarrow x^3 = \frac{c}{2} \Rightarrow x = \sqrt[3]{\frac{c}{2}}$$

$$f''(x) = \frac{2c}{x^3} + 2$$

If $c = 0$, $f(x) = x^2$ has a relative minimum, but no relative maximum.

If $c > 0$, $x = \sqrt[3]{\frac{c}{2}}$ is a relative minimum, because $f''\left(\sqrt[3]{\frac{c}{2}}\right) > 0$.

If $c < 0$, $x = \sqrt[3]{\frac{c}{2}}$ is a relative minimum too.

Answer: all c .

9. Set $\frac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = k$.

Define $F(x) = f(x) - f(a) - f'(a)(x - a) - k(x - a)^2$.

$F(a) = 0, F(b) = f(b) - f(a) - f'(a)(b - a) - k(b - a)^2 = 0$

F is continuous on $[a, b]$ and differentiable on (a, b) .

There exists $c_1, a < c_1 < b$, satisfying $F'(c_1) = 0$.

$F'(x) = f'(x) - f'(a) - 2k(x - a)$ satisfies the hypothesis of Rolle's Theorem on $[a, c_1]$:

$F'(a) = 0, F'(c_1) = 0$.

There exists $c_2, a < c_2 < c_1$ satisfying $F''(c_2) = 0$.

Finally, $F''(x) = f''(x) - 2k$ and $F''(c_2) = 0$ implies that

$$k = \frac{f''(c_2)}{2}.$$

Thus, $k = \frac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = \frac{f''(c_2)}{2} \Rightarrow f(b) = f(a) + f'(a)(b - a) + \frac{1}{2}f''(c_2)(b - a)^2$.

11. $E(\phi) = \frac{\tan \phi(1 - 0.1 \tan \phi)}{0.1 + \tan \phi} = \frac{10 \tan \phi - \tan^2 \phi}{1 + 10 \tan \phi}$

$$E'(\phi) = \frac{(1 + 10 \tan \phi)(10 \sec^2 \phi - 2 \tan \phi \sec^2 \phi) - (10 \tan \phi - \tan^2 \phi)10 \sec^2 \phi}{(1 + 10 \tan \phi)^2} = 0$$

$$\Rightarrow (1 + 10 \tan \phi)(10 \sec^2 \phi - 2 \tan \phi \sec^2 \phi) = (10 \tan \phi - \tan^2 \phi)10 \sec^2 \phi$$

$$\Rightarrow 10 \sec^2 \phi - 2 \tan \phi \sec^2 \phi + 100 \tan \phi \sec^2 \phi - 20 \tan^2 \phi \sec^2 \phi$$

$$= 100 \tan \phi \sec^2 \phi - 10 \tan^2 \phi \sec^2 \phi$$

$$\Rightarrow 10 - 2 \tan \phi = 10 \tan^2 \phi$$

$$\Rightarrow 10 \tan^2 \phi + 2 \tan \phi - 10 = 0$$

$$\tan \phi = \frac{-2 \pm \sqrt{4 + 400}}{20} \approx 0.90499, -1.10499$$

Using the positive value, $\phi \approx 0.7356$, or 42.14° .

13. $v = -2400\pi \sin \theta$

$v' = -2400\pi \cos \theta = 0$

$$\theta = \frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi, n \text{ an integer}$$

15. The line has equation $\frac{x}{3} + \frac{y}{4} = 1$ or $y = -\frac{4}{3}x + 4$.

Rectangle:

$$\text{Area} = A = xy = x\left(-\frac{4}{3}x + 4\right) = -\frac{4}{3}x^2 + 4x.$$

$$A'(x) = -\frac{8}{3}x + 4 = 0 \Rightarrow \frac{8}{3}x = 4 \Rightarrow x = \frac{3}{2}$$

Dimensions: $\frac{3}{2} \times 2$ Calculus was helpful.

Circle: The distance from the center (r, r) to the line $\frac{x}{3} + \frac{y}{4} - 1 = 0$ must be r :

$$r = \frac{\left|\frac{r}{3} + \frac{r}{4} - 1\right|}{\sqrt{\frac{1}{9} + \frac{1}{16}}} = \frac{12|7r - 12|}{5|12|} = \frac{|7r - 12|}{5}$$

$$5r = |7r - 12| \Rightarrow r = 1 \text{ or } r = 6.$$

Clearly, $r = 1$.

Semicircle: The center lies on the line $\frac{x}{3} + \frac{y}{4} = 1$ and satisfies $x = y = r$.

Thus $\frac{r}{3} + \frac{r}{4} = 1 \Rightarrow \frac{7}{12}r = 1 \Rightarrow r = \frac{12}{7}$. No calculus necessary.

17. $y = (1 + x^2)^{-1}$

$$y' = \frac{-2x}{(1 + x^2)^2}$$

$$y'' = \frac{2(3x^2 - 1)}{(x^2 + 1)^3} = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}} = \pm \frac{\sqrt{3}}{3}$$

$$y'': \frac{+++}{-\frac{\sqrt{3}}{3}} \quad 0 \quad \frac{\sqrt{3}}{3} \quad +++$$

The tangent line has greatest slope at $\left(-\frac{\sqrt{3}}{3}, \frac{3}{4}\right)$ and least slope at $\left(\frac{\sqrt{3}}{3}, \frac{3}{4}\right)$.

19. (a)

x	0.1	0.2	0.3	0.4	0.5	1.0
$\sin x$	0.09983	0.19867	0.29552	0.38942	0.47943	0.84147

$$\sin x < x$$

- (b) Let $f(x) = \sin x$. For $x \geq 1$, $\sin x < 1$ is obviously true, so assume $0 < x < 1$. Then $f'(x) = \cos x$ and on $[0, x]$ you have by the Mean Value Theorem,

$$f'(c) = \frac{f(x) - f(0)}{x - 0}, \quad 0 < c < x$$

$$\cos(c) = \frac{\sin x}{x}$$

$$\text{Hence, } \left|\frac{\sin x}{x}\right| = |\cos(c)| < 1$$

$$\Rightarrow |\sin x| < |x|$$

$$\Rightarrow \sin x < x$$

CHAPTER 4

Integration

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CHAPTER 4

Integration

Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Odd-Numbered Exercises

$$1. \frac{d}{dx}\left(\frac{3}{x^3} + C\right) = \frac{d}{dx}(3x^{-3} + C) = -9x^{-4} = \frac{-9}{x^4}$$

$$5. \frac{dy}{dt} = 3t^2$$

$$y = t^3 + C$$

$$\text{Check: } \frac{d}{dt}[t^3 + C] = 3t^2$$

$$3. \frac{d}{dx}\left(\frac{1}{3}x^3 - 4x + C\right) = x^2 - 4 = (x-2)(x+2)$$

$$7. \frac{dy}{dx} = x^{3/2}$$

$$y = \frac{2}{5}x^{5/2} + C$$

$$\text{Check: } \frac{d}{dx}\left[\frac{2}{5}x^{5/2} + C\right] = x^{3/2}$$

Given

Rewrite

Integrate

Simplify

$$9. \int \sqrt[3]{x} \, dx$$

$$\int x^{1/3} \, dx$$

$$\frac{x^{4/3}}{4/3} + C$$

$$\frac{3}{4}x^{4/3} + C$$

$$11. \int \frac{1}{x\sqrt{x}} \, dx$$

$$\int x^{-3/2} \, dx$$

$$\frac{x^{-1/2}}{-1/2} + C$$

$$-\frac{2}{\sqrt{x}} + C$$

$$13. \int \frac{1}{2x^3} \, dx$$

$$\frac{1}{2} \int x^{-3} \, dx$$

$$\frac{1}{2} \left(\frac{x^{-2}}{-2} \right) + C$$

$$-\frac{1}{4x^2} + C$$

$$15. \int (x+3) \, dx = \frac{x^2}{2} + 3x + C$$

$$\text{Check: } \frac{d}{dx}\left[\frac{x^2}{2} + 3x + C\right] = x + 3$$

$$17. \int (2x - 3x^2) \, dx = x^2 - x^3 + C$$

$$\text{Check: } \frac{d}{dx}[x^2 - x^3 + C] = 2x - 3x^2$$

$$19. \int (x^3 + 2) \, dx = \frac{1}{4}x^4 + 2x + C$$

$$\text{Check: } \frac{d}{dx}\left(\frac{1}{4}x^4 + 2x + C\right) = x^3 + 2$$

$$21. \int (x^{3/2} + 2x + 1) \, dx = \frac{2}{5}x^{5/2} + x^2 + x + C$$

$$\text{Check: } \frac{d}{dx}\left(\frac{2}{5}x^{5/2} + x^2 + x + C\right) = x^{3/2} + 2x + 1$$

$$23. \int \sqrt[3]{x^2} \, dx = \int x^{2/3} \, dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5}x^{5/3} + C$$

$$\text{Check: } \frac{d}{dx}\left(\frac{3}{5}x^{5/3} + C\right) = x^{2/3} = \sqrt[3]{x^2}$$

$$25. \int \frac{1}{x^3} \, dx = \int x^{-3} \, dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

$$\text{Check: } \frac{d}{dx}\left(-\frac{1}{2x^2} + C\right) = \frac{1}{x^3}$$

$$27. \int \frac{x^2 + x + 1}{\sqrt{x}} dx = \int (x^{3/2} + x^{1/2} + x^{-1/2}) dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C = \frac{2}{15}x^{1/2}(3x^2 + 5x + 15) + C$$

$$\text{Check: } \frac{d}{dx} \left(\frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C \right) = x^{3/2} + x^{1/2} + x^{-1/2} = \frac{x^2 + x + 1}{\sqrt{x}}$$

$$29. \int (x+1)(3x-2) dx = \int (3x^2 + x - 2) dx \\ = x^3 + \frac{1}{2}x^2 - 2x + C$$

$$\text{Check: } \frac{d}{dx} \left(x^3 + \frac{1}{2}x^2 - 2x + C \right) = 3x^2 + x - 2 \\ = (x+1)(3x-2)$$

$$31. \int y^2 \sqrt{y} dy = \int y^{5/2} dy = \frac{2}{7}y^{7/2} + C$$

$$\text{Check: } \frac{d}{dy} \left(\frac{2}{7}y^{7/2} + C \right) = y^{5/2} = y^2 \sqrt{y}$$

$$33. \int dx = \int 1 dx = x + C$$

$$\text{Check: } \frac{d}{dx}(x + C) = 1$$

$$35. \int (2 \sin x + 3 \cos x) dx = -2 \cos x + 3 \sin x + C$$

$$\text{Check: } \frac{d}{dx}(-2 \cos x + 3 \sin x + C) = 2 \sin x + 3 \cos x$$

$$37. \int (1 - \csc t \cot t) dt = t + \csc t + C$$

$$\text{Check: } \frac{d}{dt}(t + \csc t + C) = 1 - \csc t \cot t$$

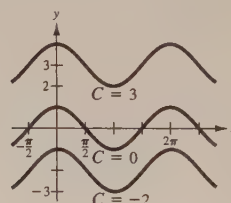
$$39. \int (\sec^2 \theta - \sin \theta) d\theta = \tan \theta + \cos \theta + C$$

$$\text{Check: } \frac{d}{d\theta}(\tan \theta + \cos \theta + C) = \sec^2 \theta - \sin \theta$$

$$41. \int (\tan^2 y + 1) dy = \int \sec^2 y dy = \tan y + C$$

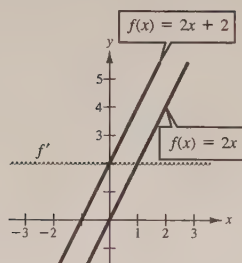
$$\text{Check: } \frac{d}{dy}(\tan y + C) = \sec^2 y = \tan^2 y + 1$$

$$43. f(x) = \cos x$$



$$45. f'(x) = 2$$

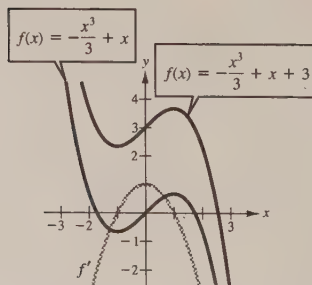
$$f(x) = 2x + C$$



Answers will vary.

$$47. f'(x) = 1 - x^2$$

$$f(x) = x - \frac{x^3}{3} + C$$



Answers will vary.

$$49. \frac{dy}{dx} = 2x - 1, (1, 1)$$

$$y = \int (2x - 1) dx = x^2 - x + C$$

$$1 = (1)^2 - (1) + C \Rightarrow C = 1$$

$$y = x^2 - x + 1$$

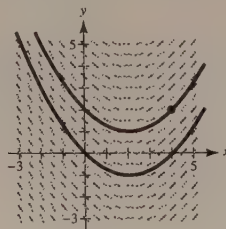
51. $\frac{dy}{dx} = \cos x, (0, 4)$

$$y = \int \cos x \, dx = \sin x + C$$

$$4 = \sin 0 + C \Rightarrow C = 4$$

$$y = \sin x + 4$$

53. (a) Answers will vary.



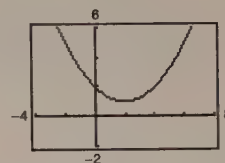
(b) $\frac{dy}{dx} = \frac{1}{2}x - 1, (4, 2)$

$$y = \frac{x^2}{4} - x + C$$

$$2 = \frac{4^2}{4} - 4 + C$$

$$2 = C$$

$$y = \frac{x^2}{4} - x + 2$$



55. $f'(x) = 4x, f(0) = 6$

$$f(x) = \int 4x \, dx = 2x^2 + C$$

$$f(0) = 6 = 2(0)^2 + C \Rightarrow C = 6$$

$$f(x) = 2x^2 + 6$$

57. $h'(t) = 8t^3 + 5, h(1) = -4$

$$h(t) = \int (8t^3 + 5) \, dt = 2t^4 + 5t + C$$

$$h(1) = -4 = 2 + 5 + C \Rightarrow C = -11$$

$$h(t) = 2t^4 + 5t - 11$$

59. $f''(x) = 2$

$$f'(2) = 5$$

$$f(2) = 10$$

$$f'(x) = \int 2 \, dx = 2x + C_1$$

$$f'(2) = 4 + C_1 = 5 \Rightarrow C_1 = 1$$

$$f'(x) = 2x + 1$$

$$f(x) = \int (2x + 1) \, dx = x^2 + x + C_2$$

$$f(2) = 6 + C_2 = 10 \Rightarrow C_2 = 4$$

$$f(x) = x^2 + x + 4$$

61. $f''(x) = x^{-3/2}$

$$f'(4) = 2$$

$$f(0) = 0$$

$$f'(x) = \int x^{-3/2} \, dx = -2x^{-1/2} + C_1 = -\frac{2}{\sqrt{x}} + C_1$$

$$f'(4) = -\frac{2}{2} + C_1 = 2 \Rightarrow C_1 = 3$$

$$f'(x) = -\frac{2}{\sqrt{x}} + 3$$

$$f(x) = \int \left(-2x^{-1/2} + 3\right) \, dx = -4x^{1/2} + 3x + C_2$$

$$f(0) = 0 + 0 + C_2 = 0 \Rightarrow C_2 = 0$$

$$f(x) = -4x^{1/2} + 3x = -4\sqrt{x} + 3x$$

63. (a) $h(t) = \int (1.5t + 5) \, dt = 0.75t^2 + 5t + C$

$$h(0) = 0 + 0 + C = 12 \Rightarrow C = 12$$

$$h(t) = 0.75t^2 + 5t + 12$$

(b) $h(6) = 0.75(6)^2 + 5(6) + 12 = 69 \text{ cm}$

65. $f(0) = -4$. Graph of f' is given.

(a) $f'(4) \approx -1.0$

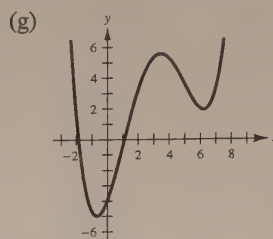
(b) No. The slopes of the tangent lines are greater than 2 on $[0, 2]$. Therefore, f must increase more than 4 units on $[0, 4]$.

(c) No, $f(5) < f(4)$ because f is decreasing on $[4, 5]$.

(d) f is an maximum at $x = 3.5$ because $f'(3.5) \approx 0$ and the first derivative test.

(e) f is concave upward when f' is increasing on $(-\infty, 1)$ and $(5, \infty)$. f is concave downward on $(1, 5)$. Points of inflection at $x = 1, 5$.

(f) f'' is a minimum at $x = 3$.



67. $a(t) = -32 \text{ ft/sec}^2$

$$v(t) = \int -32 \, dt = -32t + C_1$$

$$v(0) = 60 = C_1$$

$$s(t) = \int (-32t + 60) \, dt = -16t^2 + 60t + C_2$$

$$s(0) = 6 = C_2$$

$$s(t) = -16t^2 + 60t + 6 \text{ Position function}$$

The ball reaches its maximum height when

$$v(t) = -32t + 60 = 0$$

$$32t = 60$$

$$t = \frac{15}{8} \text{ seconds}$$

$$s\left(\frac{15}{8}\right) = -16\left(\frac{15}{8}\right)^2 + 60\left(\frac{15}{8}\right) + 6 = 62.25 \text{ feet}$$

71. $a(t) = -9.8$

$$v(t) = \int -9.8 \, dt = -9.8t + C_1$$

$$v(0) = v_0 = C_1 \Rightarrow v(t) = -9.8t + v_0$$

$$f(t) = \int (-9.8t + v_0) \, dt = -4.9t^2 + v_0t + C_2$$

$$f(0) = s_0 = C_2 \Rightarrow f(t) = -4.9t^2 + v_0t + s_0$$

75. $a = -1.6$

$$v(t) = \int -1.6 \, dt = -1.6t + v_0 = -1.6t, \text{ since the stone was dropped, } v_0 = 0.$$

$$s(t) = \int (-1.6t) \, dt = -0.8t^2 + s_0$$

$$s(20) = 0 \Rightarrow -0.8(20)^2 + s_0 = 0$$

$$s_0 = 320$$

Thus, the height of the cliff is 320 meters.

$$v(t) = -1.6t$$

$$v(20) = -32 \text{ m/sec}$$

69. From Exercise 68, we have:

$$s(t) = -16t^2 + v_0t$$

$$s'(t) = -32t + v_0 = 0 \text{ when } t = \frac{v_0}{32} = \text{time to reach maximum height.}$$

$$s\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) = 550$$

$$-\frac{v_0^2}{64} + \frac{v_0^2}{32} = 550$$

$$v_0^2 = 35,200$$

$$v_0 \approx 187.617 \text{ ft/sec}$$

73. From Exercise 71, $f(t) = -4.9t^2 + 10t + 2$.

$$v(t) = -9.8t + 10 = 0 \text{ (Maximum height when } v = 0.)$$

$$9.8t = 10$$

$$t = \frac{10}{9.8}$$

$$f\left(\frac{10}{9.8}\right) \approx 7.1 \text{ m}$$

77. $x(t) = t^3 - 6t^2 + 9t - 2 \quad 0 \leq t \leq 5$

(a) $v(t) = x'(t) = 3t^2 - 12t + 9$
 $= 3(t^2 - 4t + 3) = 3(t-1)(t-3)$

$a(t) = v'(t) = 6t - 12 = 6(t-2)$

(b) $v(t) > 0$ when $0 < t < 1$ or $3 < t < 5$.

(c) $a(t) = 6(t-2) = 0$ when $t = 2$.

$v(2) = 3(1)(-1) = -3$

81. (a) $v(0) = 25 \text{ km/hr} = 25 \cdot \frac{1000}{3600} = \frac{250}{36} \text{ m/sec}$

$v(13) = 80 \text{ km/hr} = 80 \cdot \frac{1000}{3600} = \frac{800}{36} \text{ m/sec}$

$a(t) = a$ (constant acceleration)

$v(t) = at + C$

$v(0) = \frac{250}{36} \Rightarrow v(t) = at + \frac{250}{36}$

$v(13) = \frac{800}{36} = 13a + \frac{250}{36}$

$\frac{550}{36} = 13a$

$a = \frac{550}{468} = \frac{275}{234} \approx 1.175 \text{ m/sec}^2$

(b) $s(t) = a \frac{t^2}{2} + \frac{250}{36}t \quad (s(0) = 0)$

$s(13) = \frac{275}{234} \frac{(13)^2}{2} + \frac{250}{36}(13) \approx 189.58 \text{ m}$

85. $\frac{(1 \text{ mi/hr})(5280 \text{ ft/mi})}{(3600 \text{ sec/hr})} = \frac{22}{15} \text{ ft/sec}$

t	0	5	10	15	20	25	30
$V_1(\text{ft/sec})$	0	3.67	10.27	23.47	42.53	66	95.33
$V_2(\text{ft/sec})$	0	30.8	55.73	74.8	88	93.87	95.33

(c) $S_1(t) = \int V_1(t) dt = \frac{0.1068}{3} t^3 - \frac{0.0416}{2} t^2 + 0.3679t$

$S_2(t) = \int V_2(t) dt = -\frac{0.1208t^3}{3} + \frac{6.7991t^2}{2} - 0.0707t$

[In both cases, the constant of integration is 0 because $S_1(0) = S_2(0) = 0$]

$S_1(30) \approx 953.5 \text{ feet}$

$S_2(30) \approx 1970.3 \text{ feet}$

The second car was going faster than the first until the end.

79. $v(t) = \frac{1}{\sqrt{t}} = t^{-1/2} \quad t > 0$

$x(t) = \int v(t) dt = 2t^{1/2} + C$

$x(1) = 4 = 2(1) + C \Rightarrow C = 2$

$x(t) = 2t^{1/2} + 2$ position function

$a(t) = v'(t) = -\frac{1}{2}t^{-3/2} = \frac{-1}{2t^{3/2}}$ acceleration

83. Truck: $v(t) = 30$

$s(t) = 30t$ (Let $s(0) = 0$.)

Automobile: $a(t) = 6$

$v(t) = 6t$ (Let $v(0) = 0$.)

$s(t) = 3t^2$ (Let $s(0) = 0$.)

At the point where the automobile overtakes the truck:

$30t = 3t^2$

$0 = 3t^2 - 30t$

$0 = 3t(t-10)$ when $t = 10 \text{ sec}$.

(a) $s(10) = 3(10)^2 = 300 \text{ ft}$

(b) $v(10) = 6(10) = 60 \text{ ft/sec} \approx 41 \text{ mph}$

(b) $V_1(t) = 0.1068t^2 - 0.0416t + 0.3679$

$V_2(t) = -0.1208t^2 + 6.7991t - 0.0707$

87. $a(t) = k$

$v(t) = kt$

$s(t) = \frac{k}{2}t^2$ since $v(0) = s(0) = 0$.

At the time of lift-off, $kt = 160$ and $(k/2)t^2 = 0.7$. Since $(k/2)t^2 = 0.7$,

$$t = \sqrt{\frac{1.4}{k}}$$

$$v\left(\sqrt{\frac{1.4}{k}}\right) = k\sqrt{\frac{1.4}{k}} = 160$$

$$1.4k = 160^2 \Rightarrow k = \frac{160^2}{1.4}$$

$$\approx 18,285.714 \text{ mi/hr}^2$$

$$\approx 7.45 \text{ ft/sec}^2.$$

89. True

91. True

93. False. For example, $\int x \cdot x \, dx \neq \int x \, dx \cdot \int x \, dx$ because $\frac{x^3}{3} + C \neq \left(\frac{x^2}{2} + C_1\right)\left(\frac{x^2}{2} + C_2\right)$

95. $f'(x) = \begin{cases} 1, & 0 \leq x < 2 \\ 3x, & 2 \leq x \leq 5 \end{cases}$

$$f(x) = \begin{cases} x + C_1, & 0 \leq x < 2 \\ \frac{3x^2}{2} + C_2, & 2 \leq x \leq 5 \end{cases}$$

$$f(1) = 3 \Rightarrow 1 + C_1 = 3 \Rightarrow C_1 = 2$$

f is continuous: Values must agree at $x = 2$:

$$4 = 6 + C_2 \Rightarrow C_2 = -2$$

$$f(x) = \begin{cases} x + 2, & 0 \leq x < 2 \\ \frac{3x^2}{2} - 2, & 2 \leq x \leq 5 \end{cases}$$

The left and right hand derivatives at $x = 2$ do not agree. Hence f is not differentiable at $x = 2$.

Section 4.2 Area

1. $\sum_{i=1}^5 (2i + 1) = 2 \sum_{i=1}^5 i + \sum_{i=1}^5 1 = 2(1 + 2 + 3 + 4 + 5) + 5 = 35$

3. $\sum_{k=0}^4 \frac{1}{k^2 + 1} = 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} = \frac{158}{85}$

5. $\sum_{k=1}^4 c = c + c + c + c = 4c$

7. $\sum_{i=1}^9 \frac{1}{3i}$

9. $\sum_{j=1}^8 \left[5\left(\frac{j}{8}\right) + 3 \right]$

11. $\frac{2}{n} \sum_{i=1}^n \left[\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right) \right]$

13. $\frac{3}{n} \sum_{i=1}^n \left[2\left(1 + \frac{3i}{n}\right)^2 \right]$

15. $\sum_{i=1}^{20} 2i = 2 \sum_{i=1}^{20} i = 2 \left[\frac{20(21)}{2} \right] = 420$

17. $\sum_{i=1}^{20} (i - 1)^2 = \sum_{i=1}^{19} i^2$

$$= \left[\frac{19(20)(39)}{6} \right] = 2470$$

$$\begin{aligned}
 19. \sum_{i=1}^{15} i(i-1)^2 &= \sum_{i=1}^{15} i^3 - 2 \sum_{i=1}^{15} i^2 + \sum_{i=1}^{15} i \\
 &= \frac{15^2(16)^2}{4} - 2 \frac{15(16)(31)}{6} + \frac{15(16)}{2} \\
 &= 14,400 - 2,480 + 120 \\
 &= 12,040
 \end{aligned}$$

$$21. \text{sum seq}(x \text{ [^] } 2 + 3, x, 1, 20, 1) = 2930 \quad (TI-82)$$

$$\begin{aligned}
 \sum_{i=1}^{20} (i^2 + 3) &= \frac{20(20+1)(2(20)+1)}{6} + 3(20) \\
 &= \frac{(20)(21)(41)}{6} + 60 = 2930
 \end{aligned}$$

$$\begin{aligned}
 23. S &= [3 + 4 + \frac{9}{2} + 5](1) = \frac{33}{2} = 16.5 \\
 s &= [1 + 3 + 4 + \frac{9}{2}](1) = \frac{25}{2} = 12.5
 \end{aligned}$$

$$25. S = [3 + 3 + 5](1) = 11$$

$$s = [2 + 2 + 3](1) = 7$$

$$27. S(4) = \sqrt{\frac{1}{4}\left(\frac{1}{4}\right)} + \sqrt{\frac{1}{2}\left(\frac{1}{4}\right)} + \sqrt{\frac{3}{4}\left(\frac{1}{4}\right)} + \sqrt{1\left(\frac{1}{4}\right)} = \frac{1 + \sqrt{2} + \sqrt{3} + 2}{8} \approx 0.768$$

$$s(4) = 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}\left(\frac{1}{4}\right)} + \sqrt{\frac{1}{2}\left(\frac{1}{4}\right)} + \sqrt{\frac{3}{4}\left(\frac{1}{4}\right)} = \frac{1 + \sqrt{2} + \sqrt{3}}{8} \approx 0.518$$

$$29. S(5) = 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746$$

$$s(5) = \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646$$

$$31. \lim_{n \rightarrow \infty} \left[\left(\frac{81}{n^4} \right) \frac{n^2(n+1)^2}{4} \right] = \frac{81}{4} \lim_{n \rightarrow \infty} \left[\frac{n^4 + 2n^3 + n^2}{n^4} \right] = \frac{81}{4}(1) = \frac{81}{4}$$

$$33. \lim_{n \rightarrow \infty} \left[\left(\frac{18}{n^2} \right) \frac{n(n+1)}{2} \right] = \frac{18}{2} \lim_{n \rightarrow \infty} \left[\frac{n^2 + n}{n^2} \right] = \frac{18}{2}(1) = 9$$

$$35. \sum_{i=1}^n \frac{2i+1}{n^2} = \frac{1}{n^2} \sum_{i=1}^n (2i+1) = \frac{1}{n^2} \left[2 \frac{n(n+1)}{2} + n \right] = \frac{n+2}{n} = S(n)$$

$$S(10) = \frac{12}{10} = 1.2$$

$$S(100) = 1.02$$

$$S(1000) = 1.002$$

$$S(10,000) = 1.0002$$

$$\begin{aligned}
 37. \sum_{k=1}^n \frac{6k(k-1)}{n^3} &= \frac{6}{n^3} \sum_{k=1}^n (k^2 - k) = \frac{6}{n^3} \left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right] \\
 &= \frac{6}{n^3} \left[\frac{2n^2 + 3n + 1 - 3n - 3}{6} \right] = \frac{1}{n^2} [2n^2 - 2] = S(n)
 \end{aligned}$$

$$S(10) = 1.98$$

$$S(100) = 1.9998$$

$$S(1000) = 1.999998$$

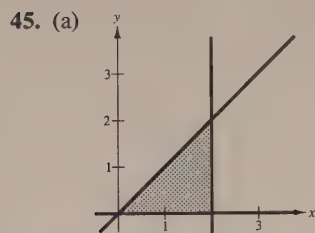
$$S(10,000) = 1.99999998$$

$$39. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{16i}{n^2} \right) = \lim_{n \rightarrow \infty} \frac{16}{n^2} \sum_{i=1}^n i = \lim_{n \rightarrow \infty} \frac{16}{n^2} \left(\frac{n(n+1)}{2} \right) = \lim_{n \rightarrow \infty} \left[8 \left(\frac{n^2 + n}{n^2} \right) \right] = 8 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = 8$$

$$41. \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n^3} (i-1)^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{i=1}^{n-1} i^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \left[\frac{(n-1)(n)(2n-1)}{6} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{6} \left[\frac{2n^3 - 3n^2 + n}{n^3} \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{6} \left(2 - \frac{3}{n} + \frac{1}{n^2} \right) \right] = \frac{1}{3}$$

$$43. \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{i}{n} \right) \left(\frac{2}{n} \right) = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sum_{i=1}^n 1 + \frac{1}{n} \sum_{i=1}^n i \right] = 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{1}{n} \left(\frac{n(n+1)}{2} \right) \right] = 2 \lim_{n \rightarrow \infty} \left[1 + \frac{n^2 + n}{2n^2} \right] = 2 \left(1 + \frac{1}{2} \right) = 3$$



(b) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

Endpoints:

$$0 < 1\left(\frac{2}{n}\right) < 2\left(\frac{2}{n}\right) < \dots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right) = 2$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^n f(x_{i-1}) \Delta x$$

$$= \sum_{i=1}^n f\left(\frac{2i-2}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[(i-1) \left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

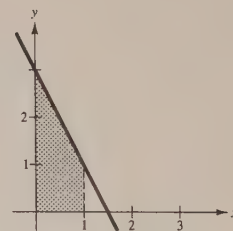
$$S(n) = \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n f\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[i \left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right)$$

47. $y = -2x + 3$ on $[0, 1]$. (Note: $\Delta x = \frac{1-0}{n} = \frac{1}{n}$)

$$s(n) = \sum_{i=1}^n f\left(\frac{i}{n}\right) \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[-2\left(\frac{i}{n}\right) + 3 \right] \left(\frac{1}{n}\right)$$

$$= 3 - \frac{2}{n^2} \sum_{i=1}^n i = 3 - \frac{2(n+1)n}{2n^2} = 2 - \frac{1}{n}$$

Area = $\lim_{n \rightarrow \infty} s(n) = 2$

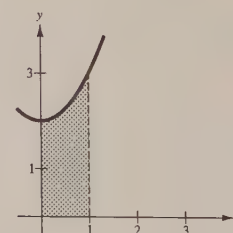


49. $y = x^2 + 2$ on $[0, 1]$. (Note: $\Delta x = \frac{1}{n}$)

$$S(n) = \sum_{i=1}^n f\left(\frac{i}{n}\right) \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[\left(\frac{i}{n}\right)^2 + 2 \right] \left(\frac{1}{n}\right)$$

$$= \left[\frac{1}{n^3} \sum_{i=1}^n i^2 \right] + 2 = \frac{n(n+1)(2n+1)}{6n^3} + 2 = \frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) + 2$$

Area = $\lim_{n \rightarrow \infty} S(n) = \frac{7}{3}$



(e)

x	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

(f) $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left[(i-1) \left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right) = \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n (i-1)$

$$= \lim_{n \rightarrow \infty} \frac{4}{n^2} \left[\frac{n(n+1)}{2} - n \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{2(n+1)}{n} - \frac{4}{n} \right] = 2$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left[i \left(\frac{2}{n}\right) \right] \left(\frac{2}{n}\right) = \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n i$$

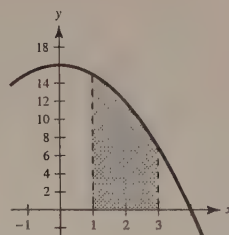
$$= \lim_{n \rightarrow \infty} \left(\frac{4}{n^2} \right) \frac{n(n+1)}{2}$$

$$= \lim_{n \rightarrow \infty} \frac{2(n+1)}{n} = 2$$

51. $y = 16 - x^2$ on $[1, 3]$. (Note: $\Delta x = \frac{2}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[16 - \left(1 + \frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right) \\ &= \frac{2}{n} \sum_{i=1}^n \left[15 - \frac{4i^2}{n^2} - \frac{4i}{n}\right] \\ &= \frac{2}{n} \left[15n - \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{4}{n} \frac{n(n+1)}{2}\right] \\ &= 30 - \frac{8}{6n^2}(n+1)(2n+1) - \frac{4}{n}(n+1) \end{aligned}$$

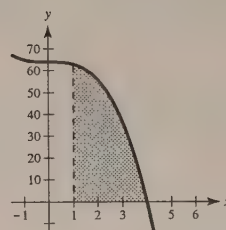
$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 30 - \frac{8}{3} - 4 = \frac{70}{3} = 23\frac{1}{3}$$



53. $y = 64 - x^3$ on $[1, 4]$. (Note: $\Delta x = \frac{4-1}{n} = \frac{3}{n}$)

$$\begin{aligned} s(n) &= \sum_{i=1}^n f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^n \left[64 - \left(1 + \frac{3i}{n}\right)^3\right]\left(\frac{3}{n}\right) \\ &= \frac{3}{n} \sum_{i=1}^n \left[63 - \frac{27i^3}{n^3} - \frac{27i^2}{n^2} - \frac{9i}{n}\right] \\ &= \frac{3}{n} \left[63n - \frac{27}{n^3} \frac{n^2(n+1)^2}{4} - \frac{27}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{9}{n} \frac{n(n+1)}{2}\right] \\ &= 189 - \frac{81}{4n^2}(n+1)^2 - \frac{81}{6n^2}(n+1)(2n+1) - \frac{27}{2} \frac{n+1}{n} \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} s(n) = 189 - \frac{81}{4} - 27 - \frac{27}{2} = \frac{513}{4} = 128.25$$

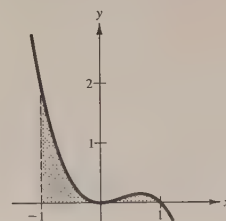


55. $y = x^2 - x^3$ on $[-1, 1]$. (Note: $\Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}$)

Again, $T(n)$ is neither an upper nor a lower sum.

$$\begin{aligned} T(n) &= \sum_{i=1}^n f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^n \left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^n \left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) = \frac{4}{n} \sum_{i=1}^n 1 - \frac{20}{n^2} \sum_{i=1}^n i + \frac{32}{n^3} \sum_{i=1}^n i^2 - \frac{16}{n^4} \sum_{i=1}^n i^3 \\ &= \frac{4}{n}(n) - \frac{20}{n^2} \cdot \frac{n(n+1)}{2} + \frac{32}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\ &= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$

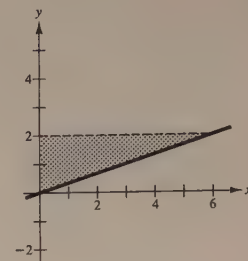


57. $f(y) = 3y, 0 \leq y \leq 2$ (Note: $\Delta y = \frac{2-0}{n} = \frac{2}{n}$)

$$S(n) = \sum_{i=1}^n f(m_i) \Delta y = \sum_{i=1}^n f\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n 3\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right)$$

$$= \frac{12}{n^2} \sum_{i=1}^n i = \left(\frac{12}{n^2}\right) \cdot \frac{n(n+1)}{2} = \frac{6(n+1)}{n} = 6 + \frac{6}{n}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \left(6 + \frac{6}{n}\right) = 6$$

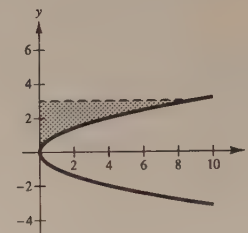


59. $f(y) = y^2, 0 \leq y \leq 3$ (Note: $\Delta y = \frac{3-0}{n} = \frac{3}{n}$)

$$S(n) = \sum_{i=1}^n f\left(\frac{3i}{n}\right) \left(\frac{3}{n}\right) = \sum_{i=1}^n \left(\frac{3i}{n}\right)^2 \left(\frac{3}{n}\right) = \frac{27}{n^3} \sum_{i=1}^n i^2$$

$$= \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{9}{n^2} \left(\frac{2n^2 + 3n + 1}{2}\right) = 9 + \frac{27}{2n} + \frac{9}{2n^2}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \left(9 + \frac{27}{2n} + \frac{9}{2n^2}\right) = 9$$



61. $g(y) = 4y^2 - y^3, 1 \leq y \leq 3$ (Note: $\Delta y = \frac{3-1}{n} = \frac{2}{n}$)

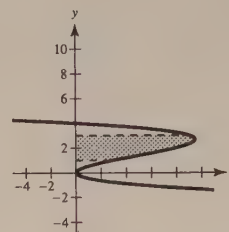
$$S(n) = \sum_{i=1}^n g\left(1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^n \left[4\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right)^3 \right] \frac{2}{n}$$

$$= \frac{2}{n} \sum_{i=1}^n \left[4\left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right) \right]$$

$$= \frac{2}{n} \sum_{i=1}^n \left[3 + \frac{10i}{n} + \frac{4i^2}{n^2} - \frac{8i^3}{n^3} \right] = \frac{2}{n} \left[3n + \frac{10}{n} \frac{n(n+1)}{2} + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2} \frac{n^2(n+1)^2}{4} \right]$$

$$\text{Area} = \lim_{n \rightarrow \infty} S(n) = 6 + 10 + \frac{8}{3} - 4 = \frac{44}{3}$$



63. $f(x) = x^2 + 3, 0 \leq x \leq 2, n = 4$

$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2}.$$

$$\Delta x = \frac{1}{2}, c_1 = \frac{1}{4}, c_2 = \frac{3}{4}, c_3 = \frac{5}{4}, c_4 = \frac{7}{4}$$

$$\text{Area} \approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 [c_i^2 + 3] \left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \left[\left(\frac{1}{16} + 3\right) + \left(\frac{9}{16} + 3\right) + \left(\frac{25}{16} + 3\right) + \left(\frac{49}{16} + 3\right) \right]$$

$$= \frac{69}{8}$$

65. $f(x) = \tan x, 0 \leq x \leq \frac{\pi}{4}, n = 4$

$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2}.$$

$$\Delta x = \frac{\pi}{16}, c_1 = \frac{\pi}{32}, c_2 = \frac{3\pi}{32}, c_3 = \frac{5\pi}{32}, c_4 = \frac{7\pi}{32}$$

$$\text{Area} \approx \sum_{i=1}^n f(c_i) \Delta x = \sum_{i=1}^4 (\tan c_i) \left(\frac{\pi}{16}\right)$$

$$= \frac{\pi}{16} \left(\tan \frac{\pi}{32} + \tan \frac{3\pi}{32} + \tan \frac{5\pi}{32} + \tan \frac{7\pi}{32} \right) \approx 0.345$$

67. $f(x) = \sqrt{x}$ on $[0, 4]$.

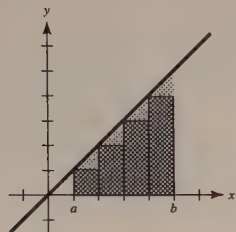
n	4	8	12	16	20
Approximate area	5.3838	5.3523	5.3439	5.3403	5.3384

(Exact value is $16/3$)

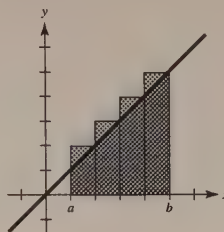
69. $f(x) = \tan\left(\frac{\pi x}{8}\right)$ on $[1, 3]$.

n	4	8	12	16	20
Approximate area	2.2223	2.2387	2.2418	2.2430	2.2435

71. We can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of these inscribed rectangles is the lower sum.



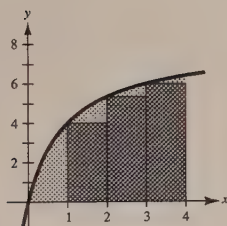
The sum of the areas of these circumscribed rectangles is the upper sum.



We can see that the rectangles do not contain all of the area in the first graph and the rectangles in the second graph cover more than the area of the region.

The exact value of the area lies between these two sums.

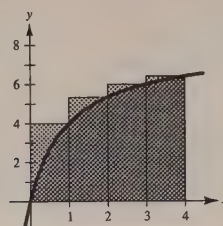
73. (a)



Lower sum:

$$s(4) = 0 + 4 + 5\frac{1}{3} + 6 = 15\frac{1}{3} = \frac{46}{3} \approx 15.333$$

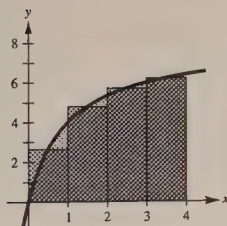
(b)



Upper sum:

$$S(4) = 4 + 5\frac{1}{3} + 6 + 6\frac{2}{3} = 21\frac{11}{15} = \frac{326}{15} \approx 21.733$$

(c)



Midpoint Rule:

$$M(4) = 2\frac{2}{3} + 4\frac{4}{5} + 5\frac{5}{7} + 6\frac{2}{9} = \frac{6112}{315} \approx 19.403$$

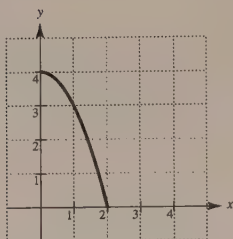
- (d) In each case, $\Delta x = 4/n$. The lower sum uses left endpoints, $(i-1)(4/n)$. The upper sum uses right endpoints, $i(4/n)$. The Midpoint Rule uses midpoints, $(i-\frac{1}{2})(4/n)$.

(e)

n	4	8	20	100	200
$s(n)$	15.333	17.368	18.459	18.995	19.06
$S(n)$	21.733	20.568	19.739	19.251	19.188
$M(n)$	19.403	19.201	19.137	19.125	19.125

- (f) $s(n)$ increases because the lower sum approaches the exact value as n increases. $S(n)$ decreases because the upper sum approaches the exact value as n increases. Because of the shape of the graph, the lower sum is always smaller than the exact value, whereas the upper sum is always larger.

75.

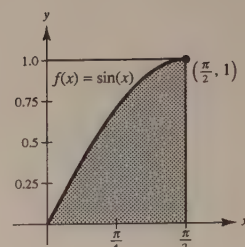
b. $A \approx 6$ square units

79. $f(x) = \sin x, \left[0, \frac{\pi}{2}\right]$

Let A_1 = area bounded by $f(x) = \sin x$, the x -axis, $x = 0$ and $x = \pi/2$. Let A_2 = area of the rectangle bounded by $y = 1$, $y = 0$, $x = 0$, and $x = \pi/2$. Thus, $A_2 = (\pi/2)(1) \approx 1.570796$. In this program, the computer is generating N_2 pairs of random points in the rectangle whose area is represented by A_2 . It is keeping track of how many of these points, N_1 , lie in the region whose area is represented by A_1 . Since the points are randomly generated, we assume that

$$\frac{A_1}{A_2} \approx \frac{N_1}{N_2} \Rightarrow A_1 \approx \frac{N_1}{N_2} A_2.$$

The larger N_2 is the better the approximation to A_1 .



81. Suppose there are n rows and $n + 1$ columns in the figure. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total, hence

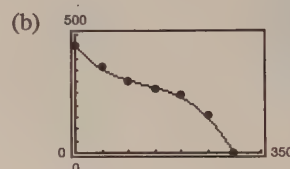
$$2[1 + 2 + \cdots + n] = n(n + 1)$$

$$1 + 2 + \cdots + n = \frac{1}{2}n(n + 1).$$

83. (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(c) Using the integration capability of a graphing utility, you obtain

$$A \approx 76,897.5 \text{ ft}^2.$$



Section 4.3 Riemann Sums and Definite Integrals

1. $f(x) = \sqrt{x}, y = 0, x = 0, x = 3, c_i = \frac{3i^2}{n^2}$

$$\Delta x_i = \frac{3i^2}{n^2} - \frac{3(i-1)^2}{n^2} = \frac{3}{n^2}(2i-1)$$

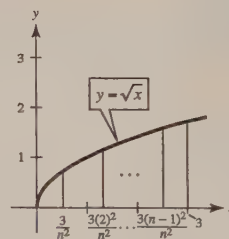
$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{\frac{3i^2}{n^2}} \frac{3}{n^2} (2i-1)$$

$$= \lim_{n \rightarrow \infty} \frac{3\sqrt{3}}{n^3} \sum_{i=1}^n (2i^2 - i)$$

$$= \lim_{n \rightarrow \infty} \frac{3\sqrt{3}}{n^3} \left[2 \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right]$$

$$= \lim_{n \rightarrow \infty} 3\sqrt{3} \left[\frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2} \right]$$

$$= 3\sqrt{3} \left[\frac{2}{3} - 0 \right] = 2\sqrt{3} \approx 3.464$$



$$3. y = 6 \text{ on } [4, 10]. \quad \left(\text{Note: } \Delta x = \frac{10-4}{n} = \frac{6}{n}, \|\Delta\| \rightarrow 0 \text{ as } n \rightarrow \infty \right)$$

$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n f\left(4 + \frac{6i}{n}\right) \left(\frac{6}{n}\right) = \sum_{i=1}^n 6 \left(\frac{6}{n}\right) = \sum_{i=1}^n \frac{36}{n} = 36$$

$$\int_4^{10} 6 \, dx = \lim_{n \rightarrow \infty} 36 = 36$$

$$5. y = x^3 \text{ on } [-1, 1]. \quad \left(\text{Note: } \Delta x = \frac{1 - (-1)}{n} = \frac{2}{n}, \|\Delta\| \rightarrow 0 \text{ as } n \rightarrow \infty \right)$$

$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n f\left(-1 + \frac{2i}{n}\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left(-1 + \frac{2i}{n}\right)^3 \left(\frac{2}{n}\right) = \sum_{i=1}^n \left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right] \left(\frac{2}{n}\right)$$

$$= -2 + \frac{12}{n^2} \sum_{i=1}^n i - \frac{24}{n^3} \sum_{i=1}^n i^2 + \frac{16}{n^4} \sum_{i=1}^n i^3$$

$$= -2 + 6\left(1 + \frac{1}{n}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n}$$

$$\int_{-1}^1 x^3 \, dx = \lim_{n \rightarrow \infty} \frac{2}{n} = 0$$

$$7. y = x^2 + 1 \text{ on } [1, 2]. \quad \left(\text{Note: } \Delta x = \frac{2-1}{n} = \frac{1}{n}, \|\Delta\| \rightarrow 0 \text{ as } n \rightarrow \infty \right)$$

$$\sum_{i=1}^n f(c_i) \Delta x_i = \sum_{i=1}^n f\left(1 + \frac{i}{n}\right) \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[\left(1 + \frac{i}{n}\right)^2 + 1\right] \left(\frac{1}{n}\right) = \sum_{i=1}^n \left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1\right] \left(\frac{1}{n}\right)$$

$$= 2 + \frac{2}{n^2} \sum_{i=1}^n i + \frac{1}{n^3} \sum_{i=1}^n i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}$$

$$\int_1^2 (x^2 + 1) \, dx = \lim_{n \rightarrow \infty} \left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{10}{3}$$

$$9. \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (3c_i + 10) \Delta x_i = \int_{-1}^5 (3x + 10) \, dx$$

on the interval $[-1, 5]$.

$$11. \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \sqrt{c_i^2 + 4} \Delta x_i = \int_0^3 \sqrt{x^2 + 4} \, dx$$

on the interval $[0, 3]$.

$$13. \int_0^5 3 \, dx$$

$$15. \int_{-4}^4 (4 - |x|) \, dx$$

$$17. \int_{-2}^2 (4 - x^2) \, dx$$

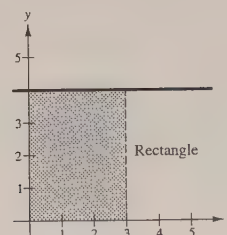
$$19. \int_0^{\pi} \sin x \, dx$$

$$21. \int_0^2 y^3 \, dy$$

23. Rectangle

$$A = bh = 3(4)$$

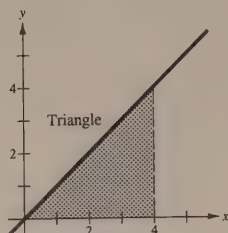
$$A = \int_0^3 4 \, dx = 12$$



25. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(4)(4)$$

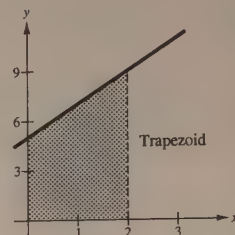
$$A = \int_0^4 x \, dx = 8$$



27. Trapezoid

$$A = \frac{b_1 + b_2}{2}h = \left(\frac{5+9}{2}\right)2$$

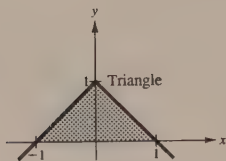
$$A = \int_0^2 (2x + 5) \, dx = 14$$



29. Triangle

$$A = \frac{1}{2}bh = \frac{1}{2}(2)(1)$$

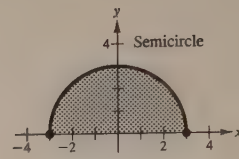
$$A = \int_{-1}^1 (1 - |x|) \, dx = 1$$



31. Semicircle

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$$

$$A = \int_{-3}^3 \sqrt{9 - x^2} \, dx = \frac{9\pi}{2}$$



In Exercises 33–39, $\int_2^4 x^3 \, dx = 60$, $\int_2^4 x \, dx = 6$, $\int_2^4 dx = 2$

$$33. \int_4^2 x \, dx = -\int_2^4 x \, dx = -6$$

$$37. \int_2^4 (x - 8) \, dx = \int_2^4 x \, dx - 8 \int_2^4 dx = 6 - 8(2) = -10$$

$$41. (a) \int_0^7 f(x) \, dx = \int_0^5 f(x) \, dx + \int_5^7 f(x) \, dx = 10 + 3 = 13$$

$$(b) \int_5^0 f(x) \, dx = -\int_0^5 f(x) \, dx = -10$$

$$(c) \int_5^5 f(x) \, dx = 0$$

$$(d) \int_0^5 3f(x) \, dx = 3 \int_0^5 f(x) \, dx = 3(10) = 30$$

$$35. \int_2^4 4x \, dx = 4 \int_2^4 x \, dx = 4(6) = 24$$

$$39. \int_2^4 \left(\frac{1}{2}x^3 - 3x + 2 \right) dx = \frac{1}{2} \int_2^4 x^3 \, dx - 3 \int_2^4 x \, dx + 2 \int_2^4 dx$$

$$= \frac{1}{2}(60) - 3(6) + 2(2) = 16$$

$$43. (a) \int_2^6 [f(x) + g(x)] \, dx = \int_2^6 f(x) \, dx + \int_2^6 g(x) \, dx$$

$$= 10 + (-2) = 8$$

$$(b) \int_2^6 [g(x) - f(x)] \, dx = \int_2^6 g(x) \, dx - \int_2^6 f(x) \, dx$$

$$= -2 - 10 = -12$$

$$(c) \int_2^6 2g(x) \, dx = 2 \int_2^6 g(x) \, dx = 2(-2) = -4$$

$$(d) \int_2^6 3f(x) \, dx = 3 \int_2^6 f(x) \, dx = 3(10) = 30$$

$$45. (a) \text{ Quarter circle below } x\text{-axis: } -\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$$

$$(b) \text{ Triangle: } \frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$$

$$(c) \text{ Triangle + Semicircle below } x\text{-axis: } -\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$$

$$(d) \text{ Sum of parts (b) and (c): } 4 - (1 + 2\pi) = 3 - 2\pi$$

$$(e) \text{ Sum of absolute values of (b) and (c): } 4 + (1 + 2\pi) = 5 + 2\pi$$

$$(f) \text{ Answer to (d) plus } 2(10) = 20: (3 - 2\pi) + 20 = 23 - 2\pi$$

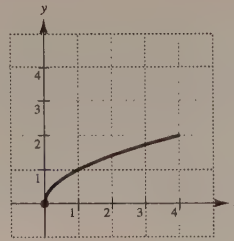
47. The left endpoint approximation will be greater than the actual area: $>$

49. Because the curve is concave upward, the midpoint approximation will be less than the actual area: $<$

51. $f(x) = \frac{1}{x-4}$

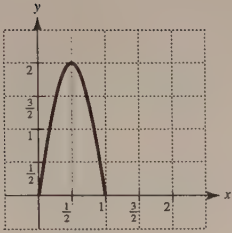
is not integrable on the interval $[3, 5]$ because f has a discontinuity at $x = 4$.

53.



a. $A \approx 5$ square units

55.



d. $\int_0^1 2 \sin \pi x \, dx \approx \frac{1}{2}(1)(2) \approx 1$

57. $\int_0^3 x\sqrt{3-x} \, dx$

n	4	8	12	16	20
$L(n)$	3.6830	3.9956	4.0707	4.1016	4.1177
$M(n)$	4.3082	4.2076	4.1838	4.1740	4.1690
$R(n)$	3.6830	3.9956	4.0707	4.1016	4.1177

59. $\int_0^{\pi/2} \sin^2 x \, dx$

n	4	8	12	16	20
$L(n)$	0.5890	0.6872	0.7199	0.7363	0.7461
$M(n)$	0.7854	0.7854	0.7854	0.7854	0.7854
$R(n)$	0.9817	0.8836	0.8508	0.8345	0.8247

61. True

63. True

65. False

$$\int_0^2 (-x) \, dx = -2$$

67. $f(x) = x^2 + 3x, [0, 8]$

$x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 7, x_4 = 8$

$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 4, \Delta x_4 = 1$

$c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 8$

$$\begin{aligned} \sum_{i=1}^4 f(c_i) \Delta x &= f(1) \Delta x_1 + f(2) \Delta x_2 + f(5) \Delta x_3 + f(8) \Delta x_4 \\ &= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272 \end{aligned}$$

$$69. f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

is not integrable on the interval $[0, 1]$. As $\|\Delta\| \rightarrow 0$, $f(c_i) = 1$ or $f(c_i) = 0$ in each subinterval since there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

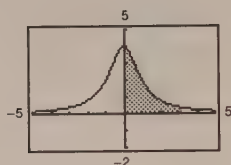
71. Let $f(x) = x^2$, $0 \leq x \leq 1$, and $\Delta x_i = 1/n$. The appropriate Riemann Sum is

$$\begin{aligned} \sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^n i^2 \\ \lim_{n \rightarrow \infty} \frac{1}{n^3} [1^2 + 2^2 + 3^2 + \cdots + n^2] &= \lim_{n \rightarrow \infty} \frac{1}{n^3} \cdot \frac{n(2n+1)(n+1)}{6} \\ &= \lim_{n \rightarrow \infty} \frac{2n^2 + 3n + 1}{6n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3} \end{aligned}$$

Section 4.4 The Fundamental Theorem of Calculus

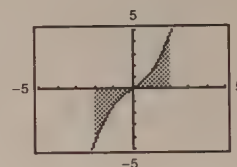
$$1. f(x) = \frac{4}{x^2 + 1}$$

$$\int_0^\pi \frac{4}{x^2 + 1} dx \text{ is positive.}$$



$$3. f(x) = x\sqrt{x^2 + 1}$$

$$\int_{-2}^2 x\sqrt{x^2 + 1} dx = 0$$



$$5. \int_0^1 2x dx = \left[x^2\right]_0^1 = 1 - 0 = 1$$

$$7. \int_{-1}^0 (x - 2) dx = \left[\frac{x^2}{2} - 2x\right]_{-1}^0 = 0 - \left(\frac{1}{2} - 2\right) = -\frac{5}{2}$$

$$9. \int_{-1}^1 (t^2 - 2) dt = \left[\frac{t^3}{3} - 2t\right]_{-1}^1 = \left(\frac{1}{3} - 2\right) - \left(-\frac{1}{3} + 2\right) = -\frac{10}{3}$$

$$11. \int_0^1 (2t - 1)^2 dt = \int_0^1 (4t^2 - 4t + 1) dt = \left[\frac{4}{3}t^3 - 2t^2 + t\right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{1}{3}$$

$$13. \int_1^2 \left(\frac{3}{x^2} - 1\right) dx = \left[-\frac{3}{x} - x\right]_1^2 = \left(-\frac{3}{2} - 2\right) - (-3 - 1) = \frac{1}{2}$$

$$15. \int_1^4 \frac{u-2}{\sqrt{u}} du = \int_1^4 (u^{1/2} - 2u^{-1/2}) du = \left[\frac{2}{3}u^{3/2} - 4u^{1/2}\right]_1^4 = \left[\frac{2}{3}(\sqrt{4})^3 - 4\sqrt{4}\right] - \left[\frac{2}{3} - 4\right] = \frac{2}{3}$$

$$17. \int_{-1}^1 (\sqrt[3]{t} - 2) dt = \left[\frac{3}{4}t^{4/3} - 2t\right]_{-1}^1 = \left(\frac{3}{4} - 2\right) - \left(\frac{3}{4} + 2\right) = -4$$

$$19. \int_0^1 \frac{x - \sqrt{x}}{3} dx = \frac{1}{3} \int_0^1 (x - x^{1/2}) dx = \frac{1}{3} \left[\frac{x^2}{2} - \frac{2}{3}x^{3/2}\right]_0^1 = \frac{1}{3} \left(\frac{1}{2} - \frac{2}{3}\right) = -\frac{1}{18}$$

$$21. \int_{-1}^0 (t^{1/3} - t^{2/3}) dt = \left[\frac{3}{4}t^{4/3} - \frac{3}{5}t^{5/3}\right]_{-1}^0 = 0 - \left(\frac{3}{4} + \frac{3}{5}\right) = -\frac{27}{20}$$

$$\begin{aligned} 23. \int_0^3 |2x - 3| dx &= \int_0^{3/2} (3 - 2x) dx + \int_{3/2}^3 (2x - 3) dx \quad \left(\text{split up the integral at the zero } x = \frac{3}{2}\right) \\ &= \left[3x - x^2\right]_0^{3/2} + \left[x^2 - 3x\right]_{3/2}^3 = \left(\frac{9}{2} - \frac{9}{4}\right) - 0 + (9 - 9) - \left(\frac{9}{4} - \frac{9}{2}\right) = 2\left(\frac{9}{2} - \frac{9}{4}\right) = \frac{9}{2} \end{aligned}$$

$$\begin{aligned}
 25. \int_0^3 |x^2 - 4| dx &= \int_0^2 (4 - x^2) dx + \int_2^3 (x^2 - 4) dx \\
 &= \left[4x - \frac{x^3}{3} \right]_0^2 + \left[\frac{x^3}{3} - 4x \right]_2^3 \\
 &= \left(8 - \frac{8}{3} \right) + (9 - 12) - \left(\frac{8}{3} - 8 \right) \\
 &= \frac{23}{3}
 \end{aligned}$$

$$27. \int_0^\pi (1 + \sin x) dx = \left[x - \cos x \right]_0^\pi = (\pi + 1) - (0 - 1) = 2 + \pi$$

$$29. \int_{-\pi/6}^{\pi/6} \sec^2 x dx = \left[\tan x \right]_{-\pi/6}^{\pi/6} = \frac{\sqrt{3}}{3} - \left(-\frac{\sqrt{3}}{3} \right) = \frac{2\sqrt{3}}{3}$$

$$31. \int_{-\pi/3}^{\pi/3} 4 \sec \theta \tan \theta d\theta = \left[4 \sec \theta \right]_{-\pi/3}^{\pi/3} = 4(2) - 4(2) = 0$$

$$33. \int_0^3 10,000(t - 6) dt = 10,000 \left[\frac{t^2}{2} - 6t \right]_0^3 = -\$135,000$$

$$35. A = \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$

$$37. A = \int_0^3 (3 - x)\sqrt{x} dx = \int_0^3 (3x^{1/2} - x^{3/2}) dx = \left[2x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^3 = \left[\frac{x\sqrt{x}}{5}(10 - 2x) \right]_0^3 = \frac{12\sqrt{3}}{5}$$

$$39. A = \int_0^{\pi/2} \cos x dx = \left[\sin x \right]_0^{\pi/2} = 1$$

41. Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (3x^2 + 1) dx = \left[x^3 + x \right]_0^2 = 8 + 2 = 10.$$

43. Since $y \geq 0$ on $[0, 2]$,

$$A = \int_0^2 (x^3 + x) dx = \left[\frac{x^4}{4} + \frac{x^2}{2} \right]_0^2 = 4 + 2 = 6.$$

$$45. \int_0^2 (x - 2\sqrt{x}) dx = \left[\frac{x^2}{2} - \frac{4x^{3/2}}{3} \right]_0^2 = 2 - \frac{8\sqrt{2}}{3}$$

$$f(c)(2 - 0) = \frac{6 - 8\sqrt{2}}{3}$$

$$c - 2\sqrt{c} = \frac{3 - 4\sqrt{2}}{3}$$

$$c - 2\sqrt{c} + 1 = \frac{3 - 4\sqrt{2}}{3} + 1$$

$$(\sqrt{c} - 1)^2 = \frac{6 - 4\sqrt{2}}{3}$$

$$\sqrt{c} - 1 = \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}}$$

$$c = \left[1 \pm \sqrt{\frac{6 - 4\sqrt{2}}{3}} \right]^2$$

$$c \approx 0.4380 \text{ or } c \approx 1.7908$$

$$47. \int_{-\pi/4}^{\pi/4} 2 \sec^2 x \, dx = \left[2 \tan x \right]_{-\pi/4}^{\pi/4} = 2(1) - 2(-1) = 4$$

$$f(c) \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right] = 4$$

$$2 \sec^2 c = \frac{8}{\pi}$$

$$\sec^2 c = \frac{4}{\pi}$$

$$\sec c = \pm \frac{2}{\sqrt{\pi}}$$

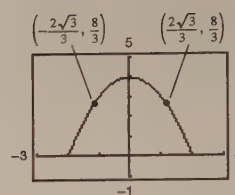
$$c = \pm \operatorname{arcsec} \left(\frac{2}{\sqrt{\pi}} \right)$$

$$= \pm \arccos \frac{\sqrt{\pi}}{2} \approx \pm 0.4817$$

$$49. \frac{1}{2 - (-2)} \int_{-2}^2 (4 - x^2) \, dx = \frac{1}{4} \left[4x - \frac{1}{3}x^3 \right]_{-2}^2 = \frac{1}{4} \left[\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right] = \frac{8}{3}$$

$$\text{Average value} = \frac{8}{3}$$

$$4 - x^2 = \frac{8}{3} \text{ when } x^2 = 4 - \frac{8}{3} \text{ or } x = \pm \frac{2\sqrt{3}}{3} \approx \pm 1.155.$$

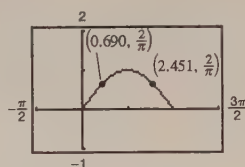


$$51. \frac{1}{\pi - 0} \int_0^{\pi} \sin x \, dx = \left[-\frac{1}{\pi} \cos x \right]_0^{\pi} = \frac{2}{\pi}$$

$$\text{Average value} = \frac{2}{\pi}$$

$$\sin x = \frac{2}{\pi}$$

$$x \approx 0.690, 2.451$$



53. If f is continuous on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$,

$$\text{then } \int_a^b f(x) \, dx = F(b) - F(a).$$

$$55. \int_0^2 f(x) \, dx = -(\text{area of region A}) = -1.5$$

$$57. \int_0^6 |f(x)| \, dx = -\int_0^2 f(x) \, dx + \int_2^6 f(x) \, dx = 1.5 + 5.0 = 6.5$$

$$59. \int_0^6 [2 + f(x)] \, dx = \int_0^6 2 \, dx + \int_0^6 f(x) \, dx \\ = 12 + 3.5 = 15.5$$

$$61. (a) F(x) = k \sec^2 x$$

$$F(0) = k = 500$$

$$F(x) = 500 \sec^2 x$$

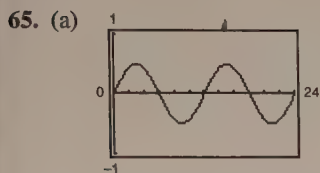
$$(b) \frac{1}{\pi/3 - 0} \int_0^{\pi/3} 500 \sec^2 x \, dx = \frac{1500}{\pi} \left[\tan x \right]_0^{\pi/3}$$

$$= \frac{1500}{\pi} (\sqrt{3} - 0)$$

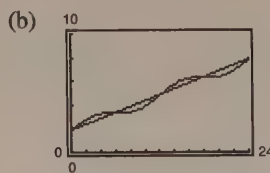
$$\approx 826.99 \text{ newtons}$$

$$\approx 827 \text{ newtons}$$

$$63. \frac{1}{5 - 0} \int_0^5 (0.1729t + 0.1522t^2 - 0.0374t^3) \, dt \approx \frac{1}{5} \left[0.08645t^2 + 0.05073t^3 - 0.00935t^4 \right]_0^5 \approx 0.5318 \text{ liter}$$

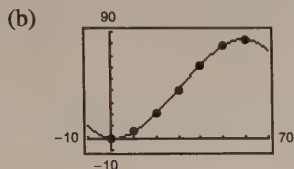


The area above the x -axis equals the area below the x -axis. Thus, the average value is zero.



The average value of S appears to be g .

67. (a) $v = -8.61 \times 10^{-4}t^3 + 0.0782t^2 - 0.208t + 0.0952$



(c) $\int_0^{60} v(t) dt = \left[\frac{-8.61 \times 10^{-4}t^4}{4} + \frac{0.0782t^3}{3} - \frac{0.208t^2}{2} + 0.0952t \right]_0^{60} \approx 2476 \text{ meters}$

69. $F(x) = \int_0^x (t - 5) dt = \left[\frac{t^2}{2} - 5t \right]_0^x = \frac{x^2}{2} - 5x$

$$F(2) = \frac{4}{2} - 5(2) = -8$$

$$F(5) = \frac{25}{2} - 5(5) = -\frac{25}{2}$$

$$F(8) = \frac{64}{2} - 5(8) = -8$$

71. $F(x) = \int_1^x \frac{10}{v^2} dv = \int_1^x 10v^{-2} dv = \left[\frac{-10}{v} \right]_1^x$

$$= -\frac{10}{x} + 10 = 10 \left(1 - \frac{1}{x} \right)$$

$$F(2) = 10 \left(\frac{1}{2} \right) = 5$$

$$F(5) = 10 \left(\frac{4}{5} \right) = 8$$

$$F(8) = 10 \left(\frac{7}{8} \right) = \frac{35}{4}$$

73. $F(x) = \int_1^x \cos \theta d\theta = \left[\sin \theta \right]_1^x = \sin x - \sin 1$

$$F(2) = \sin 2 - \sin 1 \approx 0.0678$$

$$F(5) = \sin 5 - \sin 1 \approx -1.8004$$

$$F(8) = \sin 8 - \sin 1 \approx 0.1479$$

75. (a) $\int_0^x (t + 2) dt = \left[\frac{t^2}{2} + 2t \right]_0^x = \frac{1}{2}x^2 + 2x$

(b) $\frac{d}{dx} \left[\frac{1}{2}x^2 + 2x \right] = x + 2$

77. (a) $\int_8^x \sqrt[3]{t} dt = \left[\frac{3}{4}t^{4/3} \right]_8^x = \frac{3}{4}(x^{4/3} - 16) = \frac{3}{4}x^{4/3} - 12$

(b) $\frac{d}{dx} \left[\frac{3}{4}x^{4/3} - 12 \right] = x^{1/3} = \sqrt[3]{x}$

79. (a) $\int_{x/4}^x \sec^2 t dt = \left[\tan t \right]_{x/4}^x = \tan x - 1$

(b) $\frac{d}{dx} [\tan x - 1] = \sec^2 x$

81. $F(x) = \int_{-2}^x (t^2 - 2t) dt$

$$F'(x) = x^2 - 2x$$

83. $F(x) = \int_{-1}^x \sqrt{t^4 + 1} dt$

$$F'(x) = \sqrt{x^4 + 1}$$

85. $F(x) = \int_0^x t \cos t dt$

$$F'(x) = x \cos x$$

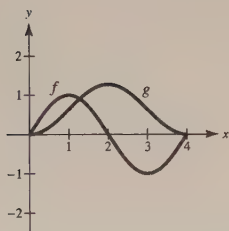
$$\begin{aligned}
 87. \quad F(x) &= \int_x^{x+2} (4t+1) dt \\
 &= \left[2t^2 + t \right]_x^{x+2} \\
 &= [2(x+2)^2 + (x+2)] - [2x^2 + x] \\
 &= 8x + 10 \\
 F'(x) &= 8
 \end{aligned}$$

$$\begin{aligned}
 89. \quad F(x) &= \int_0^{\sin x} \sqrt{t} dt = \left[\frac{2}{3} t^{3/2} \right]_0^{\sin x} = \frac{2}{3} (\sin x)^{3/2} \\
 F'(x) &= (\sin x)^{1/2} \cos x = \cos x \sqrt{\sin x}
 \end{aligned}$$

Alternate solution

$$\begin{aligned}
 F(x) &= \int_0^{\sin x} \sqrt{t} dt \\
 F'(x) &= \sqrt{\sin x} \frac{d}{dx}(\sin x) = \sqrt{\sin x}(\cos x)
 \end{aligned}$$

$$\begin{aligned}
 93. \quad g(x) &= \int_0^x f(t) dt \\
 g(0) &= 0, g(1) \approx \frac{1}{2}, g(2) \approx 1, g(3) \approx \frac{1}{2}, g(4) = 0
 \end{aligned}$$

g has a relative maximum at $x = 2$.

97. True

$$101. \quad f(x) = \int_0^{1/x} \frac{1}{t^2+1} dt + \int_0^x \frac{1}{t^2+1} dt$$

By the Second Fundamental Theorem of Calculus, we have

$$\begin{aligned}
 f'(x) &= \frac{1}{(1/x)^2+1} \left(-\frac{1}{x^2} \right) + \frac{1}{x^2+1} \\
 &= -\frac{1}{1+x^2} + \frac{1}{x^2+1} = 0.
 \end{aligned}$$

Since $f'(x) = 0$, $f(x)$ must be constant.

Alternate solution:

$$\begin{aligned}
 F(x) &= \int_x^{x+2} (4t+1) dt \\
 &= \int_x^0 (4t+1) dt + \int_0^{x+2} (4t+1) dt \\
 &= -\int_0^x (4t+1) dt + \int_0^{x+2} (4t+1) dt \\
 F'(x) &= -(4x+1) + 4(x+2) + 1 = 8
 \end{aligned}$$

$$\begin{aligned}
 91. \quad F(x) &= \int_0^{x^3} \sin t^2 dt \\
 F'(x) &= \sin(x^3)^2 \cdot 3x^2 = 3x^2 \sin x^6
 \end{aligned}$$

$$\begin{aligned}
 95. \quad (a) \quad C(x) &= 5000 \left(25 + 3 \int_0^x t^{1/4} dt \right) \\
 &= 5000 \left(25 + 3 \left[\frac{4}{5} t^{5/4} \right]_0^x \right) \\
 &= 5000 \left(25 + \frac{12}{5} x^{5/4} \right) = 1000(125 + 12x^{5/4})
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad C(1) &= 1000(125 + 12(1)) = \$137,000 \\
 C(5) &= 1000(125 + 12(5)^{5/4}) \approx \$214,721 \\
 C(10) &= 1000(125 + 12(10)^{5/4}) \approx \$338,394
 \end{aligned}$$

$$99. \quad \text{False; } \int_{-1}^1 x^{-2} dx = \int_{-1}^0 x^{-2} dx + \int_0^1 x^{-2} dx$$

Each of these integrals is infinite. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

103. $x(t) = t^3 - 6t^2 + 9t - 2$

$$x'(t) = 3t^2 - 12t + 9$$

$$= 3(t^2 - 4t + 3)$$

$$= 3(t - 3)(t - 1)$$

$$\text{Total distance} = \int_0^5 |x'(t)| dt$$

$$= \int_0^5 3|(t - 3)(t - 1)| dt$$

$$= 3 \int_0^1 (t^2 - 4t + 3) dt - 3 \int_1^3 (t^2 - 4t + 3) dt + 3 \int_3^5 (t^2 - 4t + 3) dt$$

$$= 4 + 4 + 20$$

$$= 28 \text{ units}$$

105. Total distance $= \int_1^4 |x'(t)| dt$

$$= \int_1^4 |v(t)| dt$$

$$= \int_1^4 \frac{1}{\sqrt{t}} dt$$

$$= 2t^{1/2} \Big|_1^4$$

$$= 2(2 - 1) = 2 \text{ units}$$

Section 4.5 Integration by Substitution

$\int f(g(x))g'(x) dx$	$u = g(x)$	$du = g'(x) dx$
------------------------	------------	-----------------

1. $\int (5x^2 + 1)^2(10x) dx$	$5x^2 + 1$	$10x dx$
--------------------------------	------------	----------

3. $\int \frac{x}{\sqrt{x^2 + 1}} dx$	$x^2 + 1$	$2x dx$
---------------------------------------	-----------	---------

5. $\int \tan^2 x \sec^2 x dx$	$\tan x$	$\sec^2 x dx$
--------------------------------	----------	---------------

7. $\int (1 + 2x)^4 2 dx = \frac{(1 + 2x)^5}{5} + C$

Check: $\frac{d}{dx} \left[\frac{(1 + 2x)^5}{5} + C \right] = 2(1 + 2x)^4$

9. $\int (9 - x^2)^{1/2}(-2x) dx = \frac{(9 - x^2)^{3/2}}{3/2} + C = \frac{2}{3}(9 - x^2)^{3/2} + C$

Check: $\frac{d}{dx} \left[\frac{2}{3}(9 - x^2)^{3/2} + C \right] = \frac{2}{3} \cdot \frac{3}{2}(9 - x^2)^{1/2}(-2x) = \sqrt{9 - x^2}(-2x)$

$$11. \int x^3(x^4 + 3)^2 dx = \frac{1}{4} \int (x^4 + 3)^2 (4x^3) dx = \frac{1}{4} \frac{(x^4 + 3)^3}{3} + C = \frac{(x^4 + 3)^3}{12} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^4 + 3)^3}{12} + C \right] = \frac{3(x^4 + 3)^2}{12} (4x^3) = (x^4 + 3)^2 (x^3)$$

$$13. \int x^2(x^3 - 1)^4 dx = \frac{1}{3} \int (x^3 - 1)^4 (3x^2) dx = \frac{1}{3} \left[\frac{(x^3 - 1)^5}{5} \right] + C = \frac{(x^3 - 1)^5}{15} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{(x^3 - 1)^5}{15} + C \right] = \frac{5(x^3 - 1)^4 (3x^2)}{15} = x^2(x^3 - 1)^4$$

$$15. \int t\sqrt{t^2 + 2} dt = \frac{1}{2} \int (t^2 + 2)^{1/2} (2t) dt = \frac{1}{2} \frac{(t^2 + 2)^{3/2}}{3/2} + C = \frac{(t^2 + 2)^{3/2}}{3} + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{(t^2 + 2)^{3/2}}{3} + C \right] = \frac{3/2 (t^2 + 2)^{1/2} (2t)}{3} = (t^2 + 2)^{1/2} t$$

$$17. \int 5x(1 - x^2)^{1/3} dx = -\frac{5}{2} \int (1 - x^2)^{1/3} (-2x) dx = -\frac{5}{2} \cdot \frac{(1 - x^2)^{4/3}}{4/3} + C = -\frac{15}{8} (1 - x^2)^{4/3} + C$$

$$\text{Check: } \frac{d}{dx} \left[-\frac{15}{8} (1 - x^2)^{4/3} + C \right] = -\frac{15}{8} \cdot \frac{4}{3} (1 - x^2)^{1/3} (-2x) = 5x(1 - x^2)^{1/3} = 5x\sqrt[3]{1 - x^2}$$

$$19. \int \frac{x}{(1 - x^2)^3} dx = -\frac{1}{2} \int (1 - x^2)^{-3} (-2x) dx = -\frac{1}{2} \frac{(1 - x^2)^{-2}}{-2} + C = \frac{1}{4(1 - x^2)^2} + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{1}{4(1 - x^2)^2} + C \right] = \frac{1}{4} (-2) (1 - x^2)^{-3} (-2x) = \frac{x}{(1 - x^2)^3}$$

$$21. \int \frac{x^2}{(1 + x^3)^2} dx = \frac{1}{3} \int (1 + x^3)^{-2} (3x^2) dx = \frac{1}{3} \left[\frac{(1 + x^3)^{-1}}{-1} \right] + C = -\frac{1}{3(1 + x^3)} + C$$

$$\text{Check: } \frac{d}{dx} \left[-\frac{1}{3(1 + x^3)} + C \right] = -\frac{1}{3} (-1) (1 + x^3)^{-2} (3x^2) = \frac{x^2}{(1 + x^3)^2}$$

$$23. \int \frac{x}{\sqrt{1 - x^2}} dx = -\frac{1}{2} \int (1 - x^2)^{-1/2} (-2x) dx = -\frac{1}{2} \frac{(1 - x^2)^{1/2}}{1/2} + C = -\sqrt{1 - x^2} + C$$

$$\text{Check: } \frac{d}{dx} [-\sqrt{1 - x^2} + C] = -\frac{1}{2} (1 - x^2)^{-1/2} (-2x) = \frac{x}{\sqrt{1 - x^2}}$$

$$25. \int \left(1 + \frac{1}{t}\right)^3 \left(\frac{1}{t^2}\right) dt = -\int \left(1 + \frac{1}{t}\right)^3 \left(-\frac{1}{t^2}\right) dt = -\frac{[1 + (1/t)]^4}{4} + C$$

$$\text{Check: } \frac{d}{dt} \left[-\frac{[1 + (1/t)]^4}{4} + C \right] = -\frac{1}{4} (4) \left(1 + \frac{1}{t}\right)^3 \left(-\frac{1}{t^2}\right) = \frac{1}{t^2} \left(1 + \frac{1}{t}\right)^3$$

$$27. \int \frac{1}{\sqrt{2x}} dx = \frac{1}{2} \int (2x)^{-1/2} 2 dx = \frac{1}{2} \left[\frac{(2x)^{1/2}}{1/2} \right] + C = \sqrt{2x} + C$$

$$\text{Check: } \frac{d}{dx} [\sqrt{2x} + C] = \frac{1}{2} (2x)^{-1/2} (2) = \frac{1}{\sqrt{2x}}$$

$$29. \int \frac{x^2 + 3x + 7}{\sqrt{x}} dx = \int (x^{3/2} + 3x^{1/2} + 7x^{-1/2}) dx = \frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C = \frac{2}{5}\sqrt{x}(x^2 + 5x + 35) + C$$

$$\text{Check: } \frac{d}{dx} \left[\frac{2}{5}x^{5/2} + 2x^{3/2} + 14x^{1/2} + C \right] = \frac{x^2 + 3x + 7}{\sqrt{x}}$$

$$31. \int t^2 \left(t - \frac{2}{t} \right) dt = \int (t^3 - 2t) dt = \frac{1}{4}t^4 - t^2 + C$$

$$\text{Check: } \frac{d}{dt} \left[\frac{1}{4}t^4 - t^2 + C \right] = t^3 - 2t = t^2 \left(t - \frac{2}{t} \right)$$

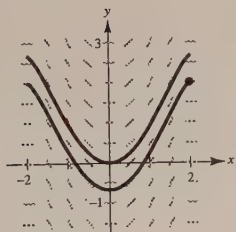
$$33. \int (9 - y)\sqrt{y} dy = \int (9y^{1/2} - y^{3/2}) dy = 9 \left(\frac{2}{3}y^{3/2} \right) - \frac{2}{5}y^{5/2} + C = \frac{2}{5}y^{3/2}(15 - y) + C$$

$$\text{Check: } \frac{d}{dy} \left[\frac{2}{5}y^{3/2}(15 - y) + C \right] = \frac{d}{dy} \left[6y^{3/2} - \frac{2}{5}y^{5/2} + C \right] = 9y^{1/2} - y^{3/2} = (9 - y)\sqrt{y}$$

$$\begin{aligned} 35. y &= \int \left[4x + \frac{4x}{\sqrt{16 - x^2}} \right] dx \\ &= 4 \int x dx - 2 \int (16 - x^2)^{-1/2} (-2x) dx \\ &= 4 \left(\frac{x^2}{2} \right) - 2 \left[\frac{(16 - x^2)^{1/2}}{1/2} \right] + C \\ &= 2x^2 - 4\sqrt{16 - x^2} + C \end{aligned}$$

$$\begin{aligned} 37. y &= \int \frac{x + 1}{(x^2 + 2x - 3)^2} dx \\ &= \frac{1}{2} \int (x^2 + 2x - 3)^{-2} (2x + 2) dx \\ &= \frac{1}{2} \left[\frac{(x^2 + 2x - 3)^{-1}}{-1} \right] + C \\ &= -\frac{1}{2(x^2 + 2x - 3)} + C \end{aligned}$$

39. (a)

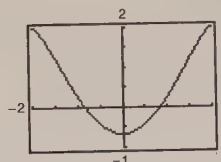


$$(b) \frac{dy}{dx} = x\sqrt{4 - x^2}, (2, 2)$$

$$\begin{aligned} y &= \int x\sqrt{4 - x^2} dx = -\frac{1}{2} \int (4 - x^2)^{1/2} (-2x) dx \\ &= -\frac{1}{2} \cdot \frac{2}{3} (4 - x^2)^{3/2} + C = -\frac{1}{3} (4 - x^2)^{3/2} + C \end{aligned}$$

$$(2, 2): 2 = -\frac{1}{3} (4 - 2^2)^{3/2} + C \Rightarrow C = 2$$

$$y = -\frac{1}{3} (4 - x^2)^{3/2} + 2$$



$$41. \int \pi \sin \pi x dx = -\cos \pi x + C$$

$$43. \int \sin 2x dx = \frac{1}{2} \int (\sin 2x)(2x) dx = -\frac{1}{2} \cos 2x + C$$

$$45. \int \frac{1}{\theta^2} \cos \frac{1}{\theta} d\theta = -\int \cos \frac{1}{\theta} \left(-\frac{1}{\theta^2} \right) d\theta = -\sin \frac{1}{\theta} + C$$

$$47. \int \sin 2x \cos 2x \, dx = \frac{1}{2} \int (\sin 2x)(2 \cos 2x) \, dx = \frac{1}{2} \frac{(\sin 2x)^2}{2} + C = \frac{1}{4} \sin^2 2x + C \quad \text{OR}$$

$$\int \sin 2x \cos 2x \, dx = -\frac{1}{2} \int (\cos 2x)(-2 \sin 2x) \, dx = -\frac{1}{2} \frac{(\cos 2x)^2}{2} + C_1 = -\frac{1}{4} \cos^2 2x + C_1 \quad \text{OR}$$

$$\int \sin 2x \cos 2x \, dx = \frac{1}{2} \int 2 \sin 2x \cos 2x \, dx = \frac{1}{2} \int \sin 4x \, dx = -\frac{1}{8} \cos 4x + C_2$$

$$49. \int \tan^4 x \sec^2 x \, dx = \frac{\tan^5 x}{5} + C = \frac{1}{5} \tan^5 x + C$$

$$\begin{aligned} 51. \int \frac{\csc^2 x}{\cot^3 x} \, dx &= -\int (\cot x)^{-3} (-\csc^2 x) \, dx \\ &= -\frac{(\cot x)^{-2}}{-2} + C = \frac{1}{2 \cot^2 x} + C = \frac{1}{2} \tan^2 x + C = \frac{1}{2} (\sec^2 x - 1) + C = \frac{1}{2} \sec^2 x + C_1 \end{aligned}$$

$$53. \int \cot^2 x \, dx = \int (\csc^2 x - 1) \, dx = -\cot x - x + C$$

$$55. f(x) = \int \cos \frac{x}{2} \, dx = 2 \sin \frac{x}{2} + C$$

Since $f(0) = 3 = 2 \sin 0 + C$, $C = 3$. Thus,

$$f(x) = 2 \sin \frac{x}{2} + 3.$$

$$57. u = x + 2, x = u - 2, dx = du$$

$$\begin{aligned} \int x \sqrt{x+2} \, dx &= \int (u-2) \sqrt{u} \, du \\ &= \int (u^{3/2} - 2u^{1/2}) \, du \\ &= \frac{2}{5} u^{5/2} - \frac{4}{3} u^{3/2} + C \\ &= \frac{2u^{3/2}}{15} (3u - 10) + C \\ &= \frac{2}{15} (x+2)^{3/2} [3(x+2) - 10] + C \\ &= \frac{2}{15} (x+2)^{3/2} (3x-4) + C \end{aligned}$$

$$59. u = 1 - x, x = 1 - u, dx = -du$$

$$\begin{aligned} \int x^2 \sqrt{1-x} \, dx &= -\int (1-u)^2 \sqrt{u} \, du \\ &= -\int (u^{1/2} - 2u^{3/2} + u^{5/2}) \, du \\ &= -\left(\frac{2}{3} u^{3/2} - \frac{4}{5} u^{5/2} + \frac{2}{7} u^{7/2} \right) + C \\ &= -\frac{2u^{3/2}}{105} (35 - 42u + 15u^2) + C \\ &= -\frac{2}{105} (1-x)^{3/2} [35 - 42(1-x) + 15(1-x)^2] + C \\ &= -\frac{2}{105} (1-x)^{3/2} (15x^2 + 12x + 8) + C \end{aligned}$$

61. $u = 2x - 1, x = \frac{1}{2}(u + 1), dx = \frac{1}{2} du$

$$\begin{aligned}
 \int \frac{x^2 - 1}{\sqrt{2x - 1}} dx &= \int \frac{[(1/2)(u + 1)]^2 - 1}{\sqrt{u}} \frac{1}{2} du \\
 &= \frac{1}{8} \int u^{-1/2} [u^2 + 2u + 1 - 4] du \\
 &= \frac{1}{8} \int (u^{3/2} + 2u^{1/2} - 3u^{-1/2}) du \\
 &= \frac{1}{8} \left(\frac{2}{5} u^{5/2} + \frac{4}{3} u^{3/2} - 6u^{1/2} \right) + C \\
 &= \frac{u^{1/2}}{60} (3u^2 + 10u - 45) + C \\
 &= \frac{\sqrt{2x - 1}}{60} [3(2x - 1)^2 + 10(2x - 1) - 45] + C \\
 &= \frac{1}{60} \sqrt{2x - 1} (12x^2 + 8x - 52) + C \\
 &= \frac{1}{15} \sqrt{2x - 1} (3x^2 + 2x - 13) + C
 \end{aligned}$$

63. $u = x + 1, x = u - 1, dx = du$

$$\begin{aligned}
 \int \frac{-x}{(x + 1) - \sqrt{x + 1}} dx &= \int \frac{-(u - 1)}{u - \sqrt{u}} du \\
 &= - \int \frac{(\sqrt{u} + 1)(\sqrt{u} - 1)}{\sqrt{u}(\sqrt{u} - 1)} du \\
 &= - \int (1 + u^{-1/2}) du \\
 &= -(u + 2u^{1/2}) + C \\
 &= -u - 2\sqrt{u} + C \\
 &= -(x + 1) - 2\sqrt{x + 1} + C \\
 &= -x - 2\sqrt{x + 1} - 1 + C \\
 &= -(x + 2\sqrt{x + 1}) + C_1
 \end{aligned}$$

where $C_1 = -1 + C$.

65. Let $u = x^2 + 1, du = 2x dx$.

$$\int_{-1}^1 x(x^2 + 1)^3 dx = \frac{1}{2} \int_{-1}^1 (x^2 + 1)^3 (2x) dx = \left[\frac{1}{8} (x^2 + 1)^4 \right]_{-1}^1 = 0$$

67. Let $u = x^3 + 1, du = 3x^2 dx$

$$\begin{aligned}
 \int_1^2 2x^2 \sqrt{x^3 + 1} dx &= 2 \cdot \frac{1}{3} \int_1^2 (x^3 + 1)^{1/2} (3x^2) dx \\
 &= \left[\frac{2}{3} \frac{(x^3 + 1)^{3/2}}{3/2} \right]_1^2 \\
 &= \frac{4}{9} \left[(x^3 + 1)^{3/2} \right]_1^2 \\
 &= \frac{4}{9} [27 - 2\sqrt{2}] = 12 - \frac{8}{9} \sqrt{2}
 \end{aligned}$$

69. Let
- $u = 2x + 1$
- ,
- $du = 2 dx$
- .

$$\int_0^4 \frac{1}{\sqrt{2x+1}} dx = \frac{1}{2} \int_0^4 (2x+1)^{-1/2} (2) dx = \left[\sqrt{2x+1} \right]_0^4 = \sqrt{9} - \sqrt{1} = 2$$

71. Let
- $u = 1 + \sqrt{x}$
- ,
- $du = \frac{1}{2\sqrt{x}} dx$
- .

$$\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx = 2 \int_1^9 (1+\sqrt{x})^{-2} \left(\frac{1}{2\sqrt{x}} \right) dx = \left[-\frac{2}{1+\sqrt{x}} \right]_1^9 = -\frac{1}{2} + 1 = \frac{1}{2}$$

- 73.
- $u = 2 - x$
- ,
- $x = 2 - u$
- ,
- $dx = -du$

When $x = 1$, $u = 1$. When $x = 2$, $u = 0$.

$$\int_1^2 (x-1)\sqrt{2-x} dx = \int_1^0 -[(2-u)-1]\sqrt{u} du = \int_1^0 (u^{3/2} - u^{1/2}) du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_1^0 = -\left[\frac{2}{5} - \frac{2}{3} \right] = \frac{4}{15}$$

- 75.
- $\int_0^{\pi/2} \cos\left(\frac{2}{3}x\right) dx = \left[\frac{3}{2} \sin\left(\frac{2}{3}x\right) \right]_0^{\pi/2} = \frac{3}{2} \left(\frac{\sqrt{3}}{2} \right) = \frac{3\sqrt{3}}{4}$

- 77.
- $u = x + 1$
- ,
- $x = u - 1$
- ,
- $dx = du$

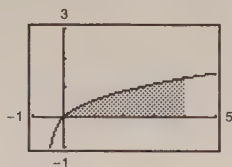
When $x = 0$, $u = 1$. When $x = 7$, $u = 8$.

$$\begin{aligned} \text{Area} &= \int_0^7 x\sqrt[3]{x+1} dx = \int_1^8 (u-1)\sqrt[3]{u} du \\ &= \int_1^8 (u^{4/3} - u^{1/3}) du = \left[\frac{3}{7}u^{7/3} - \frac{3}{4}u^{4/3} \right]_1^8 = \left(\frac{384}{7} - 12 \right) - \left(\frac{3}{7} - \frac{3}{4} \right) = \frac{1209}{28} \end{aligned}$$

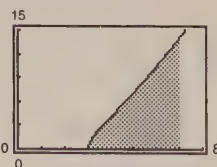
- 79.
- $A = \int_0^\pi (2 \sin x + \sin 2x) dx = -\left[2 \cos x + \frac{1}{2} \cos 2x \right]_0^\pi = 4$

- 81.
- $\text{Area} = \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) dx = 2 \int_{\pi/2}^{2\pi/3} \sec^2\left(\frac{x}{2}\right) \left(\frac{1}{2}\right) dx = \left[2 \tan\left(\frac{x}{2}\right) \right]_{\pi/2}^{2\pi/3} = 2(\sqrt{3} - 1)$

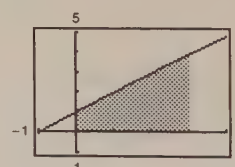
- 83.
- $\int_0^4 \frac{x}{\sqrt{2x+1}} dx \approx 3.333 = \frac{10}{3}$



- 85.
- $\int_3^7 x\sqrt{x-3} dx \approx 28.8 = \frac{144}{5}$



- 87.
- $\int_0^3 \left(\theta + \cos \frac{\theta}{6} \right) d\theta \approx 7.377$



- $$\begin{aligned} 89. \int (2x-1)^2 dx &= \frac{1}{2} \int (2x-1)^2 (2) dx = \frac{1}{6} (2x-1)^3 + C_1 = \frac{4}{3}x^3 - 2x^2 + x - \frac{1}{6} + C_1 \\ \int (2x-1)^2 dx &= \int (4x^2 - 4x + 1) dx = \frac{4}{3}x^3 - 2x^2 + x + C_2 \end{aligned}$$

They differ by a constant: $C_2 = C_1 - \frac{1}{6}$.

91. $f(x) = x^2(x^2 + 1)$ is even.

$$\begin{aligned}\int_{-2}^2 x^2(x^2 + 1) dx &= 2 \int_0^2 (x^4 + x^2) dx = 2 \left[\frac{x^5}{5} + \frac{x^3}{3} \right]_0^2 \\ &= 2 \left[\frac{32}{5} + \frac{8}{3} \right] = \frac{272}{15}\end{aligned}$$

93. $f(x) = x(x^2 + 1)^3$ is odd.

$$\int_{-2}^2 x(x^2 + 1)^3 dx = 0$$

95. $\int_0^2 x^2 dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}$; the function x^2 is an even function.

$$(a) \int_{-2}^0 x^2 dx = \int_0^2 x^2 dx = \frac{8}{3}$$

$$(c) \int_0^2 (-x^2) dx = - \int_0^2 x^2 dx = -\frac{8}{3}$$

$$(b) \int_{-2}^2 x^2 dx = 2 \int_0^2 x^2 dx = \frac{16}{3}$$

$$(d) \int_{-2}^0 3x^2 dx = 3 \int_0^2 x^2 dx = 8$$

$$97. \int_{-4}^4 (x^3 + 6x^2 - 2x - 3) dx = \int_{-4}^4 (x^3 - 2x) dx + \int_{-4}^4 (6x^2 - 3) dx = 0 + 2 \int_0^4 (6x^2 - 3) dx = 2 \left[2x^3 - 3x \right]_0^4 = 232$$

99. Answers will vary. See "Guidelines for Making a Change of Variables" on page 292.

101. $f(x) = x(x^2 + 1)^2$ is odd. Hence, $\int_{-2}^2 x(x^2 + 1)^2 dx = 0$.

$$103. \frac{dV}{dt} = \frac{k}{(t+1)^2}$$

$$V(t) = \int \frac{k}{(t+1)^2} dt = -\frac{k}{t+1} + C$$

$$V(0) = -k + C = 500,000$$

$$V(1) = -\frac{1}{2}k + C = 400,000$$

Solving this system yields $k = -200,000$ and $C = 300,000$. Thus,

$$V(t) = \frac{200,000}{t+1} + 300,000.$$

When $t = 4$, $V(4) = \$340,000$.

$$105. \frac{1}{b-a} \int_a^b \left[74.50 + 43.75 \sin \frac{\pi t}{6} \right] dt = \frac{1}{b-a} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_a^b$$

$$(a) \frac{1}{3} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_0^3 = \frac{1}{3} \left(223.5 + \frac{262.5}{\pi} \right) \approx 102.352 \text{ thousand units}$$

$$(b) \frac{1}{3} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_3^6 = \frac{1}{3} \left(447 + \frac{262.5}{\pi} - 223.5 \right) \approx 102.352 \text{ thousand units}$$

$$(c) \frac{1}{12} \left[74.50t - \frac{262.5}{\pi} \cos \frac{\pi t}{6} \right]_0^{12} = \frac{1}{12} \left(894 - \frac{262.5}{\pi} + \frac{262.5}{\pi} \right) = 74.5 \text{ thousand units}$$

107. $\frac{1}{b-a} \int_a^b [2 \sin(60\pi t) + \cos(120\pi t)] dt = \frac{1}{b-a} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_a^b$

(a) $\frac{1}{(1/60) - 0} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/60} = 60 \left[\left(\frac{1}{30\pi} + 0 \right) - \left(-\frac{1}{30\pi} \right) \right] = \frac{4}{\pi} \approx 1.273 \text{ amps}$

(b) $\frac{1}{(1/240) - 0} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/240} = 240 \left[\left(-\frac{1}{30\sqrt{2}\pi} + \frac{1}{120\pi} \right) - \left(-\frac{1}{30\pi} \right) \right]$
 $= \frac{2}{\pi} (5 - 2\sqrt{2}) \approx 1.382 \text{ amps}$

(c) $\frac{1}{(1/30) - 0} \left[-\frac{1}{30\pi} \cos(60\pi t) + \frac{1}{120\pi} \sin(120\pi t) \right]_0^{1/30} = 30 \left[\left(\frac{1}{30\pi} \right) - \left(-\frac{1}{30\pi} \right) \right] = 0 \text{ amps}$

109. False

$$\int (2x + 1)^2 dx = \frac{1}{2} \int (2x + 1)^2 2 dx = \frac{1}{6} (2x + 1)^3 + C$$

111. True

$$\int_{-10}^{10} (ax^3 + bx^2 + cx + d) dx = \int_{-10}^{10} (ax^3 + cx) dx + \int_{-10}^{10} (bx^2 + d) dx = 0 + 2 \int_0^{10} (bx^2 + d) dx$$

Odd Even

113. True

$$4 \int \sin x \cos x dx = 2 \int \sin 2x dx = -\cos 2x + C$$

115. Let $u = x + h$, then $du = dx$. When $x = a$, $u = a + h$. When $x = b$, $u = b + h$. Thus,

$$\int_a^b f(x + h) dx = \int_{a+h}^{b+h} f(u) du = \int_{a+h}^{b+h} f(x) dx.$$

Section 4.6 Numerical Integration

1. Exact: $\int_0^2 x^2 dx = \left[\frac{1}{3} x^3 \right]_0^2 = \frac{8}{3} \approx 2.6667$

Trapezoidal: $\int_0^2 x^2 dx \approx \frac{1}{4} \left[0 + 2 \left(\frac{1}{2} \right)^2 + 2(1)^2 + 2 \left(\frac{3}{2} \right)^2 + (2)^2 \right] = \frac{11}{4} = 2.7500$

Simpson's: $\int_0^2 x^2 dx \approx \frac{1}{6} \left[0 + 4 \left(\frac{1}{2} \right)^2 + 2(1)^2 + 4 \left(\frac{3}{2} \right)^2 + (2)^2 \right] = \frac{8}{3} \approx 2.6667$

3. Exact: $\int_0^2 x^3 dx = \left[\frac{x^4}{4} \right]_0^2 = 4.0000$

Trapezoidal: $\int_0^2 x^3 dx \approx \frac{1}{4} \left[0 + 2 \left(\frac{1}{2} \right)^3 + 2(1)^3 + 2 \left(\frac{3}{2} \right)^3 + (2)^3 \right] = \frac{17}{4} = 4.2500$

Simpson's: $\int_0^2 x^3 dx \approx \frac{1}{6} \left[0 + 4 \left(\frac{1}{2} \right)^3 + 2(1)^3 + 4 \left(\frac{3}{2} \right)^3 + (2)^3 \right] = \frac{24}{6} = 4.0000$

5. Exact: $\int_0^2 x^3 dx = \left[\frac{1}{4}x^4 \right]_0^2 = 4.0000$

Trapezoidal: $\int_0^2 x^3 dx \approx \frac{1}{8} \left[0 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 2\left(\frac{3}{4}\right)^3 + 2(1)^3 + 2\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 2\left(\frac{7}{4}\right)^3 + 8 \right] = 4.0625$

Simpson's: $\int_0^2 x^3 dx \approx \frac{1}{12} \left[0 + 4\left(\frac{1}{4}\right)^3 + 2\left(\frac{2}{4}\right)^3 + 4\left(\frac{3}{4}\right)^3 + 2(1)^3 + 4\left(\frac{5}{4}\right)^3 + 2\left(\frac{6}{4}\right)^3 + 4\left(\frac{7}{4}\right)^3 + 8 \right] = 4.0000$

7. Exact: $\int_4^9 \sqrt{x} dx = \left[\frac{2}{3}x^{3/2} \right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

Trapezoidal: $\int_4^9 \sqrt{x} dx \approx \frac{5}{16} \left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3 \right]$
 ≈ 12.6640

Simpson's: $\int_4^9 \sqrt{x} dx \approx \frac{5}{24} \left[2 + 4\sqrt{\frac{37}{8}} + \sqrt{21} + 4\sqrt{\frac{47}{8}} + \sqrt{26} + 4\sqrt{\frac{57}{8}} + \sqrt{31} + 4\sqrt{\frac{67}{8}} + 3 \right] \approx 12.6667$

9. Exact: $\int_1^2 \frac{1}{(x+1)^2} dx = \left[-\frac{1}{x+1} \right]_1^2 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6} \approx 0.1667$

Trapezoidal: $\int_1^2 \frac{1}{(x+1)^2} dx \approx \frac{1}{8} \left[\frac{1}{4} + 2\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 2\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9} \right]$
 $= \frac{1}{8} \left(\frac{1}{4} + \frac{32}{81} + \frac{8}{25} + \frac{32}{121} + \frac{1}{9} \right) \approx 0.1676$

Simpson's: $\int_1^2 \frac{1}{(x+1)^2} dx \approx \frac{1}{12} \left[\frac{1}{4} + 4\left(\frac{1}{((5/4)+1)^2}\right) + 2\left(\frac{1}{((3/2)+1)^2}\right) + 4\left(\frac{1}{((7/4)+1)^2}\right) + \frac{1}{9} \right]$
 $= \frac{1}{12} \left(\frac{1}{4} + \frac{64}{81} + \frac{8}{25} + \frac{64}{121} + \frac{1}{9} \right) \approx 0.1667$

11. Trapezoidal: $\int_0^2 \sqrt{1+x^3} dx \approx \frac{1}{4} [1 + 2\sqrt{1+(1/8)} + 2\sqrt{2} + 2\sqrt{1+(27/8)} + 3] \approx 3.283$

Simpson's: $\int_0^2 \sqrt{1+x^3} dx \approx \frac{1}{6} [1 + 4\sqrt{1+(1/8)} + 2\sqrt{2} + 4\sqrt{1+(27/8)} + 3] \approx 3.240$

Graphing utility: 3.241

13. $\int_0^1 \sqrt{x}\sqrt{1-x} dx = \int_0^1 \sqrt{x(1-x)} dx$

Trapezoidal: $\int_0^1 \sqrt{x(1-x)} dx \approx \frac{1}{8} \left[0 + 2\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)} \right] \approx 0.342$

Simpson's: $\int_0^1 \sqrt{x(1-x)} dx \approx \frac{1}{12} \left[0 + 4\sqrt{\frac{1}{4}\left(1-\frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1-\frac{1}{2}\right)} + 4\sqrt{\frac{3}{4}\left(1-\frac{3}{4}\right)} \right] \approx 0.372$

Graphing utility: 0.393

15. Trapezoidal: $\int_0^{\sqrt{\pi/2}} \cos(x^2) dx \approx \frac{\sqrt{\pi/2}}{8} \left[\cos 0 + 2 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2 \right]$
 ≈ 0.957

Simpson's: $\int_0^{\sqrt{\pi/2}} \cos(x^2) dx \approx \frac{\sqrt{\pi/2}}{12} \left[\cos 0 + 4 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2 \cos\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 4 \cos\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + \cos\left(\sqrt{\frac{\pi}{2}}\right)^2 \right]$
 ≈ 0.978

Graphing utility: 0.977

17. Trapezoidal: $\int_1^{1.1} \sin x^2 dx \approx \frac{1}{80} [\sin(1) + 2 \sin(1.025)^2 + 2 \sin(1.05)^2 + 2 \sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

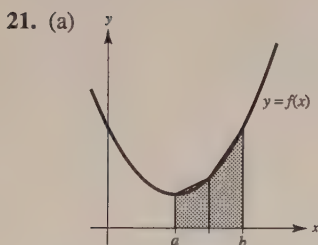
Simpson's: $\int_1^{1.1} \sin x^2 dx \approx \frac{1}{120} [\sin(1) + 4 \sin(1.025)^2 + 2 \sin(1.05)^2 + 4 \sin(1.075)^2 + \sin(1.1)^2] \approx 0.089$

Graphing utility: 0.089

19. Trapezoidal: $\int_0^{\pi/4} x \tan x dx \approx \frac{\pi}{32} \left[0 + 2 \left(\frac{\pi}{16} \right) \tan\left(\frac{\pi}{16}\right) + 2 \left(\frac{2\pi}{16} \right) \tan\left(\frac{2\pi}{16}\right) + 2 \left(\frac{3\pi}{16} \right) \tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4} \right] \approx 0.194$

Simpson's: $\int_0^{\pi/4} x \tan x dx \approx \frac{\pi}{48} \left[0 + 4 \left(\frac{\pi}{16} \right) \tan\left(\frac{\pi}{16}\right) + 2 \left(\frac{2\pi}{16} \right) \tan\left(\frac{2\pi}{16}\right) + 4 \left(\frac{3\pi}{16} \right) \tan\left(\frac{3\pi}{16}\right) + \frac{\pi}{4} \right] \approx 0.186$

Graphing utility: 0.186



The Trapezoidal Rule overestimates the area if the graph of the integrand is concave up.

23. $f(x) = x^3$
 $f'(x) = 3x^2$
 $f''(x) = 6x$
 $f'''(x) = 6$
 $f^{(4)}(x) = 0$

(a) Trapezoidal: Error $\leq \frac{(2-0)^3}{12(4^2)}(12) = 0.5$ since

$f''(x)$ is maximum in $[0, 2]$ when $x = 2$.

(b) Simpson's: Error $\leq \frac{(2-0)^5}{180(4^4)}(0) = 0$ since

$f^{(4)}(x) = 0$.

25. $f''(x) = \frac{2}{x^3}$ in $[1, 3]$.

(a) $|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| = 2$.

Trapezoidal: Error $\leq \frac{2^3}{12n^2}(2) < 0.00001$, $n^2 > 133,333.33$, $n > 365.15$; let $n = 366$.

$f^{(4)}(x) = \frac{24}{x^5}$ in $[1, 3]$

(b) $|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| = 24$.

Simpson's: Error $\leq \frac{2^5}{180n^4}(24) < 0.00001$, $n^4 > 426,666.67$, $n > 25.56$; let $n = 26$.

27. $f(x) = \sqrt{1+x}$

(a) $f''(x) = -\frac{1}{4(1+x)^{3/2}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \frac{1}{4}$.

Trapezoidal: Error $\leq \frac{8}{12n^2} \left(\frac{1}{4} \right) < 0.00001$, $n^2 > 16,666.67$, $n > 129.10$; let $n = 130$.

(b) $f^{(4)}(x) = \frac{-15}{16(1+x)^{7/2}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \frac{15}{16}$.

Simpson's: Error $\leq \frac{32}{180n^4} \left(\frac{15}{16} \right) < 0.00001$, $n^4 > 16,666.67$, $n > 11.36$; let $n = 12$.

29. $f(x) = \tan(x^2)$

(a) $f''(x) = 2 \sec^2(x^2)[1 + 4x^2 \tan(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 49.5305$.

Trapezoidal: Error $\leq \frac{(1-0)^3}{12n^2} (49.5305) < 0.00001$, $n^2 > 412,754.17$, $n > 642.46$; let $n = 643$.

(b) $f^{(4)}(x) = 8 \sec^2(x^2)[12x^2 + (3 + 32x^4) \tan(x^2) + 36x^2 \tan^2(x^2) + 48x^4 \tan^3(x^2)]$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x = 1$ and $|f^{(4)}(1)| \approx 9184.4734$.

Simpson's: Error $\leq \frac{(1-0)^5}{180n^4} (9184.4734) < 0.00001$, $n^4 > 5,102,485.22$, $n > 47.53$; let $n = 48$.

31. Let $f(x) = Ax^3 + Bx^2 + Cx + D$. Then $f^{(4)}(x) = 0$.

Simpson's: Error $\leq \frac{(b-a)^5}{180n^4} (0) = 0$

Therefore, Simpson's Rule is exact when approximating the integral of a cubic polynomial.

Example: $\int_0^1 x^3 dx = \frac{1}{6} \left[0 + 4 \left(\frac{1}{2} \right)^3 + 1 \right] = \frac{1}{6}$

This is the exact value of the integral.

33. $f(x) = \sqrt{2+3x^2}$ on $[0, 4]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	12.7771	15.3965	18.4340	15.6055	15.4845
8	14.0868	15.4480	16.9152	15.5010	15.4662
10	14.3569	15.4544	16.6197	15.4883	15.4658
12	14.5386	15.4578	16.4242	15.4814	15.4657
16	14.7674	15.4613	16.1816	15.4745	15.4657
20	14.9056	15.4628	16.0370	15.4713	15.4657

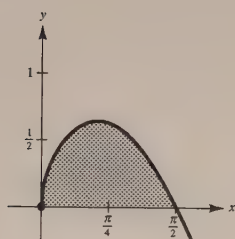
35. $f(x) = \sin\sqrt{x}$ on $[0, 4]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	2.8163	3.5456	3.7256	3.2709	3.3996
8	3.1809	3.5053	3.6356	3.4083	3.4541
10	3.2478	3.4990	3.6115	3.4296	3.4624
12	3.2909	3.4952	3.5940	3.4425	3.4674
16	3.3431	3.4910	3.5704	3.4568	3.4730
20	3.3734	3.4888	3.5552	3.4643	3.4759

37. $A = \int_0^{\pi/2} \sqrt{x} \cos x \, dx$

Simpson's Rule: $n = 14$

$$\int_0^{\pi/2} \sqrt{x} \cos x \, dx \approx \frac{\pi}{84} \left[\sqrt{0} \cos 0 + 4\sqrt{\frac{\pi}{28}} \cos \frac{\pi}{28} + 2\sqrt{\frac{\pi}{14}} \cos \frac{\pi}{14} + 4\sqrt{\frac{3\pi}{28}} \cos \frac{3\pi}{28} + \cdots + \sqrt{\frac{\pi}{2}} \cos \frac{\pi}{2} \right] \\ \approx 0.701$$



39. $W = \int_0^5 100x \sqrt{125 - x^3} \, dx$

Simpson's Rule: $n = 12$

$$\int_0^5 100x \sqrt{125 - x^3} \, dx \approx \frac{5}{3(12)} \left[0 + 400\left(\frac{5}{12}\right) \sqrt{125 - \left(\frac{5}{12}\right)^3} + 200\left(\frac{10}{12}\right) \sqrt{125 - \left(\frac{10}{12}\right)^3} \right. \\ \left. + 400\left(\frac{15}{12}\right) \sqrt{125 - \left(\frac{15}{12}\right)^3} + \cdots + 0 \right] \approx 10,233.58 \text{ ft} \cdot \text{lb}$$

41. $\int_0^{1/2} \frac{6}{\sqrt{1-x^2}} \, dx$ Simpson's Rule, $n = 6$

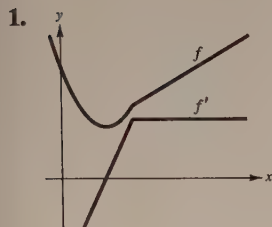
$$\pi \approx \frac{\left(\frac{1}{2} - 0\right)}{3(6)} [6 + 4(6.0209) + 2(6.0851) + 4(6.1968) + 2(6.3640) + 4(6.6002) + 6.9282] \\ \approx \frac{1}{36} [113.098] \approx 3.1416$$

43. Area $\approx \frac{1000}{2(10)} [125 + 2(125) + 2(120) + 2(112) + 2(90) + 2(90) + 2(95) + 2(88) + 2(75) + 2(35)] = 89,250 \text{ sq m}$

45. $\int_0^t \sin\sqrt{x} \, dx = 2, n = 10$

By trial and error, we obtain $t \approx 2.477$.

Review Exercises for Chapter 4



$$3. \int (2x^2 + x - 1) dx = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$$

$$5. \int \frac{x^3 + 1}{x^2} dx = \int \left(x + \frac{1}{x^2} \right) dx = \frac{1}{2}x^2 - \frac{1}{x} + C$$

$$7. \int (4x - 3 \sin x) dx = 2x^2 + 3 \cos x + C$$

$$9. f'(x) = -2x, (-1, 1)$$

$$f(x) = \int -2x dx = -x^2 + C$$

When $x = -1$:

$$y = -1 + C = 1$$

$$C = 2$$

$$y = 2 - x^2$$

$$11. a(t) = a$$

$$v(t) = \int a dt = at + C_1$$

$$v(0) = 0 + C_1 = 0 \text{ when } C_1 = 0.$$

$$v(t) = at$$

$$s(t) = \int at dt = \frac{a}{2}t^2 + C_2$$

$$s(0) = 0 + C_2 = 0 \text{ when } C_2 = 0.$$

$$s(t) = \frac{a}{2}t^2$$

$$s(30) = \frac{a}{2}(30)^2 = 3600 \text{ or}$$

$$a = \frac{2(3600)}{(30)^2} = 8 \text{ ft/sec}^2.$$

$$v(30) = 8(30) = 240 \text{ ft/sec}$$

$$13. a(t) = -32$$

$$v(t) = -32t + 96$$

$$s(t) = -16t^2 + 96t$$

$$(a) v(t) = -32t + 96 = 0 \text{ when } t = 3 \text{ sec.}$$

$$(b) s(3) = -144 + 288 = 144 \text{ ft}$$

$$(c) v(t) = -32t + 96 = \frac{96}{2} \text{ when } t = \frac{3}{2} \text{ sec.}$$

$$(d) s\left(\frac{3}{2}\right) = -16\left(\frac{9}{4}\right) + 96\left(\frac{3}{2}\right) = 108 \text{ ft}$$

$$15. (a) \sum_{i=1}^{10} (2i - 1)$$

$$(b) \sum_{i=1}^n i^3$$

$$(c) \sum_{i=1}^{10} (4i + 2)$$

17. $y = \frac{10}{x^2 + 1}, \Delta x = \frac{1}{2}, n = 4$

$$S(n) = S(4) = \frac{1}{2} \left[\frac{10}{1} + \frac{10}{(1/2)^2 + 1} + \frac{10}{(1)^2 + 1} + \frac{10}{(3/2)^2 + 1} \right]$$

$$\approx 13.0385$$

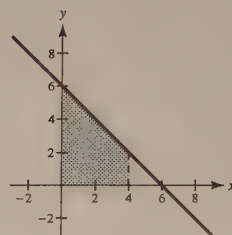
$$s(n) = s(4) = \frac{1}{2} \left[\frac{10}{(1/2)^2 + 1} + \frac{10}{1 + 1} + \frac{10}{(3/2)^2 + 1} + \frac{10}{2^2 + 1} \right]$$

$$\approx 9.0385$$

$$9.0385 < \text{Area of Region} < 13.0385$$

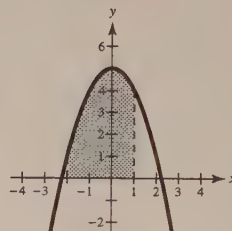
19. $y = 6 - x, \Delta x = \frac{4}{n}$, right endpoints

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(6 - \frac{4i}{n} \right) \frac{4}{n} \\ &= \lim_{n \rightarrow \infty} \frac{4}{n} \left[6n - \frac{4n(n+1)}{2} \right] \\ &= \lim_{n \rightarrow \infty} \left[24 - 8 \frac{n+1}{n} \right] = 24 - 8 = 16 \end{aligned}$$



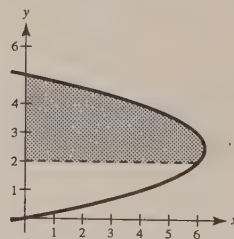
21. $y = 5 - x^2, \Delta x = \frac{3}{n}$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[5 - \left(-2 + \frac{3i}{n} \right)^2 \right] \left(\frac{3}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[1 + \frac{12i}{n} - \frac{9i^2}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[n + \frac{12n(n+1)}{2} - \frac{9n(n+1)(2n+1)}{6} \right] \\ &= \lim_{n \rightarrow \infty} \left[3 + 18 \frac{n+1}{n} - \frac{9(n+1)(2n+1)}{n^2} \right] \\ &= 3 + 18 - 9 = 12 \end{aligned}$$



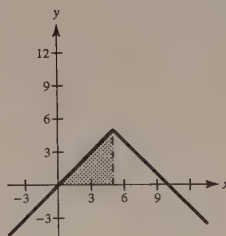
23. $x = 5y - y^2, 2 \leq y \leq 5, \Delta y = \frac{3}{n}$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[5 \left(2 + \frac{3i}{n} \right) - \left(2 + \frac{3i}{n} \right)^2 \right] \left(\frac{3}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[10 + \frac{15i}{n} - 4 - \frac{12i}{n} - \frac{9i^2}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[6 + \frac{3i}{n} - \frac{9i^2}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[6n + \frac{3n(n+1)}{2} - \frac{9n(n+1)(2n+1)}{6} \right] \\ &= \left[18 + \frac{9}{2} - 9 \right] = \frac{27}{2} \end{aligned}$$



$$25. \lim_{\|\Delta\| \rightarrow \infty} \sum_{i=1}^n (2c_i - 3) \Delta x_i = \int_4^6 (2x - 3) dx$$

27.



$$\int_0^5 (5 - |x - 5|) dx = \int_0^5 (5 - (5 - x)) dx = \int_0^5 x dx = \frac{25}{2}$$

(triangle)

$$29. (a) \int_2^6 [f(x) + g(x)] dx = \int_2^6 f(x) dx + \int_2^6 g(x) dx = 10 + 3 = 13$$

$$(b) \int_2^6 [f(x) - g(x)] dx = \int_2^6 f(x) dx - \int_2^6 g(x) dx = 10 - 3 = 7$$

$$(c) \int_2^6 [2f(x) - 3g(x)] dx = 2 \int_2^6 f(x) dx - 3 \int_2^6 g(x) dx = 2(10) - 3(3) = 11$$

$$(d) \int_2^6 5f(x) dx = 5 \int_2^6 f(x) dx = 5(10) = 50$$

$$31. \int_1^8 (\sqrt[3]{x} + 1) dx = \left[\frac{3}{4} x^{4/3} + x \right]_1^8 = \left[\frac{3}{4}(16) + 8 \right] - \left[\frac{3}{4} + 1 \right] = \frac{73}{4} \text{ (c)}$$

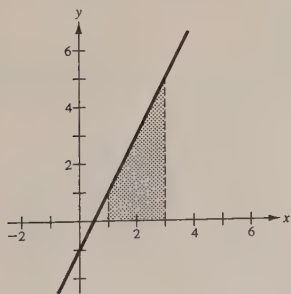
$$33. \int_0^4 (2 + x) dx = \left[2x + \frac{x^2}{2} \right]_0^4 = 8 + \frac{16}{2} = 16$$

$$35. \int_{-1}^1 (4t^3 - 2t) dt = \left[t^4 - t^2 \right]_{-1}^1 = 0$$

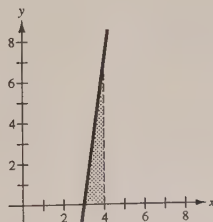
$$37. \int_4^9 x\sqrt{x} dx = \int_4^9 x^{3/2} dx = \left[\frac{2}{5} x^{5/2} \right]_4^9 = \frac{2}{5} [(\sqrt{9})^5 - (\sqrt{4})^5] = \frac{2}{5} (243 - 32) = \frac{422}{5}$$

$$39. \int_0^{3\pi/4} \sin \theta d\theta = \left[-\cos \theta \right]_0^{3\pi/4} = -\left(-\frac{\sqrt{2}}{2} \right) + 1 = 1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + 2}{2}$$

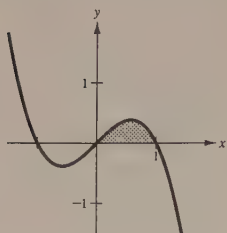
$$41. \int_1^3 (2x - 1) dx = \left[x^2 - x \right]_1^3 = 6$$



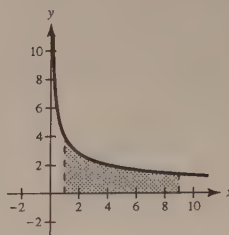
$$\begin{aligned} 43. \int_3^4 (x^2 - 9) dx &= \left[\frac{x^3}{3} - 9x \right]_3^4 \\ &= \left(\frac{64}{3} - 36 \right) - (9 - 27) \\ &= \frac{64}{3} - \frac{54}{3} = \frac{10}{3} \end{aligned}$$



$$45. \int_0^1 (x - x^3) dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$



$$47. \text{Area} = \int_1^9 \frac{4}{\sqrt{x}} dx = \left[\frac{4x^{1/2}}{(1/2)} \right]_1^9 = 8(3 - 1) = 16$$

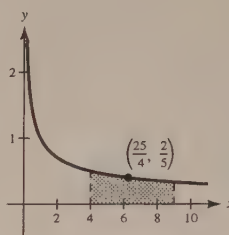


$$49. \frac{1}{9-4} \int_4^9 \frac{1}{\sqrt{x}} dx = \left[\frac{1}{5} 2\sqrt{x} \right]_4^9 = \frac{2}{5}(3 - 2) = \frac{2}{5} \text{ Average value}$$

$$\frac{2}{5} = \frac{1}{\sqrt{x}}$$

$$\sqrt{x} = \frac{5}{2}$$

$$x = \frac{25}{4}$$



$$51. F'(x) = x^2 \sqrt{1+x^3}$$

$$53. F'(x) = x^2 + 3x + 2$$

$$55. \int (x^2 + 1)^3 dx = \int (x^6 + 3x^4 + 3x^2 + 1) dx = \frac{x^7}{7} + \frac{3}{5}x^5 + x^3 + x + C$$

$$57. u = x^3 + 3, du = 3x^2 dx$$

$$\int \frac{x^2}{\sqrt{x^3 + 3}} dx = \int (x^3 + 3)^{-1/2} x^2 dx = \frac{1}{3} \int (x^3 + 3)^{-1/2} 3x^2 dx = \frac{2}{3} (x^3 + 3)^{1/2} + C$$

$$59. u = 1 - 3x^2, du = -6x dx$$

$$\int x(1 - 3x^2)^4 dx = -\frac{1}{6} \int (1 - 3x^2)^4 (-6x dx) = -\frac{1}{30} (1 - 3x^2)^5 + C = \frac{1}{30} (3x^2 - 1)^5 + C$$

$$61. \int \sin^3 x \cos x dx = \frac{1}{4} \sin^4 x + C$$

$$63. \int \frac{\sin \theta}{\sqrt{1 - \cos \theta}} d\theta = \int (1 - \cos \theta)^{-1/2} \sin \theta d\theta = 2(1 - \cos \theta)^{1/2} + C = 2\sqrt{1 - \cos \theta} + C$$

$$65. \int \tan^n x \sec^2 x dx = \frac{\tan^{n+1} x}{n+1} + C, n \neq -1$$

$$67. \int (1 + \sec \pi x)^2 \sec \pi x \tan \pi x dx = \frac{1}{\pi} \int (1 + \sec \pi x)^2 (\pi \sec \pi x \tan \pi x) dx = \frac{1}{3\pi} (1 + \sec \pi x)^3 + C$$

$$69. \int_{-1}^2 x(x^2 - 4) dx = \frac{1}{2} \int_{-1}^2 (x^2 - 4)(2x) dx = \frac{1}{2} \left[\frac{(x^2 - 4)^2}{2} \right]_{-1}^2 = \frac{1}{4} [0 - 9] = -\frac{9}{4}$$

$$71. \int_0^3 \frac{1}{\sqrt{1+x}} dx = \int_0^3 (1+x)^{-1/2} dx = \left[2(1+x)^{1/2} \right]_0^3 = 4 - 2 = 2$$

$$73. u = 1 - y, y = 1 - u, dy = -du$$

When $y = 0$, $u = 1$. When $y = 1$, $u = 0$.

$$\begin{aligned} 2\pi \int_0^1 (y+1)\sqrt{1-y} dy &= 2\pi \int_1^0 -[(1-u)+1]\sqrt{u} du \\ &= 2\pi \int_1^0 (u^{3/2} - 2u^{1/2}) du = 2\pi \left[\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} \right]_1^0 = \frac{28\pi}{15} \end{aligned}$$

$$75. \int_0^\pi \cos\left(\frac{x}{2}\right) dx = 2 \int_0^\pi \cos\left(\frac{x}{2}\right) \frac{1}{2} dx = \left[2 \sin\left(\frac{x}{2}\right) \right]_0^\pi = 2$$

$$77. u = 1 - x, x = 1 - u, dx = -du$$

When $x = a$, $u = 1 - a$. When $x = b$, $u = 1 - b$.

$$\begin{aligned} P_{a,b} &= \int_a^b \frac{15}{4} x \sqrt{1-x} dx = \frac{15}{4} \int_{1-a}^{1-b} -(1-u)\sqrt{u} du \\ &= \frac{15}{4} \int_{1-a}^{1-b} (u^{3/2} - u^{1/2}) du = \frac{15}{4} \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_{1-a}^{1-b} = \frac{15}{4} \left[\frac{2u^{3/2}}{15}(3u-5) \right]_{1-a}^{1-b} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2) \right]_a^b \end{aligned}$$

$$(a) P_{0.50, 0.75} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2) \right]_{0.50}^{0.75} = 0.353 = 35.3\%$$

$$(b) P_{0,b} = \left[-\frac{(1-x)^{3/2}}{2}(3x+2) \right]_0^b = -\frac{(1-b)^{3/2}}{2}(3b+2) + 1 = 0.5$$

$$(1-b)^{3/2}(3b+2) = 1$$

$$b \approx 0.586 = 58.6\%$$

$$79. p = 1.20 + 0.04t$$

$$C = \frac{15,000}{M} \int_t^{t+1} p ds = \frac{15,000}{M} \int_t^{t+1} (1.20 + 0.04s) ds$$

(a) 2000 corresponds to $t = 10$.

$$\begin{aligned} C &= \frac{15,000}{M} \int_{10}^{11} [1.20 + 0.04t] dt \\ &= \frac{15,000}{M} \left[1.20t + 0.02t^2 \right]_{10}^{11} = \frac{24,300}{M} \end{aligned}$$

(b) 2005 corresponds to $t = 15$.

$$C = \frac{15,000}{M} \left[1.20t + 0.02t^2 \right]_{15}^{16} = \frac{27,300}{M}$$

$$81. \text{Trapezoidal Rule } (n = 4): \int_1^2 \frac{1}{1+x^3} dx \approx \frac{1}{8} \left[\frac{1}{1+1^3} + \frac{2}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{2}{1+(1.75)^3} + \frac{1}{1+2^3} \right] \approx 0.257$$

$$\text{Simpson's Rule } (n = 4): \int_1^2 \frac{1}{1+x^3} dx \approx \frac{1}{12} \left[\frac{1}{1+1^3} + \frac{4}{1+(1.25)^3} + \frac{2}{1+(1.5)^3} + \frac{4}{1+(1.75)^3} + \frac{1}{1+2^3} \right] \approx 0.254$$

Graphing utility: 0.254

$$83. \text{Trapezoidal Rule } (n = 4): \int_0^{\pi/2} \sqrt{x} \cos x dx \approx 0.637$$

Simpson's Rule $(n = 4)$: 0.685

Graphing Utility: 0.704

$$85. (a) R < I < T < L$$

$$(b) S(4) = \frac{4-0}{3(4)} [f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)]$$

$$\approx \frac{1}{3} \left[4 + 4(2) + 2(1) + 4\left(\frac{1}{2}\right) + \frac{1}{4} \right] \approx 5.417$$

Problem Solving for Chapter 4

1. (a) $L(1) = \int_1^1 \frac{1}{t} dt = 0$

(b) $L'(x) = \frac{1}{x}$ by the Second Fundamental Theorem of Calculus.

$$L'(1) = 1$$

(c) $L(x) = 1 = \int_1^x \frac{1}{t} dt$ for $x \approx 2.718$

$$\int_1^{2.718} \frac{1}{t} dt = 0.999896 \quad (\text{Note: The exact value of } x \text{ is } e, \text{ the base of the natural logarithm function.})$$

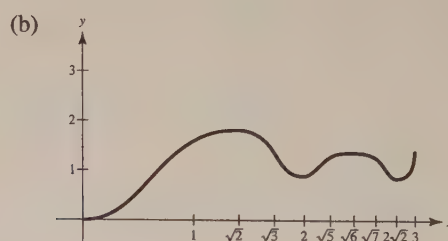
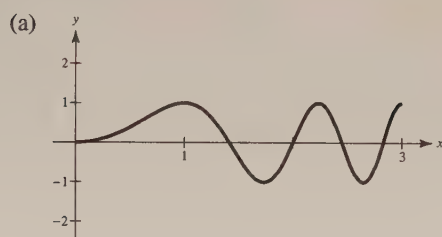
(d) We first show that $\int_1^{x_1} \frac{1}{t} dt = \int_{1/x_1}^1 \frac{1}{t} dt$.

To see this, let $u = \frac{t}{x_1}$ and $du = \frac{1}{x_1} dt$.

Then $\int_1^{x_1} \frac{1}{t} dt = \int_{1/x_1}^1 \frac{1}{ux_1} (x_1 du) = \int_{1/x_1}^1 \frac{1}{u} du = \int_{1/x_1}^1 \frac{1}{t} dt$.

Now,
$$\begin{aligned} L(x_1 x_2) &= \int_1^{x_1 x_2} \frac{1}{t} dt = \int_{1/x_1}^{x_2} \frac{1}{u} du \quad (\text{using } u = \frac{t}{x_1}) \\ &= \int_{1/x_1}^1 \frac{1}{u} du + \int_1^{x_2} \frac{1}{u} du \\ &= \int_1^{x_1} \frac{1}{u} du + \int_1^{x_2} \frac{1}{u} du \\ &= L(x_1) + L(x_2). \end{aligned}$$

3. $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$



The zeros of $y = \sin \frac{\pi x^2}{2}$ correspond to the relative extrema of $S(x)$.

(c) $S'(x) = \sin \frac{\pi x^2}{2} = 0 \Rightarrow \frac{\pi x^2}{2} = n\pi \Rightarrow x^2 = 2n \Rightarrow x = \sqrt{2n}, n \text{ integer.}$

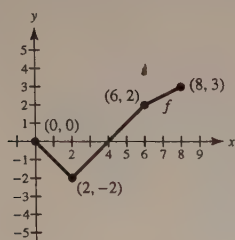
Relative maximum at $x = \sqrt{2} \approx 1.4142$ and $x = \sqrt{6} \approx 2.4495$

Relative minimum at $x = 2$ and $x = \sqrt{8} \approx 2.8284$

(d) $S''(x) = \cos\left(\frac{\pi x^2}{2}\right)(\pi x) = 0 \Rightarrow \frac{\pi x^2}{2} = \frac{\pi}{2} + n\pi \Rightarrow x^2 = 1 + 2n \Rightarrow x = \sqrt{1 + 2n}, n \text{ integer}$

Points of inflection at $x = 1, \sqrt{3}, \sqrt{5}, \text{ and } \sqrt{7}$.

5. (a)



$$(c) f(x) = \begin{cases} -x, & 0 \leq x < 2 \\ x - 4, & 2 \leq x < 6 \\ \frac{1}{2}x - 1, & 6 \leq x \leq 8 \end{cases}$$

$$F(x) = \int_0^x f(t) dt = \begin{cases} (-x^2/2), & 0 \leq x < 2 \\ (x^2/2) - 4x + 4, & 2 \leq x < 6 \\ (1/4)x^2 - x - 5, & 6 \leq x \leq 8 \end{cases}$$

$F'(x) = f(x)$. F is decreasing on $(0, 2)$ and increasing on $(2, 8)$. Therefore, the minimum is -4 at $x = 4$, and the maximum is 3 at $x = 8$.

$$7. (a) \int_{-1}^1 \cos x dx \approx \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 2 \cos\left(\frac{1}{\sqrt{3}}\right) \approx 1.6758$$

$$\int_{-1}^1 \cos x dx = \sin x \Big|_{-1}^1 = 2 \sin(1) \approx 1.6829$$

$$\text{Error: } |1.6829 - 1.6758| = 0.0071$$

$$(b) \int_{-1}^1 \frac{1}{1+x^2} dx \approx \frac{1}{1+(1/3)} + \frac{1}{1+(1/3)} = \frac{3}{2}$$

(Note: exact answer is $\pi/2 \approx 1.5708$)

9. Consider $F(x) = [f(x)]^2 \Rightarrow F'(x) = 2f(x)f'(x)$. Thus,

$$\begin{aligned} \int_a^b f(x)f'(x) dx &= \int_a^b \frac{1}{2}F'(x) dx \\ &= \left[\frac{1}{2}F(x)\right]_a^b \\ &= \frac{1}{2}[F(b) - F(a)] \\ &= \frac{1}{2}[f(b)^2 - f(a)^2] \end{aligned}$$

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{4}$	3

$$(d) F''(x) = f'(x) = \begin{cases} -1, & 0 < x < 2 \\ 1, & 2 < x < 6 \\ \frac{1}{2}, & 6 < x < 8 \end{cases}$$

$x = 2$ is a point of inflection, whereas $x = 6$ is not.

(c) Let $p(x) = ax^3 + bx^2 + cx + d$.

$$\int_{-1}^1 p(x) dx = \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{-1}^1 = \frac{2b}{3} + 2d$$

$$p\left(-\frac{1}{\sqrt{3}}\right) + p\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{b}{3} + d\right) + \left(\frac{b}{3} + d\right) = \frac{2b}{3} + 2d$$

$$11. \text{ Consider } \int_0^1 x^5 dx = \left[\frac{x^6}{6}\right]_0^1 = \frac{1}{6}.$$

The corresponding Riemann Sum using right endpoints is

$$\begin{aligned} S(n) &= \frac{1}{n} \left[\left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \cdots + \left(\frac{n}{n}\right)^5 \right] \\ &= \frac{1}{n^6} [1^5 + 2^5 + \cdots + n^5] \end{aligned}$$

$$\text{Thus, } \lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \frac{1^5 + 2^5 + \cdots + n^5}{n^6} = \frac{1}{6}.$$

13. By Theorem 4.8, $0 < f(x) \leq M \Rightarrow \int_a^b f(x) dx \leq \int_a^b M dx = M(b-a)$.

Similarly, $m \leq f(x) \Rightarrow m(b-a) = \int_a^b m dx \leq \int_a^b f(x) dx$.

Thus, $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$. On the interval $[0, 1]$, $1 \leq \sqrt{1+x^4} \leq \sqrt{2}$ and $b-a=1$.

Thus, $1 \leq \int_0^1 \sqrt{1+x^4} dx \leq \sqrt{2}$. (Note: $\int_0^1 \sqrt{1+x^4} dx \approx 1.0894$)

15. Since $-|f(x)| \leq f(x) \leq |f(x)|$,

$$-\int_a^b |f(x)| dx \leq \int_a^b f(x) dx \leq \int_a^b |f(x)| dx \Rightarrow \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

$$17. \frac{1}{365} \int_0^{365} 100,000 \left[1 + \sin \frac{2\pi(t-60)}{365} \right] dt = \frac{100,000}{365} \left[t - \frac{365}{2\pi} \cos \frac{2\pi(t-60)}{365} \right]_0^{365} = 100,000 \text{ lbs.}$$

CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

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CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

Section 5.1 The Natural Logarithmic Function: Differentiation

Solutions to Odd-Numbered Exercises

1. Simpson's Rule: $n = 10$

x	0.5	1.5	2	2.5	3	3.5	4
$\int_1^x \frac{1}{t} dt$	-0.6932	0.4055	0.6932	0.9163	1.0987	1.2529	1.3865

Note: $\int_1^{0.5} \frac{1}{t} dt = -\int_{0.5}^1 \frac{1}{t} dt$

3. (a) $\ln 45 \approx 3.8067$

(b) $\int_1^{45} \frac{1}{t} dt \approx 3.8067$

5. (a) $\ln 0.8 \approx -0.2231$

(b) $\int_1^{0.8} \frac{1}{t} dt \approx -0.2231$

7. $f(x) = \ln x + 2$

Vertical shift 2 units upward

Matches (b)

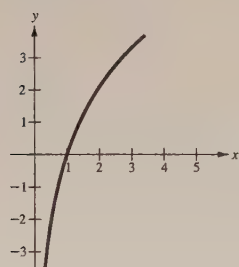
9. $f(x) = \ln(x - 1)$

Horizontal shift 1 unit to the right

Matches (a)

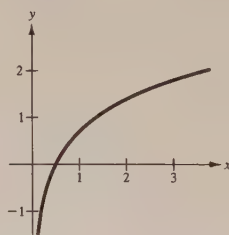
11. $f(x) = 3 \ln x$

Domain: $x > 0$



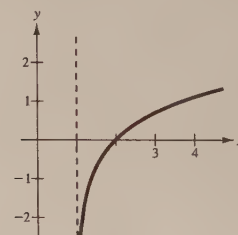
13. $f(x) = \ln 2x$

Domain: $x > 0$



15. $f(x) = \ln(x - 1)$

Domain: $x > 1$



17. (a) $\ln 6 = \ln 2 + \ln 3 \approx 1.7917$

(b) $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx -0.4055$

(c) $\ln 81 = \ln 3^4 = 4 \ln 3 \approx 4.3944$

(d) $\ln \sqrt{3} = \ln 3^{1/2} = \frac{1}{2} \ln 3 \approx 0.5493$

19. $\ln \frac{2}{3} = \ln 2 - \ln 3$

21. $\ln \frac{xy}{z} = \ln x + \ln y - \ln z$

23. $\ln \sqrt[3]{a^2 + 1} = \ln(a^2 + 1)^{1/3} = \frac{1}{3} \ln(a^2 + 1)$

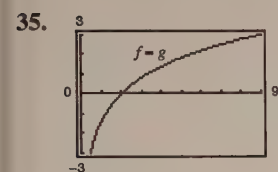
$$\begin{aligned}
 25. \ln\left(\frac{x^2-1}{x^3}\right)^3 &= 3[\ln(x^2-1) - \ln x^3] \\
 &= 3[\ln(x+1) + \ln(x-1) - 3\ln x]
 \end{aligned}$$

$$\begin{aligned}
 27. \ln z(z-1)^2 &= \ln z + \ln(z-1)^2 \\
 &= \ln z + 2\ln(z-1)
 \end{aligned}$$

$$29. \ln(x-2) - \ln(x+2) = \ln \frac{x-2}{x+2}$$

$$31. \frac{1}{3}[2\ln(x+3) + \ln x - \ln(x^2-1)] = \frac{1}{3}\ln \frac{x(x+3)^2}{x^2-1} = \ln \sqrt[3]{\frac{x(x+3)^2}{x^2-1}}$$

$$33. 2\ln 3 - \frac{1}{2}\ln(x^2+1) = \ln 9 - \ln \sqrt{x^2+1} = \ln \frac{9}{\sqrt{x^2+1}}$$



$$37. \lim_{x \rightarrow 3^+} \ln(x-3) = -\infty$$

$$39. \lim_{x \rightarrow 2^-} \ln[x^2(3-x)] = \ln 4 \approx 1.3863$$

$$41. y = \ln x^3 = 3 \ln x$$

$$y' = \frac{3}{x}$$

$$\text{At } (1, 0), y' = 3.$$

$$43. y = \ln x^2 = 2 \ln x$$

$$y' = \frac{2}{x}$$

$$\text{At } (1, 0), y' = 2.$$

$$45. g(x) = \ln x^2 = 2 \ln x$$

$$g'(x) = \frac{2}{x}$$

$$47. y = (\ln x)^4$$

$$\frac{dy}{dx} = 4(\ln x)^3 \left(\frac{1}{x}\right) = \frac{4(\ln x)^3}{x}$$

$$49. y = \ln[x\sqrt{x^2-1}] = \ln x + \frac{1}{2}\ln(x^2-1)$$

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2} \left(\frac{2x}{x^2-1}\right) = \frac{2x^2-1}{x(x^2-1)}$$

$$51. f(x) = \ln \frac{x}{x^2+1} = \ln x - \ln(x^2+1)$$

$$f'(x) = \frac{1}{x} - \frac{2x}{x^2+1} = \frac{1-x^2}{x(x^2+1)}$$

$$53. g(t) = \frac{\ln t}{t^2}$$

$$g'(t) = \frac{t^2(1/t) - 2t \ln t}{t^4} = \frac{1-2\ln t}{t^3}$$

$$55. y = \ln(\ln x^2)$$

$$\frac{dy}{dx} = \frac{1}{\ln x^2} \frac{d}{dx}(\ln x^2) = \frac{(2x/x^2)}{\ln x^2} = \frac{2}{x \ln x^2} = \frac{1}{x \ln x}$$

$$57. y = \ln \sqrt{\frac{x+1}{x-1}} = \frac{1}{2}[\ln(x+1) - \ln(x-1)]$$

$$\frac{dy}{dx} = \frac{1}{2} \left[\frac{1}{x+1} - \frac{1}{x-1} \right] = \frac{1}{1-x^2}$$

$$59. f(x) = \ln \frac{\sqrt{4+x^2}}{x} = \frac{1}{2}\ln(4+x^2) - \ln x$$

$$f'(x) = \frac{x}{4+x^2} - \frac{1}{x} = \frac{-4}{x(x^2+4)}$$

$$61. y = \frac{-\sqrt{x^2+1}}{x} + \ln(x + \sqrt{x^2+1})$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-x(\frac{x}{\sqrt{x^2+1}}) + \sqrt{x^2+1}}{x^2} + \left(\frac{1}{x + \sqrt{x^2+1}}\right)\left(1 + \frac{x}{\sqrt{x^2+1}}\right) \\ &= \frac{1}{x^2\sqrt{x^2+1}} + \left(\frac{1}{x + \sqrt{x^2+1}}\right)\left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}}\right) = \frac{1}{x^2\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} = \frac{1+x^2}{x^2\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}}{x^2} \end{aligned}$$

$$63. y = \ln|\sin x|$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

$$65. y = \ln\left|\frac{\cos x}{\cos x - 1}\right|$$

$$= \ln|\cos x| - \ln|\cos x - 1|$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1} = -\tan x + \frac{\sin x}{\cos x - 1}$$

$$67. y = \ln\left|\frac{-1 + \sin x}{2 + \sin x}\right|$$

$$= \ln|-1 + \sin x| - \ln|2 + \sin x|$$

$$\frac{dy}{dx} = \frac{\cos x}{-1 + \sin x} - \frac{\cos x}{2 + \sin x}$$

$$= \frac{3 \cos x}{(\sin x - 1)(\sin x + 2)}$$

$$69. f(x) = \sin 2x \ln x^2 = 2 \sin 2x \ln x$$

$$f'(x) = (2 \sin 2x)\left(\frac{1}{x}\right) + 4 \cos 2x \ln x$$

$$= \frac{2}{x}(\sin 2x + 2x \cos 2x \ln x)$$

$$= \frac{2}{x}(\sin 2x + x \cos 2x \ln x^2)$$

$$71. (a) y = 3x^2 - \ln x, (1, 3)$$

$$\frac{dy}{dx} = 6x - \frac{1}{x}$$

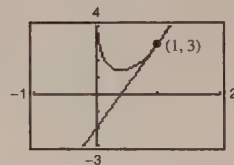
$$\text{When } x = 1, \frac{dy}{dx} = 5.$$

$$\text{Tangent line: } y - 3 = 5(x - 1)$$

$$y = 5x - 2$$

$$0 = 5x - y - 2$$

(b)



$$73. x^2 - 3 \ln y + y^2 = 10$$

$$2x - \frac{3}{y} \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx} \left(\frac{3}{y} - 2y \right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

$$75. y = 2(\ln x) + 3$$

$$y' = \frac{2}{x}$$

$$y'' = -\frac{2}{x^2}$$

$$xy'' + y' = x\left(-\frac{2}{x^2}\right) + \frac{2}{x} = 0$$

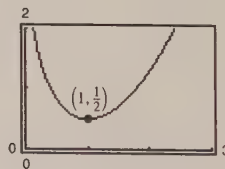
$$77. y = \frac{x^2}{2} - \ln x$$

$$\text{Domain: } x > 0$$

$$y' = x - \frac{1}{x} = \frac{(x+1)(x-1)}{x} = 0 \text{ when } x = 1.$$

$$y'' = 1 + \frac{1}{x^2} > 0$$

$$\text{Relative minimum: } \left(1, \frac{1}{2}\right)$$

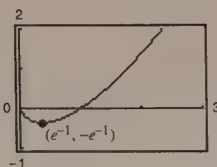


79. $y = x \ln x$

 Domain: $x > 0$

$$y' = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x = 0 \text{ when } x = e^{-1}.$$

$$y'' = \frac{1}{x} > 0$$

 Relative minimum: $(e^{-1}, -e^{-1})$


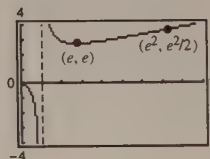
81. $y = \frac{x}{\ln x}$

 Domain: $0 < x < 1, x > 1$

$$y' = \frac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{(\ln x)^2(1/x) - (\ln x - 1)(2/x) \ln x}{(\ln x)^4} = \frac{2 - \ln x}{x(\ln x)^3} = 0 \text{ when } x = e^2.$$

 Relative minimum: (e, e)

 Point of inflection: $(e^2, e^2/2)$


83. $f(x) = \ln x, f(1) = 0$

$$f'(x) = \frac{1}{x}, f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2}, f''(1) = -1$$

$$P_1(x) = f(1) + f'(1)(x-1) = x-1, P_1(1) = 0$$

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$$

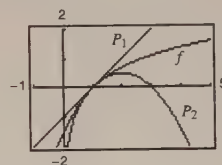
$$= (x-1) - \frac{1}{2}(x-1)^2, P_2(1) = 0$$

$$P_1'(x) = 1, P_1'(1) = 1$$

$$P_2'(x) = 1 - (x-1) = 2-x, P_2'(1) = 1$$

$$P_2''(x) = -1, P_2''(1) = -1$$

The values of f, P_1, P_2 , and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



85. Find x such that $\ln x = -x$.

$$f(x) = (\ln x) + x = 0$$

$$f'(x) = \frac{1}{x} + 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \left[\frac{1 - \ln x_n}{1 + x_n} \right]$$

n	1	2	3
x_n	0.5	0.5644	0.5671
$f(x_n)$	-0.1931	-0.0076	-0.0001

 Approximate root: $x = 0.567$

87. $y = x\sqrt{x^2 - 1}$

$$\ln y = \ln x + \frac{1}{2} \ln(x^2 - 1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} + \frac{x}{x^2 - 1}$$

$$\frac{dy}{dx} = y \left[\frac{2x^2 - 1}{x(x^2 - 1)} \right] = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$$

$$89. \quad y = \frac{x^2 \sqrt{3x-2}}{(x-1)^2}$$

$$\ln y = 2 \ln x + \frac{1}{2} \ln(3x-2) - 2 \ln(x-1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}$$

$$\frac{dy}{dx} = y \left[\frac{3x^2 - 15x + 8}{2x(3x-2)(x-1)} \right]$$

$$= \frac{3x^3 - 15x^2 + 8x}{2(x-1)^3 \sqrt{3x-2}}$$

$$91. \quad y = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}$$

$$\ln y = \ln x + \frac{3}{2} \ln(x-1) - \frac{1}{2} \ln(x+1)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} + \frac{3}{2} \left(\frac{1}{x-1} \right) - \frac{1}{2} \left(\frac{1}{x+1} \right)$$

$$\frac{dy}{dx} = \frac{y}{2} \left[\frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1} \right]$$

$$= \frac{y}{2} \left[\frac{4x^2 + 4x - 2}{x(x^2-1)} \right] = \frac{(2x^2 + 2x - 1)\sqrt{x-1}}{(x+1)^{3/2}}$$

93. Answers will vary. See Theorem 5.1 and 5.2.

95. $\ln e^x = x$ because $f(x) = \ln x$ and $g(x) = e^x$ are inverse functions.

97. (a) $f(1) \neq f(3)$

(b) $f'(x) = 1 - \frac{2}{x} = 0$ for $x = 2$.

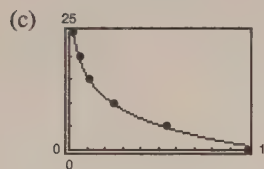
$$99. \quad \beta = 10 \log_{10} \left(\frac{I}{10^{-16}} \right) = \frac{10}{\ln 10} \ln \left(\frac{I}{10^{-16}} \right) = \frac{10}{\ln 10} [\ln I + 16 \ln 10] = 160 + 10 \log_{10} I$$

$$\beta(10^{-10}) = \frac{10}{\ln 10} [\ln 10^{-10} + 16 \ln 10] = \frac{10}{\ln 10} [-10 \ln 10 + 16 \ln 10] = \frac{10}{\ln 10} [6 \ln 10] = 60 \text{ decibels}$$

101. (a) You get an error message because $\ln h$ does not exist for $h = 0$.

(b) Reversing the data, you obtain

$$h = 0.8627 - 6.4474 \ln p.$$



(d) If $p = 0.75$, $h \approx 2.72$ km.

(e) If $h = 13$ km, $p \approx 0.15$ atmospheres.

(f) $h = 0.8627 - 6.4474 \ln p$

$$1 = -6.4474 \frac{1}{p} \frac{dp}{dh} \quad (\text{implicit differentiation})$$

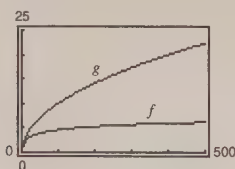
$$\frac{dp}{dh} = \frac{p}{-6.4474}$$

For $h = 5$, $p = 0.55$ and $dp/dh = -0.0853$ atmos/km.

For $h = 20$, $p = 0.06$ and $dp/dh = -0.00931$ atmos/km.

As the altitude increases, the rate of change of pressure decreases.

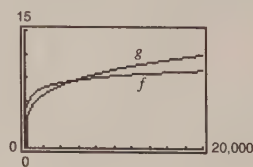
103. (a) $f(x) = \ln x$, $g(x) = \sqrt{x}$



$$f'(x) = \frac{1}{x}, \quad g'(x) = \frac{1}{2\sqrt{x}}$$

For $x > 4$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for "large" values of x .

(b) $f(x) = \ln x$, $g(x) = \sqrt[4]{x}$



$$f'(x) = \frac{1}{x}, \quad g'(x) = \frac{1}{4\sqrt[3]{x^3}}$$

For $x > 256$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for "large" values of x . $f(x) = \ln x$ increases very slowly for "large" values of x .

105. False

$$\ln x + \ln 25 = \ln(25x) \neq \ln(x + 25)$$

Section 5.2 The Natural Logarithmic Function: Integration

$$1. \int \frac{5}{x} dx = 5 \int \frac{1}{x} dx = 5 \ln |x| + C$$

$$3. u = x + 1, du = dx$$

$$\int \frac{1}{x+1} dx = \ln|x+1| + C$$

$$5. u = 3 - 2x, du = -2 dx$$

$$\begin{aligned} \int \frac{1}{3-2x} dx &= -\frac{1}{2} \int \frac{1}{3-2x} (-2) dx \\ &= -\frac{1}{2} \ln|3-2x| + C \end{aligned}$$

$$7. u = x^2 + 1, du = 2x dx$$

$$\begin{aligned} \int \frac{x}{x^2+1} dx &= \frac{1}{2} \int \frac{1}{x^2+1} (2x) dx \\ &= \frac{1}{2} \ln(x^2+1) + C \\ &= \ln\sqrt{x^2+1} + C \end{aligned}$$

$$\begin{aligned} 9. \int \frac{x^2-4}{x} dx &= \int \left(x - \frac{4}{x}\right) dx \\ &= \frac{x^2}{2} - 4 \ln|x| + C \end{aligned}$$

$$11. u = x^3 + 3x^2 + 9x, du = 3(x^2 + 2x + 3) dx$$

$$\begin{aligned} \int \frac{x^2+2x+3}{x^3+3x^2+9x} dx &= \frac{1}{3} \int \frac{3(x^2+2x+3)}{x^3+3x^2+9x} dx \\ &= \frac{1}{3} \ln|x^3+3x^2+9x| + C \end{aligned}$$

$$\begin{aligned} 13. \int \frac{x^2-3x+2}{x+1} dx &= \int \left(x - 4 + \frac{6}{x+1}\right) dx \\ &= \frac{x^2}{2} - 4x + 6 \ln|x+1| + C \end{aligned}$$

$$\begin{aligned} 15. \int \frac{x^3-3x^2+5}{x-3} dx &= \int \left(x^2 + \frac{5}{x-3}\right) dx \\ &= \frac{x^3}{3} + 5 \ln|x-3| + C \end{aligned}$$

$$\begin{aligned} 17. \int \frac{x^4+x-4}{x^2+2} dx &= \int \left(x^2 - 2 + \frac{x}{x^2+2}\right) dx \\ &= \frac{x^3}{3} - 2x + \frac{1}{2} \ln(x^2+2) + C \end{aligned}$$

$$19. u = \ln x, du = \frac{1}{x} dx$$

$$\int \frac{(\ln x)^2}{x} dx = \frac{1}{3} (\ln x)^3 + C$$

$$21. u = x + 1, du = dx$$

$$\begin{aligned} \int \frac{1}{\sqrt{x+1}} dx &= \int (x+1)^{-1/2} dx \\ &= 2(x+1)^{1/2} + C \\ &= 2\sqrt{x+1} + C \end{aligned}$$

$$\begin{aligned} 23. \int \frac{2x}{(x-1)^2} dx &= \int \frac{2x-2+2}{(x-1)^2} dx \\ &= \int \frac{2(x-1)}{(x-1)^2} dx + 2 \int \frac{1}{(x-1)^2} dx \\ &= 2 \int \frac{1}{x-1} dx + 2 \int \frac{1}{(x-1)^2} dx \\ &= 2 \ln|x-1| - \frac{2}{(x-1)} + C \end{aligned}$$

$$25. u = 1 + \sqrt{2x}, du = \frac{1}{\sqrt{2x}} dx \Rightarrow (u - 1) du = dx$$

$$\begin{aligned} \int \frac{1}{1 + \sqrt{2x}} dx &= \int \frac{(u - 1)}{u} du = \int \left(u - \frac{1}{u}\right) du \\ &= u - \ln|u| + C_1 \\ &= (1 + \sqrt{2x}) - \ln|1 + \sqrt{2x}| + C_1 \\ &= \sqrt{2x} - \ln(1 + \sqrt{2x}) + C \end{aligned}$$

where $C = C_1 + 1$.

$$27. u = \sqrt{x} - 3, du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2(u + 3) du = dx$$

$$\begin{aligned} \int \frac{\sqrt{x}}{\sqrt{x} - 3} dx &= 2 \int \frac{(u + 3)^2}{u} du = 2 \int \frac{u^2 + 6u + 9}{u} du = 2 \int \left(u + 6 + \frac{9}{u}\right) du \\ &= 2 \left[\frac{u^2}{2} + 6u + 9 \ln|u| \right] + C_1 = u^2 + 12u + 18 \ln|u| + C_1 \\ &= (\sqrt{x} - 3)^2 + 12(\sqrt{x} - 3) + 18 \ln|\sqrt{x} - 3| + C_1 \\ &= x + 6\sqrt{x} + 18 \ln|\sqrt{x} - 3| + C \text{ where } C = C_1 - 27. \end{aligned}$$

$$29. \int \frac{\cos \theta}{\sin \theta} d\theta = \ln|\sin \theta| + C$$

$(u = \sin \theta, du = \cos \theta d\theta)$

$$31. \int \csc 2x dx = \frac{1}{2} \int (\csc 2x)(2) dx$$

$$= -\frac{1}{2} \ln|\csc 2x + \cot 2x| + C$$

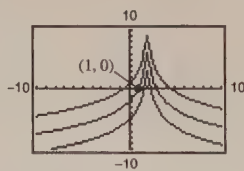
$$33. \int \frac{\cos t}{1 + \sin t} dt = \ln|1 + \sin t| + C$$

$$35. \int \frac{\sec x \tan x}{\sec x - 1} dx = \ln|\sec x - 1| + C$$

$$37. y = \int \frac{3}{2 - x} dx$$

$$= -3 \int \frac{1}{x - 2} dx$$

$$= -3 \ln|x - 2| + C$$



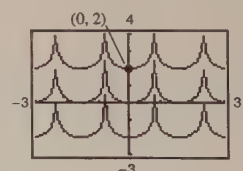
$$(1, 0): 0 = -3 \ln|1 - 2| + C \Rightarrow C = 0$$

$$y = -3 \ln|x - 2|$$

$$39. s = \int \tan(2\theta) d\theta$$

$$= \frac{1}{2} \int \tan(2\theta)(2 d\theta)$$

$$= -\frac{1}{2} \ln|\cos 2\theta| + C$$

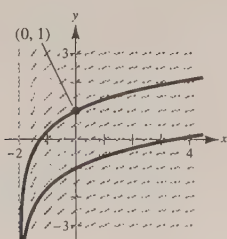


$$(0, 2): 2 = -\frac{1}{2} \ln|\cos(0)| + C \Rightarrow C = 2$$

$$s = -\frac{1}{2} \ln|\cos 2\theta| + 2$$

$$41. \frac{dy}{dx} = \frac{1}{x + 2}, (0, 1)$$

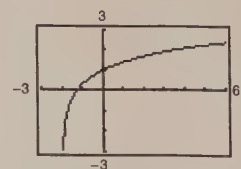
(a)



$$(b) y = \int \frac{1}{x + 2} dx = \ln|x + 2| + C$$

$$y(0) = 1 \Rightarrow 1 = \ln 2 + C \Rightarrow C = 1 - \ln 2$$

$$\text{Hence, } y = \ln|x + 2| + 1 - \ln 2 = \ln \left| \frac{x + 2}{2} \right| + 1.$$



$$43. \int_0^4 \frac{5}{3x+1} dx = \left[\frac{5}{3} \ln|3x+1| \right]_0^4 \\ = \frac{5}{3} \ln 13 \approx 4.275$$

$$47. \int_0^2 \frac{x^2-2}{x+1} dx = \int_0^2 \left(x-1 - \frac{1}{x+1} \right) dx \\ = \left[\frac{1}{2}x^2 - x - \ln|x+1| \right]_0^2 = -\ln 3$$

$$51. -\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$$

$$53. \ln|\sec x + \tan x| + C = \ln\left| \frac{(\sec x + \tan x)(\sec x - \tan x)}{(\sec x - \tan x)} \right| + C = \ln\left| \frac{\sec^2 x - \tan^2 x}{\sec x - \tan x} \right| + C \\ = \ln\left| \frac{1}{\sec x - \tan x} \right| + C = -\ln|\sec x - \tan x| + C$$

$$55. \int \frac{1}{1+\sqrt{x}} dx = 2(1+\sqrt{x}) - 2\ln(1+\sqrt{x}) + C_1 \\ = 2[\sqrt{x} - \ln(1+\sqrt{x})] + C \text{ where } C = C_1 + 2.$$

$$57. \int \cos(1-x) dx = -\sin(1-x) + C$$

$$59. \int_{\pi/4}^{\pi/2} (\csc x - \sin x) dx = \left[-\ln|\csc x + \cot x| + \cos x \right]_{\pi/4}^{\pi/2} = \ln(\sqrt{2}+1) - \frac{\sqrt{2}}{2} \approx 0.174$$

Note: In Exercises 61 and 63, you can use the Second Fundamental Theorem of Calculus or integrate the function.

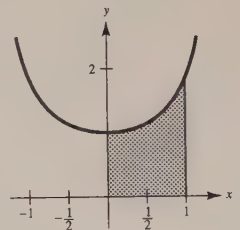
$$61. F(x) = \int_1^x \frac{1}{t} dt$$

$$F'(x) = \frac{1}{x}$$

$$63. F(x) = \int_x^{3x} \frac{1}{t} dt = \int_1^{3x} \frac{1}{t} dt - \int_1^x \frac{1}{t} dt$$

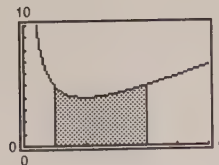
$$F'(x) = \frac{3}{3x} - \frac{1}{x} = 0$$

65.

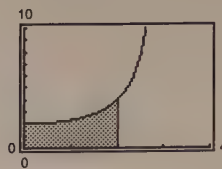

 $A \approx 1.25$

Matches (d)

$$67. A = \int_1^4 \frac{x^2+4}{x} dx = \int_1^4 \left(x + \frac{4}{x} \right) dx \\ = \left[\frac{x^2}{2} + 4 \ln x \right]_1^4 = (8 + 4 \ln 4) - \frac{1}{2} \\ = \frac{15}{2} + 8 \ln 2 \approx 13.045 \text{ square units}$$



$$\begin{aligned}
 69. \int_0^2 2 \sec \frac{\pi x}{6} dx &= \frac{12}{\pi} \int_0^2 \sec \left(\frac{\pi x}{6} \right) \frac{\pi}{6} dx \\
 &= \left[\frac{12}{\pi} \ln \left| \sec \frac{\pi x}{6} + \tan \frac{\pi x}{6} \right| \right]_0^2 \\
 &= \frac{12}{\pi} \ln \left| \sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right| - \frac{12}{\pi} \ln |1 + 0| \\
 &= \frac{12}{\pi} \ln(2 + \sqrt{3}) \approx 5.03041
 \end{aligned}$$



71. Power Rule

73. Substitution: ($u = x^2 + 4$)
and Log Rule

75. Divide the polynomials:

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

$$\begin{aligned}
 77. \text{Average value} &= \frac{1}{4-2} \int_2^4 \frac{8}{x^2} dx = 4 \int_2^4 x^{-2} dx \\
 &= \left[-4 \frac{1}{x} \right]_2^4 \\
 &= -4 \left(\frac{1}{4} - \frac{1}{2} \right) = 1
 \end{aligned}$$

$$\begin{aligned}
 79. \text{Average value} &= \frac{1}{e-1} \int_1^e \frac{\ln x}{x} dx = \frac{1}{e-1} \left[\frac{(\ln x)^2}{2} \right]_1^e \\
 &= \frac{1}{e-1} \left(\frac{1}{2} \right) \\
 &= \frac{1}{2e-2} \approx 0.291
 \end{aligned}$$

$$81. P(t) = \int \frac{3000}{1+0.25t} dt = (3000)(4) \int \frac{0.25}{1+0.25t} dt = 12,000 \ln|1+0.25t| + C$$

$$P(0) = 12,000 \ln|1+0.25(0)| + C = 1000$$

$$C = 1000$$

$$P(t) = 12,000 \ln|1+0.25t| + 1000 = 1000[12 \ln|1+0.25t| + 1]$$

$$P(3) = 1000[12(\ln 1.75) + 1] \approx 7715$$

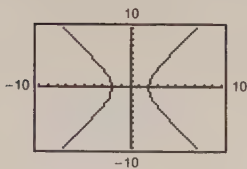
$$83. \frac{1}{50-40} \int_{40}^{50} \frac{90,000}{400+3x} dx = \left[3000 \ln|400+3x| \right]_{40}^{50} \approx \$168.27$$

$$85. (a) 2x^2 - y^2 = 8$$

$$y^2 = 2x^2 - 8$$

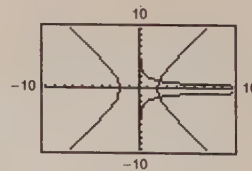
$$y_1 = \sqrt{2x^2 - 8}$$

$$y_2 = -\sqrt{2x^2 - 8}$$



$$(b) y^2 = e^{-\int (1/x) dx} = e^{-\ln x + C} = e^{\ln(1/x)} (e^C) = \frac{1}{x} k$$

$$\text{Let } k = 4 \text{ and graph } y^2 = \frac{4}{x} \quad \left(\begin{array}{l} y_1 = 2/\sqrt{x} \\ y_2 = -2/\sqrt{x} \end{array} \right)$$



$$(c) \text{ In part (a), } 2x^2 - y^2 = 8$$

$$4x - 2yy' = 0$$

$$y' = \frac{2x}{y}$$

$$\text{In part (b), } y^2 = \frac{4}{x} = 4x^{-1}$$

$$2yy' = \frac{-4}{x^2}$$

$$y' = \frac{-2}{yx^2} = \frac{-2y}{y^2 x^2} = \frac{-2y}{4x} = \frac{-y}{2x}$$

Using a graphing utility the graphs intersect at (2.214, 1.344). The slopes are 3.295 and $-0.304 = (-1)/3.295$, respectively.

87. False

$$\frac{1}{2}(\ln x) = \ln(x^{1/2}) \neq (\ln x)^{1/2}$$

89. True

$$\begin{aligned}\int \frac{1}{x} dx &= \ln|x| + C_1 \\ &= \ln|x| + \ln|C| = \ln|Cx|, \quad C \neq 0\end{aligned}$$

Section 5.3 Inverse Functions

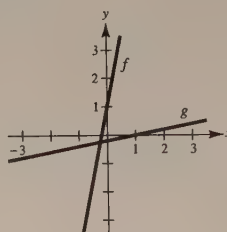
1. (a) $f(x) = 5x + 1$

$$g(x) = \frac{x-1}{5}$$

$$f(g(x)) = f\left(\frac{x-1}{5}\right) = 5\left(\frac{x-1}{5}\right) + 1 = x$$

$$g(f(x)) = g(5x+1) = \frac{(5x+1)-1}{5} = x$$

(b)



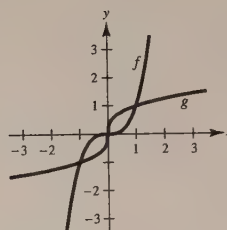
3. (a) $f(x) = x^3$

$$g(x) = \sqrt[3]{x}$$

$$f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

(b)



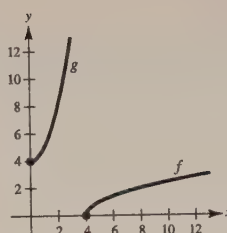
5. (a) $f(x) = \sqrt{x-4}$

$$g(x) = x^2 + 4, \quad x \geq 0$$

$$\begin{aligned}f(g(x)) &= f(x^2 + 4) \\ &= \sqrt{(x^2 + 4) - 4} = \sqrt{x^2} = x\end{aligned}$$

$$\begin{aligned}g(f(x)) &= g(\sqrt{x-4}) \\ &= (\sqrt{x-4})^2 + 4 = x - 4 + 4 = x\end{aligned}$$

(b)



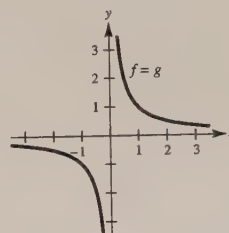
7. (a) $f(x) = \frac{1}{x}$

$$g(x) = \frac{1}{x}$$

$$f(g(x)) = \frac{1}{1/x} = x$$

$$g(f(x)) = \frac{1}{1/x} = x$$

(b)

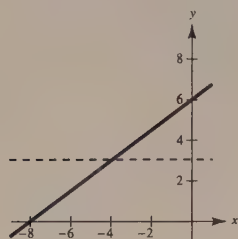


9. Matches (c)

11. Matches (a)

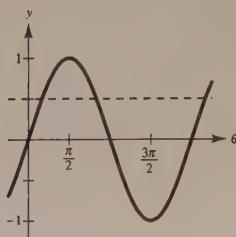
13. $f(x) = \frac{3}{4}x + 6$

One-to-one; has an inverse



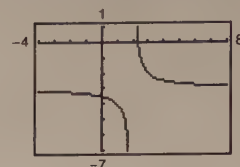
15. $f(\theta) = \sin \theta$

Not one-to-one; does not have an inverse



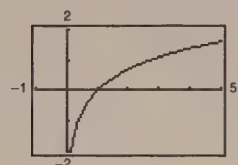
17. $h(s) = \frac{1}{s-2} - 3$

One-to-one; has an inverse



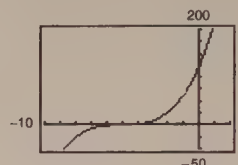
19. $f(x) = \ln x$

One-to-one; has an inverse



21. $g(x) = (x + 5)^3$

One-to-one; has an inverse



23. $f(x) = (x + a)^3 + b$

$$f'(x) = 3(x + a)^2 \geq 0 \text{ for all } x.$$

 f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

25. $f(x) = \frac{x^4}{4} - 2x^2$

$$f'(x) = x^3 - 4x = 0 \text{ when } x = 0, 2, -2.$$

 f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

27. $f(x) = 2 - x - x^3$

$$f'(x) = -1 - 3x^2 < 0 \text{ for all } x.$$

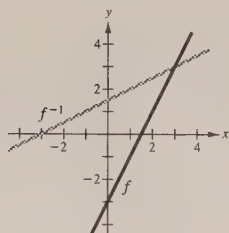
 f is decreasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

29. $f(x) = 2x - 3 = y$

$$x = \frac{y + 3}{2}$$

$$y = \frac{x + 3}{2}$$

$$f^{-1}(x) = \frac{x + 3}{2}$$

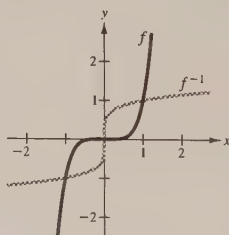


31. $f(x) = x^5 = y$

$$x = \sqrt[5]{y}$$

$$y = \sqrt[5]{x}$$

$$f^{-1}(x) = \sqrt[5]{x} = x^{1/5}$$

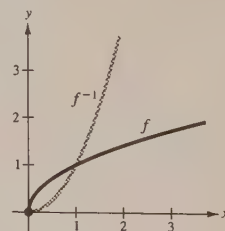


33. $f(x) = \sqrt{x} = y$

$$x = y^2$$

$$y = x^2$$

$$f^{-1}(x) = x^2, \quad x \geq 0$$

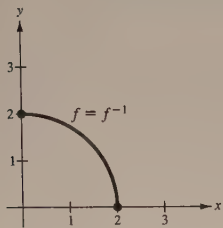


35. $f(x) = \sqrt{4 - x^2} = y, \quad 0 \leq x \leq 2$

$$x = \sqrt{4 - y^2}$$

$$y = \sqrt{4 - x^2}$$

$$f^{-1}(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$$

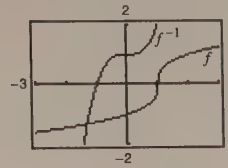


37. $f(x) = \sqrt[3]{x - 1} = y$

$$x = y^3 + 1$$

$$y = x^3 + 1$$

$$f^{-1}(x) = x^3 + 1$$



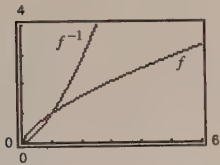
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

39. $f(x) = x^{2/3} = y, \quad x \geq 0$

$$x = y^{3/2}$$

$$y = x^{3/2}$$

$$f^{-1}(x) = x^{3/2}, \quad x \geq 0$$



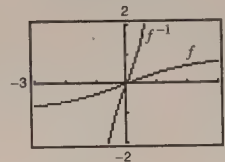
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

41. $f(x) = \frac{x}{\sqrt{x^2 + 7}} = y$

$$x = \frac{\sqrt{7}y}{\sqrt{1 - y^2}}$$

$$y = \frac{\sqrt{7}x}{\sqrt{1 - x^2}}$$

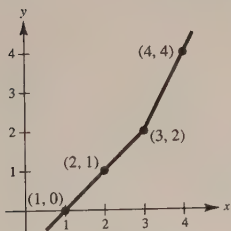
$$f^{-1}(x) = \frac{\sqrt{7}x}{\sqrt{1 - x^2}}, \quad -1 < x < 1$$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

 43.

x	1	2	3	4
$f^{-1}(x)$	0	1	2	4



45. (a) Let
- x
- be the number of pounds of the commodity costing 1.25 per pound. Since there are 50 pounds total, the amount of the second commodity is
- $50 - x$
- . The total cost is

$$y = 1.25x + 1.60(50 - x)$$

$$= -0.35x + 80 \quad 0 \leq x \leq 50.$$

- (b) We find the inverse of the original function:

$$y = -0.35x + 80$$

$$0.35x = 80 - y$$

$$x = \frac{100}{35}(80 - y)$$

$$\text{Inverse: } y = \frac{100}{35}(80 - x) = \frac{20}{7}(80 - x).$$

x represents cost and y represents pounds.

- (c) Domain of inverse is
- $62.5 \leq x \leq 80$
- .

- (d) If
- $x = 73$
- in the inverse function,
-
- $y = \frac{100}{35}(80 - 73) = \frac{100}{5} = 20$
- pounds.

47. $f(x) = (x - 4)^2$ on $[4, \infty)$

$$f'(x) = 2(x - 4) > 0 \text{ on } (4, \infty)$$

f is increasing on $[4, \infty)$. Therefore, f is strictly monotonic and has an inverse.

51. $f(x) = \cos x$ on $[0, \pi]$

$$f'(x) = -\sin x < 0 \text{ on } (0, \pi)$$

f is decreasing on $[0, \pi]$. Therefore, f is strictly monotonic and has an inverse.

53. $f(x) = \frac{x}{x^2 - 4} = y$ on $(-2, 2)$

$$x^2y - 4y = x$$

$$x^2y - x - 4y = 0$$

$$a = y, b = -1, c = -4y$$

$$x = \frac{1 \pm \sqrt{1 - 4(y)(-4y)}}{2y} = \frac{1 \pm \sqrt{1 + 16y^2}}{2y}$$

$$y = f^{-1}(x) = \begin{cases} (1 - \sqrt{1 + 16x^2})/2x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

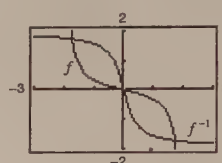
49. $f(x) = \frac{4}{x^2}$ on $(0, \infty)$

$$f'(x) = -\frac{8}{x^3} < 0 \text{ on } (0, \infty)$$

f is decreasing on $(0, \infty)$. Therefore, f is strictly monotonic and has an inverse.

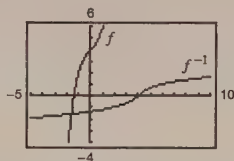
Domain of f^{-1} : all x

Range of f^{-1} : $-2 < y < 2$



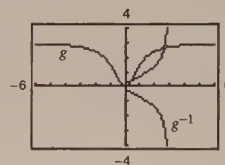
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

55. (a), (b)



(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

57. (a), (b)



(c) g is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

59. $f(x) = \sqrt{x - 2}$, Domain: $x \geq 2$

$$f'(x) = \frac{1}{2\sqrt{x - 2}} > 0 \text{ for } x > 2.$$

f is one-to-one; has an inverse

$$\sqrt{x - 2} = y$$

$$x - 2 = y^2$$

$$x = y^2 + 2$$

$$y = x^2 + 2$$

$$f^{-1}(x) = x^2 + 2, x \geq 0$$

61. $f(x) = |x - 2|, x \leq 2$

$$= -(x - 2)$$

$$= 2 - x$$

f is one-to-one; has an inverse

$$2 - x = y$$

$$2 - y = x$$

$$f^{-1}(x) = 2 - x, x \geq 0$$

63. $f(x) = (x - 3)^2$ is one-to-one for $x \geq 3$.

$$(x - 3)^2 = y$$

$$x - 3 = \sqrt{y}$$

$$x = \sqrt{y} + 3$$

$$y = \sqrt{x} + 3$$

$$f^{-1}(x) = \sqrt{x} + 3, x \geq 0$$

(Answer is not unique)

65. $f(x) = |x + 3|$ is one-to-one for $x \geq -3$.

$$x + 3 = y$$

$$x = y - 3$$

$$y = x - 3$$

$$f^{-1}(x) = x - 3, x \geq 0$$

(Answer is not unique)

67. Yes, the volume is an increasing function, and hence one-to-one. The inverse function gives the time t corresponding to the volume V .

71. $f(x) = x^3 + 2x - 1, f(1) = 2 = a$

$$f'(x) = 3x^2 + 2$$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 2} = \frac{1}{5}$$

75. $f(x) = x^3 - \frac{4}{x}, f(2) = 6 = a$

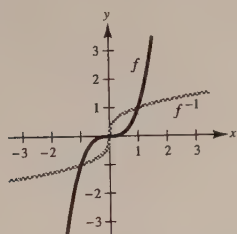
$$f'(x) = 3x^2 + \frac{4}{x^2}$$

$$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{3(2)^2 + (4/2^2)} = \frac{1}{13}$$

77. (a) Domain $f = \text{Domain } f^{-1} = (-\infty, \infty)$

(b) Range $f = \text{Range } f^{-1} = (-\infty, \infty)$

(c)



(d) $f(x) = x^3, \left(\frac{1}{2}, \frac{1}{8}\right)$

$$f'(x) = 3x^2$$

$$f'\left(\frac{1}{2}\right) = \frac{3}{4}$$

$$f^{-1}(x) = \sqrt[3]{x}, \left(\frac{1}{8}, \frac{1}{2}\right)$$

$$(f^{-1})'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

$$(f^{-1})'\left(\frac{1}{8}\right) = \frac{4}{3}$$

69. No, $C(t)$ is not one-to-one because long distance costs are step functions. A call lasting 2.1 minutes costs the same as one lasting 2.2 minutes.

73. $f(x) = \sin x, f\left(\frac{\pi}{6}\right) = \frac{1}{2} = a$

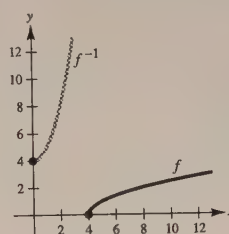
$$f'(x) = \cos x$$

$$\begin{aligned} (f^{-1})'\left(\frac{1}{2}\right) &= \frac{1}{f'(f^{-1}(1/2))} = \frac{1}{f'(\pi/6)} = \frac{1}{\cos(\pi/6)} \\ &= \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3} \end{aligned}$$

79. (a) Domain $f = [4, \infty), \text{Domain } f^{-1} = [0, \infty)$

(b) Range $f = [0, \infty), \text{Range } f^{-1} = [4, \infty)$

(c)



(d) $f(x) = \sqrt{x-4}, (5, 1)$

$$f'(x) = \frac{1}{2\sqrt{x-4}}$$

$$f'(5) = \frac{1}{2}$$

$$f^{-1}(x) = x^2 + 4, (1, 5)$$

$$(f^{-1})'(x) = 2x$$

$$(f^{-1})'(1) = 2$$

81. $x = y^3 - 7y^2 + 2$

$$1 = 3y^2 \frac{dy}{dx} - 14y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3y^2 - 14y}. \text{ At } (-4, 1), \frac{dy}{dx} = \frac{1}{3 - 14} = \frac{-1}{11}.$$

Alternate solution: let $f(x) = x^3 - 7x^2 + 2$.Then $f'(x) = 3x^2 - 14x$ and $f'(1) = -11$.

$$\text{Hence, } \frac{dy}{dx} = \frac{1}{-11} = \frac{-1}{11}.$$

In Exercises 83 and 85, use the following.

$$f(x) = \frac{1}{8}x - 3 \text{ and } g(x) = x^3$$

$$f^{-1}(x) = 8(x + 3) \text{ and } g^{-1}(x) = \sqrt[3]{x}$$

83. $(f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}(1) = 32$

85. $(f^{-1} \circ f^{-1})(6) = f^{-1}(f^{-1}(6)) = f^{-1}(72) = 600$

In Exercises 87 and 89, use the following.

$$f(x) = x + 4 \text{ and } g(x) = 2x - 5$$

$$f^{-1}(x) = x - 4 \text{ and } g^{-1}(x) = \frac{x + 5}{2}$$

$$\begin{aligned}
 87. (g^{-1} \circ f^{-1})(x) &= g^{-1}(f^{-1}(x)) \\
 &= g^{-1}(x - 4) \\
 &= \frac{(x - 4) + 5}{2} \\
 &= \frac{x + 1}{2}
 \end{aligned}$$

$$\begin{aligned}
 89. (f \circ g)(x) &= f(g(x)) \\
 &= f(2x - 5) \\
 &= (2x - 5) + 4 \\
 &= 2x - 1 \\
 \text{Hence, } (f \circ g)^{-1}(x) &= \frac{x + 1}{2} \\
 (\text{Note: } (f \circ g)^{-1} &= g^{-1} \circ f^{-1})
 \end{aligned}$$

91. Answers will vary. See page 335 and Example 3.

93. $y = x^2$ on $(-\infty, \infty)$ does not have an inverse.95. f is not one-to-one because many different x -values yield the same y -value.

Example: $f(0) = f(\pi) = 0$

Not continuous at $\frac{(2n-1)\pi}{2}$, where n is an integer97. Let $(f \circ g)(x) = y$ then $x = (f \circ g)^{-1}(y)$. Also,

$$\begin{aligned}
 (f \circ g)(x) &= y \\
 f(g(x)) &= y \\
 g(x) &= f^{-1}(y) \\
 x &= g^{-1}(f^{-1}(y)) \\
 &= (g^{-1} \circ f^{-1})(y)
 \end{aligned}$$

Since f and g are one-to-one functions,
 $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.99. Suppose $g(x)$ and $h(x)$ are both inverses of $f(x)$. Then the graph of $f(x)$ contains the point (a, b) if and only if the graphs of $g(x)$ and $h(x)$ contain the point (b, a) . Since the graphs of $g(x)$ and $h(x)$ are the same, $g(x) = h(x)$. Therefore, the inverse of $f(x)$ is unique.

101. False

$$\text{Let } f(x) = x^2.$$

105. Not true

$$\text{Let } f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1 - x, & 1 < x \leq 2 \end{cases}$$

f is one-to-one, but not strictly monotonic.

103. True

$$107. \quad f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}, \quad f(2) = 0$$

$$f'(x) = \frac{1}{\sqrt{1+x^4}}$$

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{1/\sqrt{17}} = \sqrt{17}$$

Section 5.4 Exponential Functions: Differentiation and Integration

1. $e^0 = 1$

$\ln 1 = 0$

3. $\ln 2 = 0.6931 \dots$

$e^{0.6931 \dots} = 2$

5. $e^{\ln x} = x$

$x = e^4$

7. $e^x = 12$

$x = \ln 12 \approx 2.485$

9. $9 - 2e^x = 7$

$2e^x = 2$

$e^x = 1$

$x = 0$

11. $50e^{-x} = 30$

$e^{-x} = \frac{3}{5}$

$-x = \ln\left(\frac{3}{5}\right)$

$x = \ln\left(\frac{5}{3}\right) \approx 0.511$

13. $\ln x = 2$

$x = e^2 \approx 7.389$

15. $\ln(x - 3) = 2$

$x - 3 = e^2$

$x = 3 + e^2 \approx 10.389$

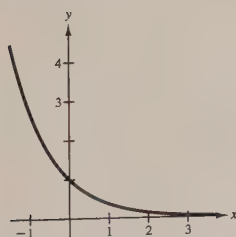
17. $\ln\sqrt{x+2} = 1$

$\sqrt{x+2} = e^1 = e$

$x + 2 = e^2$

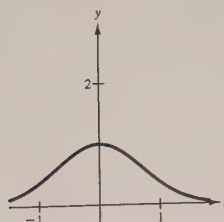
$x = e^2 - 2 \approx 5.389$

19. $y = e^{-x}$

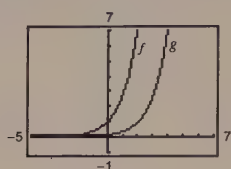


21. $y = e^{-x^2}$

Symmetric with respect to the y-axis

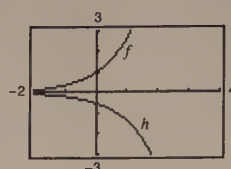
Horizontal asymptote: $y = 0$ 

23. (a)



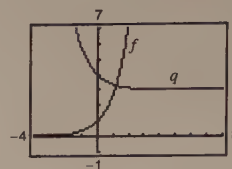
Horizontal shift 2 units to the right

(b)



A reflection in the x -axis and a vertical shrink

(c)



Vertical shift 3 units upward and a reflection in the y -axis

25. $y = Ce^{ax}$

Horizontal asymptote: $y = 0$

Matches (c)

27. $y = C(1 - e^{-ax})$

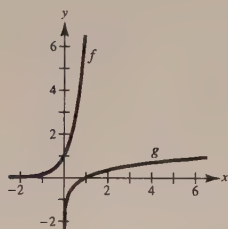
Vertical shift C units

Reflection in both the x - and y -axes

Matches (a)

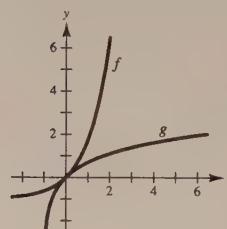
29. $f(x) = e^{2x}$

$$g(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$$

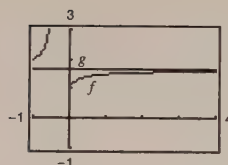


31. $f(x) = e^x - 1$

$$g(x) = \ln(x + 1)$$



33.



As $x \rightarrow \infty$, the graph of f approaches the graph of g .

$$\lim_{x \rightarrow \infty} \left(1 + \frac{0.5}{x}\right)^x = e^{0.5}$$

35. $\left(1 + \frac{1}{1,000,000}\right)^{1,000,000} \approx 2.718280469$

$$e \approx 2.718281828$$

$$e > \left(1 + \frac{1}{1,000,000}\right)^{1,000,000}$$

37. (a) $y = e^{3x}$

$$y' = 3e^{3x}$$

$$\text{At } (0, 1), y' = 3.$$

(b) $y = e^{-3x}$

$$y' = -3e^{-3x}$$

$$\text{At } (0, 1), y' = -3.$$

39. $f(x) = e^{2x}$

$$f'(x) = 2e^{2x}$$

41. $f(x) = e^{-2x+x^2}$

$$\frac{dy}{dx} = 2(x - 1)e^{-2x+x^2}$$

43. $y = e^{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

45. $g(t) = (e^{-t} + e^t)^3$

$$g'(t) = 3(e^{-t} + e^t)^2(e^t - e^{-t})$$

47. $y = \ln e^{x^2} = x^2$

$$\frac{dy}{dx} = 2x$$

49. $y = \ln(1 + e^{2x})$

$$\frac{dy}{dx} = \frac{2e^{2x}}{1 + e^{2x}}$$

$$51. y = \frac{2}{e^x + e^{-x}} = 2(e^x + e^{-x})^{-1}$$

$$\begin{aligned}\frac{dy}{dx} &= -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) \\ &= \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2}\end{aligned}$$

$$55. f(x) = e^{-x} \ln x$$

$$f'(x) = e^{-x} \left(\frac{1}{x} \right) - e^{-x} \ln x = e^{-x} \left(\frac{1}{x} - \ln x \right)$$

$$59. xe^y - 10x + 3y = 0$$

$$xe^y \frac{dy}{dx} + e^y - 10 + 3 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} (xe^y + 3) = 10 - e^y$$

$$\frac{dy}{dx} = \frac{10 - e^y}{xe^y + 3}$$

$$63. y = e^x (\cos \sqrt{2}x + \sin \sqrt{2}x)$$

$$y' = e^x (-\sqrt{2} \sin \sqrt{2}x + \sqrt{2} \cos \sqrt{2}x) + e^x (\cos \sqrt{2}x + \sin \sqrt{2}x)$$

$$= e^x [(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x]$$

$$y'' = e^x [-(\sqrt{2} + 2) \sin \sqrt{2}x + (\sqrt{2} - 2) \cos \sqrt{2}x] + e^x [(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x]$$

$$= e^x [(-1 - 2\sqrt{2}) \sin \sqrt{2}x + (-1 + 2\sqrt{2}) \cos \sqrt{2}x]$$

$$-2y' + 3y = -2e^x [(1 + \sqrt{2}) \cos \sqrt{2}x + (1 - \sqrt{2}) \sin \sqrt{2}x] + 3e^x [\cos \sqrt{2}x + \sin \sqrt{2}x]$$

$$= e^x [(1 - 2\sqrt{2}) \cos \sqrt{2}x + (1 + 2\sqrt{2}) \sin \sqrt{2}x] = -y''$$

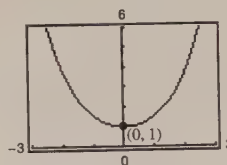
$$\text{Therefore, } -2y' + 3y = -y'' \Rightarrow y'' - 2y' + 3y = 0.$$

$$65. f(x) = \frac{e^x + e^{-x}}{2}$$

$$f'(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{e^x + e^{-x}}{2} > 0$$

Relative minimum: (0, 1)



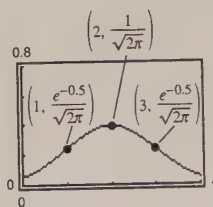
$$67. g(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-2)^2/2}$$

$$g'(x) = \frac{-1}{\sqrt{2\pi}} (x-2) e^{-(x-2)^2/2}$$

$$g''(x) = \frac{1}{\sqrt{2\pi}} (x-1)(x-3) e^{-(x-2)^2/2}$$

Relative maximum: $\left(2, \frac{1}{\sqrt{2\pi}}\right) \approx (2, 0.399)$

Points of inflection: $\left(1, \frac{1}{\sqrt{2\pi}} e^{-1/2}\right), \left(3, \frac{1}{\sqrt{2\pi}} e^{-1/2}\right) \approx (1, 0.242), (3, 0.242)$



$$53. y = x^2 e^x - 2x e^x + 2e^x = e^x (x^2 - 2x + 2)$$

$$\frac{dy}{dx} = e^x (2x - 2) + e^x (x^2 - 2x + 2) = x^2 e^x$$

$$57. y = e^x (\sin x + \cos x)$$

$$\frac{dy}{dx} = e^x (\cos x - \sin x) + (\sin x + \cos x) (e^x)$$

$$= e^x (2 \cos x) = 2e^x \cos x$$

$$61. f(x) = (3 + 2x)e^{-3x}$$

$$f'(x) = (3 + 2x)(-3e^{-3x}) + 2e^{-3x}$$

$$= (-7 - 6x)e^{-3x}$$

$$f''(x) = (-7 - 6x)(-3e^{-3x}) - 6e^{-3x}$$

$$= 3(6x + 5)e^{-3x}$$

69. $f(x) = x^2 e^{-x}$

$$f'(x) = -x^2 e^{-x} + 2x e^{-x} = x e^{-x}(2 - x) = 0 \text{ when } x = 0, 2.$$

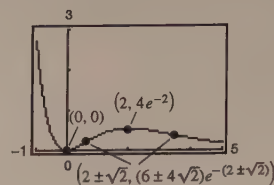
$$f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x)$$

$$= e^{-x}(x^2 - 4x + 2) = 0 \text{ when } x = 2 \pm \sqrt{2}.$$

Relative minimum: $(0, 0)$ Relative maximum: $(2, 4e^{-2})$

$$x = 2 \pm \sqrt{2}$$

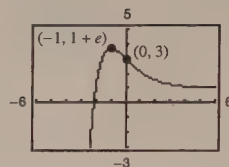
$$y = (2 \pm \sqrt{2})^2 e^{-(2 \pm \sqrt{2})}$$

Points of inflection: $(3.414, 0.384), (0.586, 0.191)$ 

71. $g(t) = 1 + (2 + t)e^{-t}$

$$g'(t) = -(1 + t)e^{-t}$$

$$g''(t) = te^{-t}$$

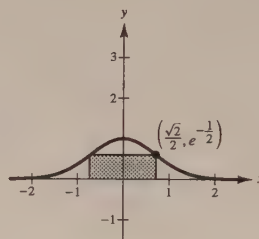
Relative maximum: $(-1, 1 + e) \approx (-1, 3.718)$ Point of inflection: $(0, 3)$ 

73. $A = (\text{base})(\text{height}) = 2xe^{-x^2}$

$$\frac{dA}{dx} = -4x^2 e^{-x^2} + 2e^{-x^2}$$

$$= 2e^{-x^2}(1 - 2x^2) = 0 \text{ when } x = \frac{\sqrt{2}}{2}.$$

$$A = \sqrt{2}e^{-1/2}$$



75. $y = \frac{L}{1 + ae^{-x/b}}, a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^2} = \frac{\frac{aL}{b}e^{-x/b}}{(1 + ae^{-x/b})^2}$$

$$y'' = \frac{(1 + ae^{-x/b})^2 \left(\frac{-aL}{b^2}e^{-x/b}\right) - \left(\frac{aL}{b}e^{-x/b}\right) 2(1 + ae^{-x/b})\left(\frac{-a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^4}$$

$$= \frac{(1 + ae^{-x/b})\left(\frac{-aL}{b^2}e^{-x/b}\right) + 2\left(\frac{aL}{b}e^{-x/b}\right)\left(\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^3}$$

$$= \frac{La e^{-x/b} [ae^{-x/b} - 1]}{(1 + ae^{-x/b})^3 b^2}$$

$$y'' = 0 \text{ if } ae^{-x/b} = 1 \Rightarrow \frac{-x}{b} = \ln\left(\frac{1}{a}\right) \Rightarrow x = b \ln a$$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

Therefore, the y -coordinate of the inflection point is $L/2$.

77. $e^{-x} = x \Rightarrow f(x) = x - e^{-x}$

$$f'(x) = 1 + e^{-x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}$$

$$x_1 = 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 0.5379$$

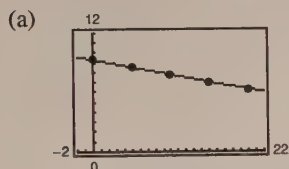
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.5670$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 0.5671$$

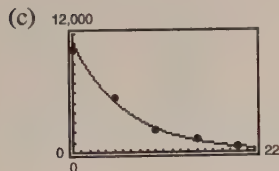
We approximate the root of f to be $x = 0.567$.

81.

h	0	5	10	15	20
P	10,332	5,583	2,376	1,240	517
$\ln P$	9.243	8.627	7.773	7.123	6.248



$y = -0.1499h + 9.3018$ is the regression line for data $(h, \ln P)$.



83. $f(x) = e^{x/2}, f(0) = 1$

$$f'(x) = \frac{1}{2}e^{x/2}, f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{x/2}, f''(0) = \frac{1}{4}$$

$$P_1(x) = 1 + \frac{1}{2}(x - 0) = \frac{x}{2} + 1, P_1(0) = 1$$

$$P_1'(x) = \frac{1}{2}, P_1'(0) = \frac{1}{2}$$

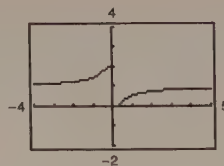
$$P_2(x) = 1 + \frac{1}{2}(x - 0) + \frac{1}{8}(x - 0)^2 = \frac{x^2}{8} + \frac{x}{2} + 1, P_2(0) = 1$$

$$P_2'(x) = \frac{1}{4}x + \frac{1}{2}, P_2'(0) = \frac{1}{2}$$

$$P_2''(x) = \frac{1}{4}, P_2''(0) = \frac{1}{4}$$

The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.

79. (a)



(b) When x increases without bound, $1/x$ approaches zero, and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. Thus, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞ , $e^{1/x}$ approaches ∞ and $f(x)$ approaches zero. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist since the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

(b) $\ln P = ah + b$

$$P = e^{ah+b} = e^b e^{ah}$$

$$P = Ce^{ah}, C = e^b$$

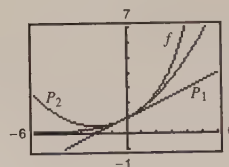
For our data, $a = -0.1499$ and $C = e^{9.3018} = 10,957.7$

$$P = 10,957.7e^{-0.1499h}$$

(d) $\frac{dP}{dh} = (10,957.71)(-0.1499)e^{-0.1499h}$

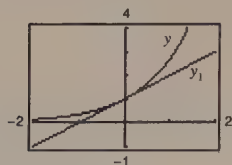
$$= -1642.56e^{-0.1499h}$$

For $h = 5$, $\frac{dP}{dh} = -776.3$. For $h = 18$, $\frac{dP}{dh} \approx -110.6$.



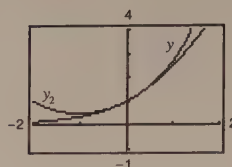
85. (a) $y = e^x$

$y_1 = 1 + x$



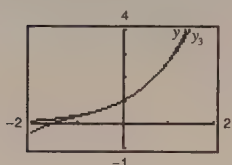
(b) $y = e^x$

$y_2 = 1 + x + \left(\frac{x^2}{2}\right)$



(c) $y = e^x$

$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$



87. Let $u = 5x$, $du = 5 dx$.

$$\int e^{5x} 5 dx = e^{5x} + C$$

89. Let $u = -2x$, $du = -2 dx$.

$$\begin{aligned} \int_0^1 e^{-2x} dx &= -\frac{1}{2} \int_0^1 e^{-2x} (-2) dx = \left[-\frac{1}{2} e^{-2x} \right]_0^1 \\ &= \frac{1}{2} (1 - e^{-2}) = \frac{e^2 - 1}{2e^2} \end{aligned}$$

91. $\int x e^{-x^2} dx = -\frac{1}{2} \int e^{-x^2} (-2x) dx = -\frac{1}{2} e^{-x^2} + C$

93. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2 \int e^{\sqrt{x}} \left(\frac{1}{2\sqrt{x}} \right) dx = 2e^{\sqrt{x}} + C$

95. Let $u = 1 + e^{-x}$, $du = -e^{-x} dx$.

$$\int \frac{e^{-x}}{1 + e^{-x}} dx = -\int \frac{-e^{-x}}{1 + e^{-x}} dx = -\ln(1 + e^{-x}) + C = \ln\left(\frac{e^x}{e^x + 1}\right) + C = x - \ln(e^x + 1) + C$$

97. Let $u = \frac{3}{x}$, $du = -\frac{3}{x^2} dx$.

$$\begin{aligned} \int_1^3 \frac{e^{3/x}}{x^2} dx &= -\frac{1}{3} \int_1^3 e^{3/x} \left(-\frac{3}{x^2} \right) dx \\ &= \left[-\frac{1}{3} e^{3/x} \right]_1^3 = \frac{e}{3} (e^2 - 1) \end{aligned}$$

99. Let $u = 1 - e^x$, $du = -e^x dx$.

$$\begin{aligned} \int e^x \sqrt{1 - e^x} dx &= -\int (1 - e^x)^{1/2} (-e^x) dx \\ &= -\frac{2}{3} (1 - e^x)^{3/2} + C \end{aligned}$$

101. Let $u = e^x - e^{-x}$, $du = (e^x + e^{-x}) dx$.

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \ln|e^x - e^{-x}| + C$$

$$\begin{aligned} 103. \int \frac{5 - e^x}{e^{2x}} dx &= \int 5e^{-2x} dx - \int e^{-x} dx \\ &= -\frac{5}{2} e^{-2x} + e^{-x} + C \end{aligned}$$

$$\begin{aligned} 105. \int e^{\sin \pi x} \cos \pi x dx &= \frac{1}{\pi} \int e^{\sin \pi x} (\pi \cos \pi x) dx \\ &= \frac{1}{\pi} e^{\sin \pi x} + C \end{aligned}$$

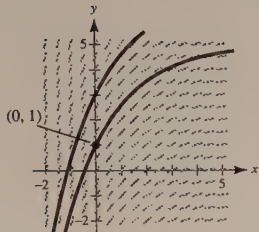
$$\begin{aligned} 107. \int e^{-x} \tan(e^{-x}) dx &= -\int [\tan(e^{-x})] (-e^{-x}) dx \\ &= \ln|\cos(e^{-x})| + C \end{aligned}$$

109. Let $u = ax^2$, $du = 2ax \, dx$. (Assume $a \neq 0$)

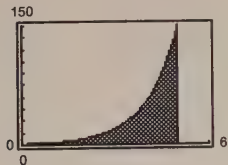
$$y = \int x e^{ax^2} dx$$

$$= \frac{1}{2a} \int e^{ax^2} (2ax) dx = \frac{1}{2a} e^{ax^2} + C$$

113. (a)



115. $\int_0^5 e^x dx = [e^x]_0^5 = e^5 - 1 \approx 147.413$



119. (a) $f(u - v) = e^{u-v} = (e^u)(e^{-v}) = \frac{e^u}{e^v} = \frac{f(u)}{f(v)}$

(b) $f(kx) = e^{kx} = (e^x)^k = [f(x)]^k$

123. $\int_0^x e^t dt \geq \int_0^x 1 dt$

$$\left[e^t \right]_0^x \geq \left[t \right]_0^x$$

$$e^x - 1 \geq x \Rightarrow e^x \geq 1 + x \text{ for } x \geq 0$$

127. Yes. $f(x) = Ce^x$, C a constant.

111. $f'(x) = \int \frac{1}{2} (e^x + e^{-x}) dx = \frac{1}{2} (e^x - e^{-x}) + C_1$

$$f'(0) = C_1 = 0$$

$$f(x) = \int \frac{1}{2} (e^x - e^{-x}) dx = \frac{1}{2} (e^x + e^{-x}) + C_2$$

$$f(0) = 1 + C_2 = 1 \Rightarrow C_2 = 0$$

$$f(x) = \frac{1}{2} (e^x + e^{-x})$$

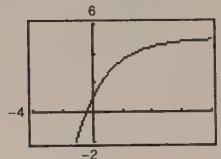
(b) $\frac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$

$$y = \int 2e^{-x/2} dx = -4 \int e^{-x/2} \left(-\frac{1}{2} dx \right)$$

$$= -4e^{-x/2} + C$$

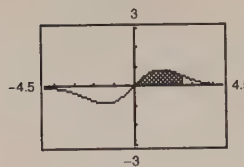
$$(0, 1): 1 = -4e^0 + C = -4 + C \Rightarrow C = 5$$

$$y = -4e^{-x/2} + 5$$



117. $\int_0^{\sqrt{6}} x e^{-x^2/4} dx = \left[-2e^{-x^2/4} \right]_0^{\sqrt{6}}$

$$= -2e^{-3/2} + 2 \approx 1.554$$



121. $0.0665 \int_{48}^{60} e^{-0.0139(t-48)^2} dt$

Graphing Utility: $0.4772 = 47.72\%$

125. $f(x) = e^x$. Domain is $(-\infty, \infty)$ and range is $(0, \infty)$. f is continuous, increasing, one-to-one, and concave upwards on its entire domain.

$$\lim_{x \rightarrow -\infty} e^x = 0 \text{ and } \lim_{x \rightarrow \infty} e^x = \infty$$

129. $e^{-x} > 0 \Rightarrow \int_0^2 e^{-x} dx > 0$

131. $f(x) = \frac{\ln x}{x}$

(a) $f'(x) = \frac{1 - \ln x}{x^2} = 0$ when $x = e$.

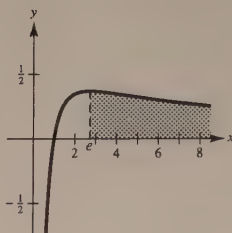
On $(0, e)$, $f'(x) > 0 \Rightarrow f$ is increasing.On (e, ∞) , $f'(x) < 0 \Rightarrow f$ is decreasing.(b) For $e \leq A < B$, we have:

$$\frac{\ln A}{A} > \frac{\ln B}{B}$$

$$B \ln A > A \ln B$$

$$\ln A^B > \ln B^A$$

$$A^B > B^A.$$

(c) Since $e < \pi$, from part (b) we have $e^\pi > \pi^e$.Section 5.5 Bases Other than e and Applications

1. $y = \left(\frac{1}{2}\right)^{t/3}$

At $t_0 = 6$, $y = \left(\frac{1}{2}\right)^{6/3} = \frac{1}{4}$

3. $y = \left(\frac{1}{2}\right)^{t/7}$

At $t_0 = 10$, $y = \left(\frac{1}{2}\right)^{10/7} \approx 0.3715$

5. $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

7. $\log_7 1 = 0$

9. (a) $2^3 = 8$

$\log_2 8 = 3$

(b) $3^{-1} = \frac{1}{3}$

$\log_3 \frac{1}{3} = -1$

11. (a) $\log_{10} 0.01 = -2$

$10^{-2} = 0.01$

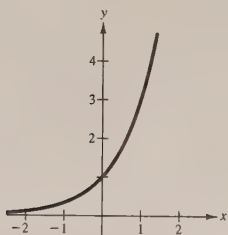
(b) $\log_{0.5} 8 = -3$

$0.5^{-3} = 8$

$\left(\frac{1}{2}\right)^{-3} = 8$

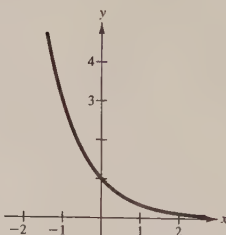
13. $y = 3^x$

x	-2	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



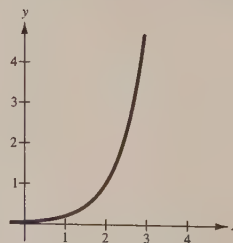
15. $y = \left(\frac{1}{3}\right)^x = 3^{-x}$

x	-2	-1	0	1	2
y	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$



17. $h(x) = 5^{x-2}$

x	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5



19. (a) $\log_{10} 1000 = x$

$$10^x = 1000$$

$$x = 3$$

(b) $\log_{10} 0.1 = x$

$$10^x = 0.1$$

$$x = -1$$

21. (a) $\log_3 x = -1$

$$3^{-1} = x$$

$$x = \frac{1}{3}$$

(b) $\log_2 x = -4$

$$2^{-4} = x$$

$$x = \frac{1}{16}$$

23. (a) $x^2 - x = \log_5 25$

$$x^2 - x = \log_5 5^2 = 2$$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

$$x = -1 \text{ OR } x = 2$$

(b) $3x + 5 = \log_2 64$

$$3x + 5 = \log_2 2^6 = 6$$

$$3x = 1$$

$$x = \frac{1}{3}$$

25. $3^{2x} = 75$

$$2x \ln 3 = \ln 75$$

$$x = \left(\frac{1}{2}\right) \frac{\ln 75}{\ln 3} \approx 1.965$$

27. $2^{3-z} = 625$

$$(3-z) \ln 2 = \ln 625$$

$$3-z = \frac{\ln 625}{\ln 2}$$

$$z = 3 - \frac{\ln 625}{\ln 2} \approx -6.288$$

29. $\left(1 + \frac{0.09}{12}\right)^{12t} = 3$

$$12t \ln\left(1 + \frac{0.09}{12}\right) = \ln 3$$

$$t = \left(\frac{1}{12}\right) \frac{\ln 3}{\ln\left(1 + \frac{0.09}{12}\right)} \approx 12.253$$

31. $\log_2(x-1) = 5$

$$x-1 = 2^5 = 32$$

$$x = 33$$

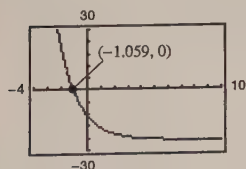
33. $\log_3 x^2 = 4.5$

$$x^2 = 3^{4.5}$$

$$x = \pm \sqrt{3^{4.5}} \approx \pm 11.845$$

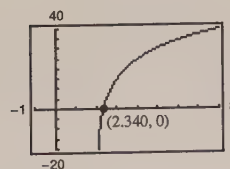
35. $g(x) = 6(2^{1-x}) - 25$

$$\text{Zero: } x \approx -1.059$$



37. $h(s) = 32 \log_{10}(s-2) + 15$

$$\text{Zero: } s \approx 2.340$$

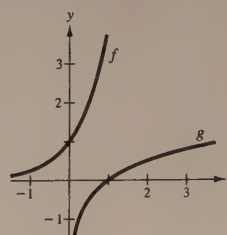


39. $f(x) = 4^x$

$g(x) = \log_4 x$

x	-2	-1	0	$\frac{1}{2}$	1
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4

x	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4
$g(x)$	-2	-1	0	$\frac{1}{2}$	1



41. $f(x) = 4^x$

$f'(x) = (\ln 4) 4^x$

43. $y = 5^{x-2}$

$\frac{dy}{dx} = (\ln 5) 5^{x-2}$

45. $g(t) = t^2 2^t$

$g'(t) = t^2 (\ln 2) 2^t + (2t) 2^t$

$= t 2^t (t \ln 2 + 2)$

$= 2^t t(2 + t \ln 2)$

47. $h(\theta) = 2^{-\theta} \cos \pi \theta$

$h'(\theta) = 2^{-\theta}(-\pi \sin \pi \theta) - (\ln 2) 2^{-\theta} \cos \pi \theta$

$= -2^{-\theta}[(\ln 2) \cos \pi \theta + \pi \sin \pi \theta]$

49. $y = \log_3 x$

$\frac{dy}{dx} = \frac{1}{x \ln 3}$

51. $f(x) = \log_2 \frac{x^2}{x-1}$

$= 2 \log_2 x - \log_2 (x-1)$

$f'(x) = \frac{2}{x \ln 2} - \frac{1}{(x-1) \ln 2}$

$= \frac{x-2}{(\ln 2)x(x-1)}$

53. $y = \log_5 \sqrt{x^2 - 1} = \frac{1}{2} \log_5 (x^2 - 1)$

$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{2x}{(x^2 - 1) \ln 5} = \frac{x}{(x^2 - 1) \ln 5}$

55. $g(t) = \frac{10 \log_4 t}{t} = \frac{10}{\ln 4} \left(\frac{\ln t}{t} \right)$

$g'(t) = \frac{10}{\ln 4} \left[\frac{t(1/t) - \ln t}{t^2} \right]$

$= \frac{10}{t^2 \ln 4} [1 - \ln t] = \frac{5}{t^2 \ln 2} (1 - \ln t)$

57. $y = x^{2/x}$

$\ln y = \frac{2}{x} \ln x$

$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{2}{x} \left(\frac{1}{x} \right) + \ln x \left(-\frac{2}{x^2} \right) = \frac{2}{x^2} (1 - \ln x)$

$\frac{dy}{dx} = \frac{2y}{x^2} (1 - \ln x) = 2x^{(2/x)-2} (1 - \ln x)$

59. $y = (x-2)^{x+1}$

$\ln y = (x+1) \ln(x-2)$

$\frac{1}{y} \left(\frac{dy}{dx} \right) = (x+1) \left(\frac{1}{x-2} \right) + \ln(x-2)$

$\frac{dy}{dx} = y \left[\frac{x+1}{x-2} + \ln(x-2) \right]$

$= (x-2)^{x+1} \left[\frac{x+1}{x-2} + \ln(x-2) \right]$

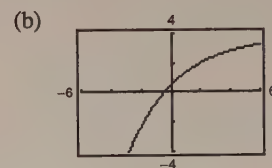
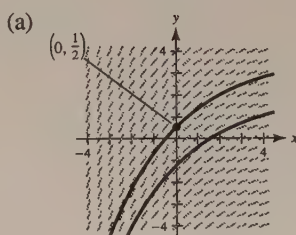
61. $\int 3^x dx = \frac{3^x}{\ln 3} + C$

$$\begin{aligned}
 63. \int_{-1}^2 2^x dx &= \left[\frac{2^x}{\ln 2} \right]_{-1}^2 \\
 &= \frac{1}{\ln 2} \left[4 - \frac{1}{2} \right] \\
 &= \frac{7}{2 \ln 2} = \frac{7}{\ln 4}
 \end{aligned}$$

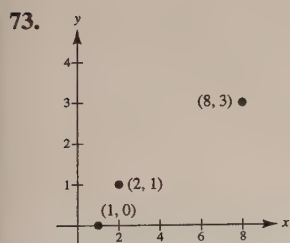
$$\begin{aligned}
 67. \int \frac{3^{2x}}{1 + 3^{2x}} dx, u = 1 + 3^{2x}, du = 2(\ln 3)3^{2x} dx \\
 \frac{1}{2 \ln 3} \int \frac{(2 \ln 3)3^{2x}}{1 + 3^{2x}} dx = \frac{1}{2 \ln 3} \ln(1 + 3^{2x}) + C
 \end{aligned}$$

$$\begin{aligned}
 69. \frac{dy}{dx} &= 0.4^{x/3}, \left(0, \frac{1}{2}\right) \\
 y &= \int 0.4^{x/3} dx = 3 \int 0.4^{x/3} \left(\frac{1}{3} dx\right) \\
 &= \frac{3}{\ln 0.4} 0.4^{x/3} + C = 3(\ln 2.5)(0.4)^{x/3} + C \\
 \frac{1}{2} &= 3(\ln 2.5) + C \Rightarrow C = \frac{1}{2} - 3 \ln 2.5 \\
 y &= 3(\ln 2.5)(0.4)^{x/3} + \frac{1}{2} - 3 \ln 2.5 \\
 &= \frac{3(1 - 0.4^{x/3})}{\ln 2.5} + \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 65. \int x(5^{-x^2}) dx &= -\frac{1}{2} \int 5^{-x^2} (-2x) dx \\
 &= -\left(\frac{1}{2}\right) \frac{5^{-x^2}}{\ln 5} + C \\
 &= \frac{-1}{2 \ln 5} (5^{-x^2}) + C
 \end{aligned}$$



71. Answers will vary. Example: Growth and decay problems



x	1	2	8
y	0	1	3

- (a) y is an exponential function of x : False
 (b) y is a logarithmic function of x : True; $y = \log_2 x$
 (c) x is an exponential function of y : True, $2^y = x$
 (d) y is a linear function of x : False

$$75. f(x) = \log_2 x \Rightarrow f'(x) = \frac{1}{x \ln 2}$$

$$g(x) = x^x \Rightarrow g'(x) = x^x(1 + \ln x)$$

Note: Let $y = g(x)$. Then: $\ln y = \ln x^x = x \ln x$

$$\frac{1}{y} y' = x \cdot \frac{1}{x} + \ln x$$

$$y' = y(1 + \ln x)$$

$$y' = x^x(1 + \ln x) = g'(x)$$

$$h(x) = x^2 \Rightarrow h'(x) = 2x$$

$$k(x) = 2^x \Rightarrow k'(x) = (\ln 2)2^x$$

From greatest to smallest rate of growth:

$$g(x), k(x), h(x), f(x)$$

$$77. C(t) = P(1.05)^t$$

$$\begin{aligned}
 (a) C(10) &= 24.95(1.05)^{10} \\
 &\approx \$40.64
 \end{aligned}$$

$$\begin{aligned}
 (b) \frac{dC}{dt} &= P(\ln 1.05)(1.05)^t \\
 \text{When } t = 1, \frac{dC}{dt} &\approx 0.051P. \\
 \text{When } t = 8, \frac{dC}{dt} &\approx 0.072P.
 \end{aligned}$$

$$\begin{aligned}
 (c) \frac{dC}{dt} &= (\ln 1.05)[P(1.05)^t] \\
 &= (\ln 1.05)C(t)
 \end{aligned}$$

The constant of proportionality is $\ln 1.05$.

79. $P = \$1000$, $r = 3\frac{1}{2}\% = 0.035$, $t = 10$

$$A = 1000 \left(1 + \frac{0.035}{n} \right)^{10n}$$

$$A = 1000e^{(0.035)(10)} = 1419.07$$

n	1	2	4	12	365	Continuous
A	1410.60	1414.78	1416.91	1418.34	1419.04	1419.07

81. $P = \$1000$, $r = 5\% = 0.05$, $t = 30$

$$A = 1000 \left(1 + \frac{0.05}{n} \right)^{30n}$$

$$A = 1000e^{(0.05)(30)} = 4481.69$$

n	1	2	4	12	365	Continuous
A	4321.94	4399.79	4440.21	4467.74	4481.23	4481.69

83. $100,000 = Pe^{0.05t} \Rightarrow P = 100,000e^{-0.05t}$

t	1	10	20	30	40	50
P	95,122.94	60,653.07	36,787.94	22,313.02	13,533.53	8208.50

85. $100,000 = P \left(1 + \frac{0.05}{12} \right)^{12t} \Rightarrow P = 100,000 \left(1 + \frac{0.05}{12} \right)^{-12t}$

t	1	10	20	30	40	50
P	95,132.82	60,716.10	36,864.45	22,382.66	13,589.88	8251.24

87. (a) $A = 20,000 \left(1 + \frac{0.06}{365} \right)^{(365)(8)} \approx \$32,320.21$

(b) $A = \$30,000$

(c) $A = 8000 \left(1 + \frac{0.06}{365} \right)^{(365)(8)} + 20,000 \left(1 + \frac{0.06}{365} \right)^{(365)(4)}$
 $\approx \$12,928.09 + 25,424.48 = \$38,352.57$

(d) $A = 9000 \left[\left(1 + \frac{0.06}{365} \right)^{(365)(8)} + \left(1 + \frac{0.06}{365} \right)^{(365)(4)} + 1 \right]$
 $\approx \$34,985.11$

Take option (c).

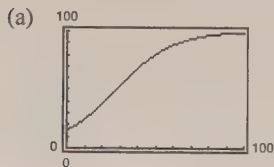
89. (a) $\lim_{t \rightarrow \infty} 6.7e^{-(48.1)/t} = 6.7e^0 = 6.7$ million ft^3

(b) $V' = \frac{322.27}{t^2} e^{-(48.1)/t}$

$V'(20) \approx 0.073$ million ft^3/yr

$V'(60) \approx 0.040$ million ft^3/yr

91. $y = \frac{300}{3 + 17e^{-0.0625x}}$



(b) If $x = 2$ (2000 egg masses), $y \approx 16.67 \approx 16.7\%$.

(c) If $y = 66.67\%$, then $x \approx 38.8$ or 38,800 egg masses.

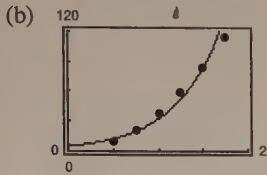
(d) $y = 300(3 + 17e^{-0.0625x})^{-1}$

$$y' = \frac{318.75e^{-0.0625x}}{(3 + 17e^{-0.0625x})^2}$$

$$y'' = \frac{19.921875e^{-0.0625x}(17e^{-0.0625x} - 3)}{(3 + 17e^{-0.0625x})^3}$$

$17e^{-0.0625x} - 3 = 0 \Rightarrow x \approx 27.8$ or 27,800 egg masses.

93. (a) $B = 4.7539(6.7744)^d = 4.7539e^{1.9132d}$



(c) $B'(d) = 9.0952e^{1.9132d}$

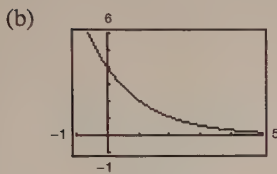
$B'(0.8) \approx 42.03$ tons/inch

$B'(1.5) \approx 160.38$ tons/inch

95. (a) $\int_0^4 f(t) dt \approx 5.67$

$\int_0^4 g(t) dt \approx 5.67$

$\int_0^4 h(t) dt \approx 5.67$



97. $P = \int_0^{10} 2000e^{-0.06t} dt$

$= \left[\frac{2000}{-0.06} e^{-0.06t} \right]_0^{10}$

$\approx \$15,039.61$

(c) The functions appear to be equal: $f(t) = g(t) = h(t)$

Analytically,

$f(t) = 4\left(\frac{3}{8}\right)^{2t/3} = 4\left[\left(\frac{3}{8}\right)^{2/3}\right]^t = 4\left(\frac{9^{1/3}}{4}\right)^t = g(t)$

$h(t) = 4e^{-0.653886t} = 4[e^{-0.653886}]^t \approx 4(0.52002)^t$

$g(t) = 4\left(\frac{9^{1/3}}{4}\right)^t \approx 4(0.52002)^t$

No. The definite integrals over a given interval may be equal when the functions are not equal.

99.

t	0	1	2	3	4
y	1200	720	432	259.20	155.52

$y = C(k^t)$

When $t = 0, y = 1200 \Rightarrow C = 1200$.

$y = 1200(k^t)$

$\frac{720}{1200} = 0.6, \frac{432}{720} = 0.6, \frac{259.20}{432} = 0.6, \frac{155.52}{259.20} = 0.6$

Let $k = 0.6$.

$y = 1200(0.6)^t$

101. False. e is an irrational number.

103. True.

$f(g(x)) = 2 + e^{\ln(x-2)}$
 $= 2 + x - 2 = x$

$g(f(x)) = \ln(2 + e^x - 2)$
 $= \ln e^x = x$

105. True.

$\frac{d}{dx}[e^x] = e^x$ and $\frac{d}{dx}[e^{-x}] = -e^{-x}$

$e^x = e^{-x}$ when $x = 0$.

$(e^0)(-e^{-0}) = -1$

$$107. \quad \frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), y(0) = 1$$

$$\frac{dy}{y[(5/4) - y]} = \frac{8}{25} dt \Rightarrow \frac{4}{5} \int \left(\frac{1}{y} + \frac{1}{(5/4) - y} \right) dy = \int \frac{8}{25} dt \Rightarrow$$

$$\ln y - \ln\left(\frac{5}{4} - y\right) = \frac{2}{5}t + C$$

$$\ln\left(\frac{y}{(5/4) - y}\right) = \frac{2}{5}t + C$$

$$\frac{y}{(5/4) - y} = e^{(2/5)t+C} = C_1 e^{(2/5)t}$$

$$y(0) = 1 \Rightarrow C_1 = 4 \Rightarrow 4e^{(2/5)t} = \frac{y}{(5/4) - y}$$

$$\Rightarrow 4e^{(2/5)t}\left(\frac{5}{4} - y\right) = y \Rightarrow 5e^{(2/5)t} = 4e^{(2/5)t}y + y = (4e^{(2/5)t} + 1)y$$

$$\Rightarrow y = \frac{5e^{(2/5)t}}{4e^{(2/5)t} + 1} = \frac{5}{4 + e^{-0.4t}} = \frac{1.25}{1 + 0.25e^{-0.4t}}$$

Section 5.6 Differential Equations: Growth and Decay

$$1. \quad \frac{dy}{dx} = x + 2$$

$$y = \int (x + 2)dx = \frac{x^2}{2} + 2x + C$$

$$3. \quad \frac{dy}{dx} = y + 2$$

$$\frac{dy}{y + 2} = dx$$

$$\int \frac{1}{y + 2} dy = \int dx$$

$$\ln|y + 2| = x + C_1$$

$$y + 2 = e^{x+C_1} = Ce^x$$

$$y = Ce^x - 2$$

$$5. \quad y' = \frac{5x}{y}$$

$$yy' = 5x$$

$$\int yy' dx = \int 5x dx$$

$$\int y dy = \int 5x dx$$

$$\frac{1}{2}y^2 = \frac{5}{2}x^2 + C_1$$

$$y^2 - 5x^2 = C$$

$$7. \quad y' = \sqrt{x}y$$

$$\frac{y'}{y} = \sqrt{x}$$

$$\int \frac{y'}{y} dx = \int \sqrt{x} dx$$

$$\int \frac{dy}{y} = \int \sqrt{x} dx$$

$$\ln|y| = \frac{2}{3}x^{3/2} + C_1$$

$$y = e^{(2/3)x^{3/2}+C_1}$$

$$= e^{C_1} e^{(2/3)x^{3/2}}$$

$$= Ce^{(2/3)x^{3/2}}$$

9. $(1 + x^2)y' - 2xy = 0$

$$y' = \frac{2xy}{1 + x^2}$$

$$\frac{y'}{y} = \frac{2x}{1 + x^2}$$

$$\int \frac{y'}{y} dx = \int \frac{2x}{1 + x^2} dx$$

$$\int \frac{dy}{y} = \int \frac{2x}{1 + x^2} dx$$

$$\ln|y| = \ln(1 + x^2) + C_1$$

$$\ln|y| = \ln(1 + x^2) + \ln C$$

$$\ln|y| = \ln[C(1 + x^2)]$$

$$y = C(1 + x^2)$$

11. $\frac{dQ}{dt} = \frac{k}{t^2}$

$$\int \frac{dQ}{dt} dt = \int \frac{k}{t^2} dt$$

$$\int dQ = -\frac{k}{t} + C$$

$$Q = -\frac{k}{t} + C$$

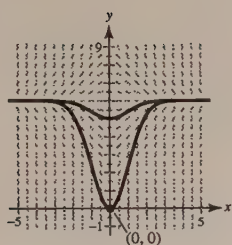
13. $\frac{dN}{ds} = k(250 - s)$

$$\int \frac{dN}{ds} ds = \int k(250 - s) ds$$

$$\int dN = -\frac{k}{2}(250 - s)^2 + C$$

$$N = -\frac{k}{2}(250 - s)^2 + C$$

15. (a)



(b) $\frac{dy}{dx} = x(6 - y), (0, 0)$

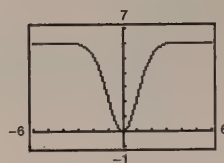
$$\frac{dy}{y - 6} = -x dx$$

$$\ln|y - 6| = \frac{-x^2}{2} + C$$

$$y - 6 = e^{-x^2/2 + C} = C_1 e^{-x^2/2}$$

$$y = 6 + C_1 e^{-x^2/2}$$

$$(0, 0): 0 = 6 + C_1 \Rightarrow C_1 = -6 \Rightarrow y = 6 - 6e^{-x^2/2}$$



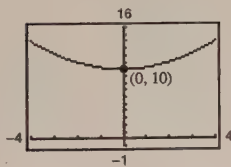
17. $\frac{dy}{dt} = \frac{1}{2}t, (0, 10)$

$$\int dy = \int \frac{1}{2}t dt$$

$$y = \frac{1}{4}t^2 + C$$

$$10 = \frac{1}{4}(0)^2 + C \Rightarrow C = 10$$

$$y = \frac{1}{4}t^2 + 10$$



19. $\frac{dy}{dt} = -\frac{1}{2}y, (0, 10)$

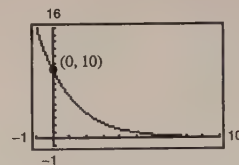
$$\int \frac{dy}{y} = \int -\frac{1}{2} dt$$

$$\ln|y| = -\frac{1}{2}t + C_1$$

$$y = e^{-(t/2) + C_1} = e^{C_1} e^{-t/2} = C e^{-t/2}$$

$$10 = C e^0 \Rightarrow C = 10$$

$$y = 10e^{-t/2}$$



21. $\frac{dy}{dx} = ky$

$$y = Ce^{kx} \text{ (Theorem 5.16)}$$

$$(0, 4): 4 = Ce^0 = C$$

$$(3, 10): 10 = 4e^{3k} \Rightarrow k = \frac{1}{3} \ln\left(\frac{5}{2}\right)$$

$$\text{When } x = 6, y = 4e^{1/3 \ln(5/2)(6)} = 4e^{\ln(5/2)^2}$$

$$= 4\left(\frac{5}{2}\right)^2 = 25$$

23. $\frac{dV}{dt} = kV$

$$V = Ce^{kt} \text{ (Theorem 5.16)}$$

$$(0, 20,000): C = 20,000$$

$$(4, 12,500): 12,500 = 20,000e^{4k} \Rightarrow k = \frac{1}{4} \ln\left(\frac{5}{8}\right)$$

$$\text{When } t = 6, V = 20,000e^{1/4 \ln(5/8)(6)} = 20,000e^{\ln(5/8)^{3/2}}$$

$$= 20,000\left(\frac{5}{8}\right)^{3/2} \approx 9882.118$$

$$25. y = Ce^{kt}, \left(0, \frac{1}{2}\right), (5, 5)$$

$$C = \frac{1}{2}$$

$$y = \frac{1}{2}e^{kt}$$

$$5 = \frac{1}{2}e^{5k}$$

$$k = \frac{\ln 10}{5} \approx 0.4605$$

$$y = \frac{1}{2}e^{0.4605t}$$

$$27. y = Ce^{kt}, (1, 1), (5, 5)$$

$$1 = Ce^k$$

$$5 = Ce^{5k}$$

$$5Ce^k = Ce^{5k}$$

$$5e^k = e^{5k}$$

$$5 = e^{4k}$$

$$k = \frac{\ln 5}{4} \approx 0.4024$$

$$y = Ce^{0.4024t}$$

$$1 = Ce^{0.4024}$$

$$C \approx 0.6687$$

$$y = 0.6687e^{0.4024t}$$

29. A differential equation in x and y is an equation that involves x , y and derivatives of y .

$$31. \frac{dy}{dx} = \frac{1}{2}xy$$

$$\frac{dy}{dx} > 0 \text{ when } xy > 0. \text{ Quadrants I and III.}$$

33. Since the initial quantity is 10 grams, $y = 10e^{[\ln(1/2)/1620]t}$. When $t = 1000$, $y = 10e^{[\ln(1/2)/1620](1000)} \approx 6.52$ grams. When $t = 10,000$, $y = 10e^{[\ln(1/2)/1620](10,000)} \approx 0.14$ gram.

35. Since $y = Ce^{[\ln(1/2)/1620]t}$, we have $0.5 = Ce^{[\ln(1/2)/1620](10,000)} \Rightarrow C \approx 36.07$.

Initial quantity: 36.07 grams.

When $t = 1000$, we have $y = Ce^{[\ln(1/2)/1620](1000)} \approx 23.51$ grams.

37. Since the initial quantity is 5 grams, we have $y = 5.0e^{[\ln(1/2)/5730]t}$.

When $t = 1000$, $y \approx 4.43$ g.

When $t = 10,000$, $y \approx 1.49$ g.

39. Since $y = Ce^{[\ln(1/2)/24,360]t}$, we have $2.1 = Ce^{[\ln(1/2)/24,360](1000)} \Rightarrow C \approx 2.16$. Thus, the initial quantity is 2.16 grams. When $t = 10,000$, $y = 2.16e^{[\ln(1/2)/24,360](10,000)} \approx 1.63$ grams.

41. Since $\frac{dy}{dx} = ky$, $y = Ce^{kt}$ or $y = y_0e^{kt}$.

$$\frac{1}{2}y_0 = y_0e^{1620k}$$

$$k = \frac{-\ln 2}{1620}$$

$$y = y_0e^{-(\ln 2)t/1620}$$

When $t = 100$, $y = y_0e^{-(\ln 2)/16.2} \approx y_0(0.9581)$.

Therefore, 95.81% of the present amount still exists.

43. Since $A = 1000e^{0.06t}$, the time to double is given by $2000 = 1000e^{0.06t}$ and we have

$$2 = e^{0.06t}$$

$$\ln 2 = 0.06t$$

$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years.}$$

Amount after 10 years: $A = 1000e^{(0.06)(10)} \approx \1822.12

45. Since $A = 750e^{rt}$ and $A = 1500$ when $t = 7.75$, we have the following.

$$1500 = 750e^{7.75r}$$

$$r = \frac{\ln 2}{7.75} \approx 0.0894 = 8.94\%$$

Amount after 10 years: $A = 750e^{0.0894(10)} \approx \1833.67

47. Since $A = 500e^{rt}$ and $A = 1292.85$ when $t = 10$, we have the following.

$$1292.85 = 500e^{10r}$$

$$r = \frac{\ln(1292.85/500)}{10} \approx 0.0950 = 9.50\%$$

The time to double is given by

$$1000 = 500e^{0.0950t}$$

$$t = \frac{\ln 2}{0.095} \approx 7.30 \text{ years.}$$

49. $500,000 = P\left(1 + \frac{0.075}{12}\right)^{(12)(20)}$

$$P = 500,000\left(1 + \frac{0.075}{12}\right)^{-240}$$

$$\approx \$112,087.09$$

51. $500,000 = P\left(1 + \frac{0.08}{12}\right)^{(12)(35)}$

$$P = 500,000\left(1 + \frac{0.08}{12}\right)^{-420}$$

$$= \$30,688.87$$

53. (a) $2000 = 1000(1 + 0.07)^t$

$$2 = 1.07^t$$

$$\ln 2 = t \ln 1.07$$

$$t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.07}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.007}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.07}{12}\right)$$

$$t = \frac{\ln 2}{12 \ln(1 + (0.07/12))} \approx 9.93 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.07}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.07}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.07}{365}\right)$$

$$t = \frac{\ln 2}{365 \ln(1 + (0.07/365))} \approx 9.90 \text{ years}$$

(d) $2000 = 1000e^{(0.07)t}$

$$2 = e^{0.07t}$$

$$\ln 2 = 0.07t$$

$$t = \frac{\ln 2}{0.07} \approx 9.90 \text{ years}$$

55. (a) $2000 = 1000(1 + 0.085)^t$

$$2 = 1.085^t$$

$$\ln 2 = t \ln 1.085$$

$$t = \frac{\ln 2}{\ln 1.085} \approx 8.50 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.085}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.085}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.085}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \frac{0.085}{12}\right)} \approx 8.18 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.085}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.085}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.085}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.085}{365}\right)} \approx 8.16 \text{ years}$$

(d) $2000 = 1000e^{0.085t}$

$$2 = e^{0.085t}$$

$$\ln 2 = 0.085t$$

$$t = \frac{\ln 2}{0.085} \approx 8.15 \text{ years}$$

57. $P = Ce^{kt} = Ce^{-0.009t}$

$$P(-1) = 8.2 = Ce^{-0.009(-1)} \Rightarrow C = 8.1265$$

$$P = 8.1265e^{-0.009t}$$

$$P(10) \approx 7.43 \quad \text{or} \quad 7,430,000 \text{ people in 2010}$$

61. If $k < 0$, the population decreases.If $k > 0$, the population increases.

65. (a) $19 = 30(1 - e^{20k})$

$$30e^{20k} = 11$$

$$k = \frac{\ln(11/30)}{20} \approx -0.0502$$

$$N \approx 30(1 - e^{-0.0502t})$$

67. $S = Ce^{k/t}$

(a) $S = 5$ when $t = 1$

$$5 = Ce^k$$

$$\lim_{t \rightarrow \infty} Ce^{k/t} = C = 30$$

$$5 = 30e^k$$

$$k = \ln \frac{1}{6} \approx -1.7918$$

$$S \approx 30e^{-1.7918/t}$$

69. $A(t) = V(t)e^{-0.10t} = 100,000e^{0.8\sqrt{t}}e^{-0.10t} = 100,000e^{0.8\sqrt{t}-0.10t}$

$$\frac{dA}{dt} = 100,000 \left(\frac{0.4}{\sqrt{t}} - 0.10 \right) e^{0.8\sqrt{t}-0.10t} = 0 \text{ when } 16.$$

The timber should be harvested in the year 2014, (1998 + 16). **Note:** You could also use a graphing utility to graph $A(t)$ and find the maximum of $A(t)$. Use the viewing rectangle $0 \leq x \leq 30$ and $0 \leq y \leq 600,000$.

71. $\beta(I) = 10 \log_{10} \frac{I}{I_0}, I_0 = 10^{-16}$

(a) $\beta(10^{-14}) = 10 \log_{10} \frac{10^{-14}}{10^{-16}} = 20 \text{ decibels}$

(b) $\beta(10^{-9}) = 10 \log_{10} \frac{10^{-9}}{10^{-16}} = 70 \text{ decibels}$

(c) $\beta(10^{-6.5}) = 10 \log_{10} \frac{10^{-6.5}}{10^{-16}} = 95 \text{ decibels}$

(d) $\beta(10^{-4}) = 10 \log_{10} \frac{10^{-4}}{10^{-16}} = 120 \text{ decibels}$

59. $P = Ce^{kt} = Ce^{0.036t}$

$$P(-1) = 4.6 = Ce^{0.036(-1)} \Rightarrow C = 4.7686$$

$$P = 4.7686e^{0.036t}$$

$$P(10) \approx 6.83 \quad \text{or} \quad 6,830,000 \text{ people in 2010}$$

63. $P = Ce^{kx}, (0, 760), (1000, 672.71)$

$$C = 760$$

$$672.71 = 760e^{1000k}$$

$$x = \frac{\ln(672.71/760)}{1000} \approx -0.000122$$

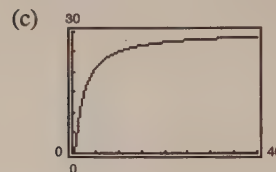
$$P \approx 760e^{-0.000122x}$$

When $x = 3000$, $P \approx 527.06$ mm Hg.

(b) $25 = 30(1 - e^{-0.0502t})$

$$e^{-0.0502t} = \frac{1}{6}$$

$$t = \frac{-\ln 6}{-0.0502} \approx 36 \text{ days}$$

(b) When $t = 5$, $S \approx 20.9646$ which is 20,965 units.

73. $R = \frac{\ln I - \ln I_0}{\ln 10}, I = e^{R \ln 10} = 10^R$

(a) $8.3 = \frac{\ln I - \ln I_0}{\ln 10}$

$$I = 10^{8.3} \approx 199,526,231.5$$

(b) $2R = \frac{\ln I - \ln I_0}{\ln 10}$

$$I = e^{2R \ln 10} = e^{2R \ln 10} = (e^{R \ln 10})^2 = (10^R)^2$$

Increases by a factor of $e^{2R \ln 10}$ or 10^{2R} .

(c) $\frac{dR}{dI} = \frac{\ln I - \ln I_0}{\ln 10}$

75. False. If $y = Ce^{kt}$, $y' = Cke^{kt} \neq \text{constant}$.

77. True

Section 5.7 Differential Equations: Separation of Variables

1. Differential equation: $y' = 4y$

Solution: $y = Ce^{4x}$

Check: $y' = 4Ce^{4x} = 4y$

3. Differential equation: $y'' + y = 0$

Solution: $y = C_1 \cos x + C_2 \sin x$

Check: $y' = -C_1 \sin x + C_2 \cos x$

$$y'' = -C_1 \cos x - C_2 \sin x$$

$$y'' + y = -C_1 \cos x - C_2 \sin x + C_1 \cos x + C_2 \sin x = 0$$

5. Differential Equation:

$$y'' + y = \tan x$$

$$y = -\cos x \ln|\sec x + \tan x|$$

$$y' = (-\cos x) \frac{1}{\sec x + \tan x} (\sec x \cdot \tan x + \sec^2 x) + \sin x \ln|\sec x + \tan x|$$

$$= \frac{(-\cos x)}{\sec x + \tan x} (\sec x)(\tan x + \sec x) + \sin x \ln|\sec x + \tan x|$$

$$= -1 + \sin x \ln|\sec x + \tan x|$$

$$y'' = (\sin x) \frac{1}{\sec x + \tan x} (\sec x \cdot \tan x + \sec^2 x) + \cos x \ln|\sec x + \tan x|$$

$$= (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x|$$

Substituting,

$$y'' + y = (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x| - \cos x \ln|\sec x + \tan x|$$

$$= \tan x.$$

In Exercises 7–11, the differential equation is $y^{(4)} - 16y = 0$.

7. $y = 3 \cos x$

$$y^{(4)} = 3 \cos x$$

$$y^{(4)} - 16y = -45 \cos x \neq 0,$$

No.

9. $y = e^{-2x}$

$$y^{(4)} = 16e^{-2x}$$

$$y^{(4)} - 16y = 16e^{-2x} - 16e^{-2x} = 0,$$

Yes.

11. $y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \sin 2x + C_4 \cos 2x$

$$y^{(4)} = 16C_1 e^{2x} + 16C_2 e^{-2x} + 16C_3 \sin 2x + 16C_4 \cos 2x$$

$$y^{(4)} - 16y = 0,$$

Yes.

In 13–17, the differential equation is $xy' - 2y = x^3e^x$.

13. $y = x^2, y' = 2x$

$$xy' - 2y = x(2x) - 2(x^2) = 0 \neq x^3e^x$$

No.

17. $y = \ln x, y' = \frac{1}{x}$

$$xy' - 2y = x\left(\frac{1}{x}\right) - 2\ln x \neq x^3e^x, \quad \text{No.}$$

19. $y = Ce^{kx}$

$$\frac{dy}{dx} = Cke^{kx}$$

Since $dy/dx = 0.07y$, we have $Cke^{kx} = 0.07Ce^{kx}$.
Thus, $k = 0.07$.

C cannot be determined.

15. $y = x^2(2 + e^x), y' = x^2(e^x) + 2x(2 + e^x)$

$$xy' - 2y = x[x^2e^x + 2xe^x + 4x] - 2[x^2e^x + 2x^2] = x^3e^x,$$

Yes.

21. $y^2 = Cx^3$ passes through $(4, 4)$

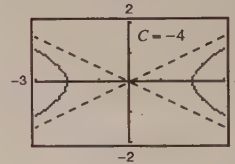
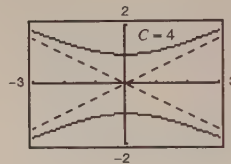
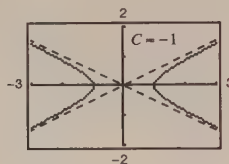
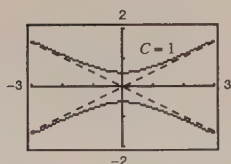
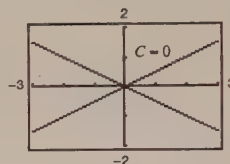
$$16 = C(64) \Rightarrow C = \frac{1}{4}$$

$$\text{Particular solution: } y^2 = \frac{1}{4}x^3 \text{ or } 4y^2 = x^3$$

23. Differential equation: $4yy' - x = 0$

$$\text{General solution: } 4y^2 - x^2 = C$$

Particular solutions: $C = 0$, Two intersecting lines
 $C = \pm 1$, $C = \pm 4$, Hyperbolas



25. Differential equation: $y' + 2y = 0$

$$\text{General Solution: } y = Ce^{-2x}$$

$$y' + 2y = C(-2)e^{-2x} + 2(Ce^{-2x}) = 0$$

$$\text{Initial condition: } y(0) = 3, 3 = Ce^0 = C$$

$$\text{Particular solution: } y = 3e^{-2x}$$

27. Differential equation: $y'' + 9y = 0$

$$\text{General solution: } y = C_1 \sin 3x + C_2 \cos 3x$$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x,$$

$$y'' = -9C_1 \sin 3x - 9C_2 \cos 3x$$

$$y'' + 9y = (-9C_1 \sin 3x - 9C_2 \cos 3x) + 9(C_1 \sin 3x + C_2 \cos 3x) = 0$$

$$\text{Initial conditions: } y\left(\frac{\pi}{6}\right) = 2, y'\left(\frac{\pi}{6}\right) = 1$$

$$2 = C_1 \sin\left(\frac{\pi}{2}\right) + C_2 \cos\left(\frac{\pi}{2}\right) \Rightarrow C_1 = 2$$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$$

$$1 = 3C_1 \cos\left(\frac{\pi}{2}\right) - 3C_2 \sin\left(\frac{\pi}{2}\right)$$

$$= -3C_2 \Rightarrow C_2 = -\frac{1}{3}$$

$$\text{Particular solution: } y = 2 \sin 3x - \frac{1}{3} \cos 3x$$

29. Differential equation: $x^2y'' - 3xy' + 3y = 0$

General solution: $y^4 = C_1x + C_2x^3$

$$y' = C_1 + 3C_2x^2, y'' = 6C_2x$$

$$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) +$$

$$3(C_1x + C_2x^3) = 0$$

Initial conditions: $y(2) = 0, y'(2) = 4$

$$0 = 2C_1 + 8C_2$$

$$y' = C_1 + 3C_2x^2$$

$$4 = C_1 + 12C_2$$

$$\left. \begin{array}{l} C_1 + 4C_2 = 0 \\ C_1 + 12C_2 = 4 \end{array} \right\} C_2 = \frac{1}{2}, C_1 = -2$$

Particular solution: $y = -2x + \frac{1}{2}x^3$

31. $\frac{dy}{dx} = 3x^2$

$$y = \int 3x^2 dx = x^3 + C$$

35. $\frac{dy}{dx} = \frac{x-2}{x} = 1 - \frac{2}{x}$

$$y = \int \left[1 - \frac{2}{x} \right] dx$$

$$= x - 2 \ln|x| + C = x - \ln x^2 + C$$

39. $\frac{dy}{dx} = x\sqrt{x-3}$ Let $u = \sqrt{x-3}$, then $x = u^2 + 3$ and $dx = 2u du$.

$$y = \int x\sqrt{x-3} dx = \int (u^2 + 3)(u)(2u) du$$

$$= 2 \int (u^4 + 3u^2) du = 2 \left(\frac{u^5}{5} + u^3 \right) + C = \frac{2}{5}(x-3)^{5/2} + 2(x-3)^{3/2} + C$$

41. $\frac{dy}{dx} = xe^{x^2}$

$$y = \int xe^{x^2} dx = \frac{1}{2}e^{x^2} + C$$

$$(u = x^2, du = 2x dx)$$

43. $\frac{dy}{dx} = \frac{x}{y}$

$$\int y dy = \int x dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$y^2 - x^2 = C$$

45. $\frac{dr}{ds} = 0.05r$

$$\int \frac{dr}{r} = \int 0.05 ds$$

$$\ln|r| = 0.05s + C_1$$

$$r = e^{0.05s+C_1} = Ce^{0.05s}$$

47. $(2+x)y' = 3y$

$$\int \frac{dy}{y} = \int \frac{3}{2+x} dx$$

$$\ln|y| = 3 \ln|2+x| + \ln C = \ln|C(2+x)^3|$$

$$y = C(2+x)^3$$

49. $yy' = \sin x$

$$\int y dy = \int \sin x dx$$

$$\frac{y^2}{2} = -\cos x + C_1$$

$$y^2 = -2 \cos x + C$$

53. $y \ln x - xy' = 0$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x} dx \quad \left(u = \ln x, du = \frac{dx}{x} \right)$$

$$\ln|y| = \frac{1}{2}(\ln x)^2 + C_1$$

$$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$$

57. $y(x+1) + y' = 0$

$$\int \frac{dy}{y} = -\int (x+1) dx$$

$$\ln|y| = -\frac{(x+1)^2}{2} + C_1$$

$$y = Ce^{-(x+1)^2/2}$$

Initial condition: $y(-2) = 1$, $1 = Ce^{-1/2}$, $C = e^{1/2}$

Particular solution: $y = e^{[1-(x+1)^2]/2} = e^{-(x^2+2x)/2}$

61. $\frac{du}{dv} = uv \sin v^2$

$$\int \frac{du}{u} = \int v \sin v^2 dv$$

$$\ln|u| = -\frac{1}{2} \cos v^2 + C_1$$

$$u = Ce^{-(\cos v^2)/2}$$

Initial condition: $u(0) = 1$, $C = \frac{1}{e^{-1/2}} = e^{1/2}$

Particular solution: $u = e^{(1-\cos v^2)/2}$

51. $\sqrt{1-4x^2} \frac{dy}{dx} = x$

$$dy = \frac{x}{\sqrt{1-4x^2}} dx$$

$$\int dy = \int \frac{x}{\sqrt{1-4x^2}} dx$$

$$= -\frac{1}{8} \int (1-4x^2)^{-1/2} (-8x dx)$$

$$y = -\frac{1}{4}(1-4x^2)^{1/2} + C$$

55. $yy' - e^x = 0$

$$\int y dy = \int e^x dx$$

$$\frac{y^2}{2} = e^x + C_1$$

$$y^2 = 2e^x + C$$

Initial condition: $y(0) = 4$, $16 = 2 + C$, $C = 14$

Particular solution: $y^2 = 2e^x + 14$

59. $y(1+x^2) \frac{dy}{dx} = x(1+y^2)$

$$\frac{y}{1+y^2} dy = \frac{x}{1+x^2} dx$$

$$\frac{1}{2} \ln(1+y^2) = \frac{1}{2} \ln(1+x^2) + C_1$$

$$\ln(1+y^2) = \ln(1+x^2) + \ln C = \ln[C(1+x^2)]$$

$$1+y^2 = C(1+x^2)$$

$$y(0) = \sqrt{3}: 1+3 = C \Rightarrow C = 4$$

$$1+y^2 = 4(1+x^2)$$

$$y^2 = 3+4x^2$$

63. $dP - kP dt = 0$

$$\int \frac{dP}{P} = k \int dt$$

$$\ln|P| = kt + C_1$$

$$P = Ce^{kt}$$

Initial condition: $P(0) = P_0$, $P_0 = Ce^0 = C$

Particular solution: $P = P_0 e^{kt}$

65. $\frac{dy}{dx} = \frac{-9x}{16y}$

$$\int 16y \, dy = -\int 9x \, dx$$

$$8y^2 = \frac{-9}{2}x^2 + C$$

Initial condition: $y(1) = 1$, $8 = -\frac{9}{2} + C$, $C = \frac{25}{2}$

Particular solution: $8y^2 = \frac{-9}{2}x^2 + \frac{25}{2}$,

$$16y^2 + 9x^2 = 25$$

69. $f(x, y) = x^3 - 4xy^2 + y^3$

$$\begin{aligned} f(tx, ty) &= t^3 x^3 - 4t x t^2 y^2 + t^3 y^3 \\ &= t^3(x^3 - 4xy^2 + y^3) \end{aligned}$$

Homogeneous of degree 3

73. $f(x, y) = 2 \ln xy$

$$\begin{aligned} f(tx, ty) &= 2 \ln[tx \, ty] \\ &= 2 \ln[t^2 xy] = 2(\ln t^2 + \ln xy) \end{aligned}$$

Not homogeneous

77. $y' = \frac{x+y}{2x}$, $y = vx$

$$v + x \frac{dv}{dx} = \frac{x+vx}{2x}$$

$$x \frac{dv}{dx} = \frac{1+v}{2} - v = \frac{1-v}{2}$$

$$2 \int \frac{dv}{1-v} = \int \frac{dx}{x}$$

$$-\ln(1-v)^2 = \ln|x| + \ln C = \ln[Cx]$$

$$\frac{1}{(1-v)^2} = |Cx|$$

$$\frac{1}{[1-(y/x)]^2} = |Cx|$$

$$\frac{x^2}{(x-y)^2} = |Cx|$$

$$|x| = C(x-y)^2$$

67. $m = \frac{dy}{dx} = \frac{0-y}{(x+2)-x} = -\frac{y}{2}$

$$\int \frac{dy}{y} = \int -\frac{1}{2} dx$$

$$\ln|y| = -\frac{1}{2}x + C_1$$

$$y = Ce^{-x/2}$$

71. $f(x, y) = \frac{x^2 y^2}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{t^4 x^2 y^2}{\sqrt{t^2 x^2 + t^2 y^2}} = t^3 \frac{x^2 y^2}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 3

75. $f(x, y) = 2 \ln \frac{x}{y}$

$$f(tx, ty) = 2 \ln \frac{tx}{ty} = 2 \ln \frac{x}{y}$$

Homogeneous degree 0

79. $y' = \frac{x-y}{x+y}$, $y = vx$

$$v + x \frac{dv}{dx} = \frac{x-vx}{x+vx}$$

$$v \, dx + x \, dv = \frac{1-v}{1+v} dx$$

$$x \, dv = \left(\frac{1-v}{1+v} - v \right) dx = \frac{1-2v-v^2}{1+v} dx$$

$$\int \frac{v+1}{v^2+2v-1} dv = -\int \frac{dx}{x}$$

$$\frac{1}{2} \ln|v^2+2v-1| = -\ln|x| + \ln C_1 = \ln \left| \frac{C_1}{x} \right|$$

$$|v^2+2v-1| = \frac{C}{x^2}$$

$$\left| \frac{y^2}{x^2} + 2\frac{y}{x} - 1 \right| = \frac{C}{x^2}$$

$$|y^2 + 2xy - x^2| = C$$

81. $y' = \frac{xy}{x^2 - y^2}, y = vx$

$$v + x \frac{dv}{dx} = \frac{x^2 v}{x^2 - x^2 v^2}$$

$$v dx + x dv = \frac{v}{1 - v^2} dx$$

$$x dv = \left(\frac{v}{1 - v^2} - v \right) dx = \left(\frac{v^3}{1 - v^2} \right) dx$$

$$\int \frac{1 - v^2}{v^3} dv = \int \frac{dx}{x}$$

$$-\frac{1}{2v^2} - \ln|v| = \ln|x| + \ln C_1 = \ln|C_1 x|$$

$$\frac{-1}{2v^2} = \ln|C_1 x v|$$

$$\frac{-x^2}{2y^2} = \ln|C_1 y|$$

$$y = C e^{-x^2/2y^2}$$

85. $\left(x \sec \frac{y}{x} + y \right) dx - x dy = 0, y = vx$

$$(x \sec v + xv) dx - x(v dx + x dv) = 0$$

$$(\sec v + v) dx = v dx + x dv$$

$$\int \cos v dv = \int \frac{dx}{x}$$

$$\sin v = \ln x + \ln C_1$$

$$x = C e^{\sin v}$$

$$= C e^{\sin(y/x)}$$

Initial condition: $y(1) = 0, 1 = C e^0 = C$

Particular solution: $x = e^{\sin(y/x)}$

83. $x dy - (2xe^{-y/x} + y) dx = 0, y = vx$

$$x(v dx + x dv) - (2xe^{-v} + vx) dx = 0$$

$$\int e^v dv = \int \frac{2}{x} dx$$

$$e^v = \ln C_1 x^2$$

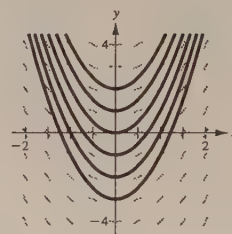
$$e^{y/x} = \ln C_1 + \ln x^2$$

$$e^{y/x} = C + \ln x^2$$

Initial condition: $y(1) = 0, 1 = C$

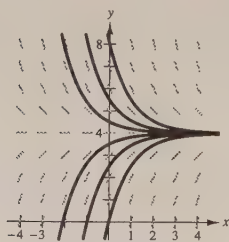
Particular solution: $e^{y/x} = 1 + \ln x^2$

87. $\frac{dy}{dx} = x$



$$y = \int x dx = \frac{1}{2} x^2 + C$$

89. $\frac{dy}{dx} = 4 - y$



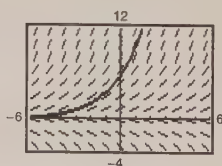
$$\int \frac{dy}{4 - y} = \int dx$$

$$\ln |4 - y| = -x + C_1$$

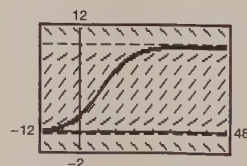
$$4 - y = e^{-x + C_1}$$

$$y = 4 + C e^{-x}$$

91. $\frac{dy}{dx} = 0.5y, y(0) = 6$



93. $\frac{dy}{dx} = 0.02y(10 - y), y(0) = 2$



95. $\frac{dy}{dt} = ky, y = Ce^{kt}$

Initial conditions: $y(0) = y_0$

$$y(1620) = \frac{y_0}{2}$$

$$C = y_0$$

$$\frac{y_0}{2} = y_0 e^{1620k}$$

$$k = \frac{\ln(1/2)}{1620}$$

Particular solution: $y = y_0 e^{-t(\ln 2)/1620}$

When $t = 25$, $y \approx 0.989y_0$, $y = 98.9\%$ of y_0 .

99. $\frac{dy}{dx} = ky(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$ and $y = 4$. Matches (c).

101. $\frac{dw}{dt} = k(1200 - w)$

$$\int \frac{dw}{1200 - w} = \int k dt$$

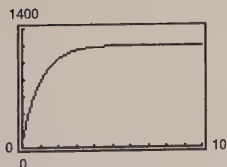
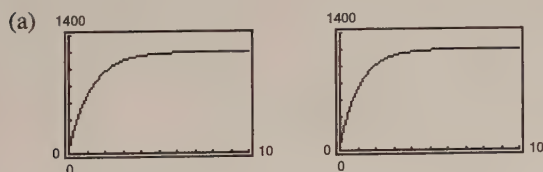
$$\ln|1200 - w| = -kt + C_1$$

$$1200 - w = e^{-kt+C_1} = Ce^{-kt}$$

$$w = 1200 - Ce^{-kt}$$

$$w(0) = 60 = 1200 - C \Rightarrow C = 1200 - 60 = 1140$$

$$w = 1200 - 1140e^{-kt}$$



(b) $k = 0.8$: $t = 1.31$ years

$k = 0.9$: $t = 1.16$ years

$k = 1.0$: $t = 1.05$ years

(c) Maximum weight: 1200 pounds

$$\lim_{t \rightarrow \infty} w = 1200$$

97. $\frac{dy}{dx} = k(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 4$; but not along $y = 0$. Matches (a).

103. (a) $\frac{dv}{dt} = k(W - v)$

$$\int \frac{dv}{W - v} = \int k dt$$

$$-\ln|W - v| = kt + C_1$$

$$v = W - Ce^{-kt}$$

Initial conditions:

$$W = 20, v = 0 \text{ when } t = 0, \text{ and}$$

$$v = 5 \text{ when } t = 1.$$

$$C = 20, k = -\ln(3/4)$$

Particular solution:

$$v = 20(1 - e^{\ln(3/4)t}) = 20\left(1 - \left(\frac{3}{4}\right)^t\right)$$

or

$$v \approx 20(1 - e^{-0.2877t})$$

$$(b) s = \int 20(1 - e^{-0.2877t}) dt$$

$$\approx 20[t + 3.4761e^{-0.2877t}] + C$$

Since $s(0) = 0$, $C \approx -69.5$ and we have
 $s \approx 20t + 69.5(e^{-0.2877t} - 1)$.

105. Given family (circles):
- $x^2 + y^2 = C$

$$2x + 2yy' = 0$$

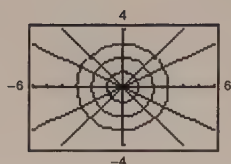
$$y' = -\frac{x}{y}$$

Orthogonal trajectory (lines): $y' = \frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + \ln K$$

$$y = Kx$$



107. Given family (parabolas):
- $x^2 = Cy$

$$2x = Cy'$$

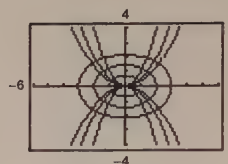
$$y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{x}{2y}$

$$2 \int y \, dy = - \int x \, dx$$

$$y^2 = -\frac{x^2}{2} + K_1$$

$$x^2 + 2y^2 = K$$



109. Given family:
- $y^2 = Cx^3$

$$2yy' = 3Cx^2$$

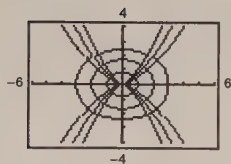
$$y' = \frac{3Cx^2}{2y} = \frac{3x^2}{2y} \left(\frac{y^2}{x^3} \right) = \frac{3y}{2x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{2x}{3y}$

$$3 \int y \, dy = -2 \int x \, dx$$

$$\frac{3y^2}{2} = -x^2 + K_1$$

$$3y^2 + 2x^2 = K$$



111. A general solution of order
- n
- has
- n
- arbitrary constants while in a particular solution initial conditions are given in order to solve for all these constants.

- 113.
- $M(x, y)dx + N(x, y)dy = 0$
- , where
- M
- and
- N
- are homogeneous functions of the same degree.

See Example 7a.

115. False. Consider Example 2.
- $y = x^3$
- is a solution to
- $xy' - 3y = 0$
- , but
- $y = x^3 + 1$
- is not a solution.

117. False

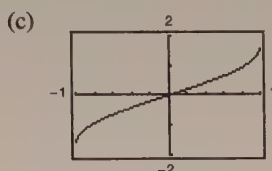
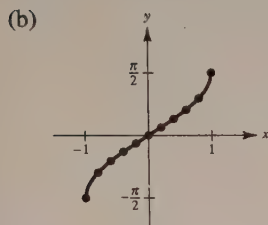
$$\begin{aligned} f(tx, ty) &= t^2x^2 + t^2xy + 2 \\ &\neq t^2f(x, y) \end{aligned}$$

Section 5.8 Inverse Trigonometric Functions: Differentiation

1. $y = \arcsin x$

(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	-1.571	-0.927	-0.644	-0.412	-0.201	0	0.201	0.412	0.644	0.927	1.571



(d) Symmetric about origin:
 $\arcsin(-x) = -\arcsin x$
 Intercept: $(0, 0)$

3. False.

$$\arccos \frac{1}{2} = \frac{\pi}{3}$$

since the range is $[0, \pi]$.

5. $\arcsin \frac{1}{2} = \frac{\pi}{6}$

7. $\arccos \frac{1}{2} = \frac{\pi}{3}$

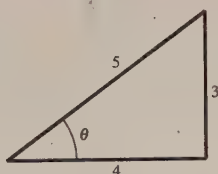
9. $\arctan \frac{\sqrt{3}}{3} = \frac{\pi}{6}$

11. $\operatorname{arccsc}(-\sqrt{2}) = -\frac{\pi}{4}$

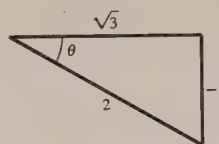
13. $\arccos(-0.8) \approx 2.50$

15. $\operatorname{arcsec}(1.269) = \arccos\left(\frac{1}{1.269}\right) \approx 0.66$

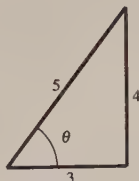
17. (a) $\sin(\arctan \frac{3}{4}) = \frac{3}{5}$



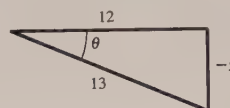
19. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right] = \cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$



(b) $\sec(\arcsin \frac{4}{5}) = \frac{5}{3}$



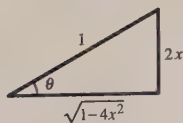
(b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right] = -\frac{13}{5}$



21. $y = \cos(\arcsin 2x)$

$$\theta = \arcsin 2x$$

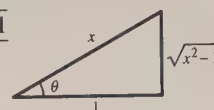
$$y = \cos \theta = \sqrt{1 - 4x^2}$$



23. $y = \sin(\operatorname{arcsec} x)$

$$\theta = \operatorname{arcsec} x, 0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2}$$

$$y = \sin \theta = \frac{\sqrt{x^2 - 1}}{|x|}$$

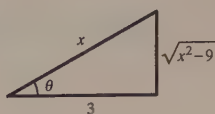


The absolute value bars on x are necessary because of the restriction $0 \leq \theta \leq \pi, \theta \neq \pi/2$, and $\sin \theta$ for this domain must always be nonnegative.

$$25. y = \tan\left(\operatorname{arcsec} \frac{x}{3}\right)$$

$$\theta = \operatorname{arcsec} \frac{x}{3}$$

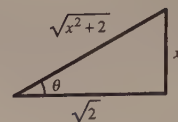
$$y = \tan \theta = \frac{\sqrt{x^2 - 9}}{3}$$



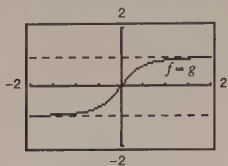
$$27. y = \csc\left(\arctan \frac{x}{\sqrt{2}}\right)$$

$$\theta = \arctan \frac{x}{\sqrt{2}}$$

$$y = \csc \theta = \frac{\sqrt{x^2 + 2}}{x}$$



$$29. \sin(\arctan 2x) = \frac{2x}{\sqrt{1 + 4x^2}}$$

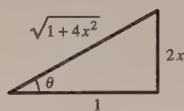


Asymptotes: $y = \pm 1$

$$\arctan 2x = \theta$$

$$\tan \theta = 2x$$

$$\sin \theta = \frac{2x}{\sqrt{1 + 4x^2}}$$



$$33. \arcsin \sqrt{2x} = \arccos \sqrt{x}$$

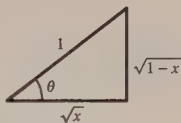
$$\sqrt{2x} = \sin(\arccos \sqrt{x})$$

$$\sqrt{2x} = \sqrt{1 - x}, 0 \leq x \leq 1$$

$$2x = 1 - x$$

$$3x = 1$$

$$x = \frac{1}{3}$$



$$35. (a) \operatorname{arccsc} x = \arcsin \frac{1}{x}, x \geq 1$$

Let $y = \operatorname{arccsc} x$. Then for

$$-\frac{\pi}{2} \leq y < 0 \text{ and } 0 < y \leq \frac{\pi}{2},$$

$$\csc y = x \Rightarrow \sin y = 1/x. \text{ Thus, } y = \arcsin(1/x).$$

Therefore, $\operatorname{arccsc} x = \arcsin(1/x)$.

$$(b) \arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}, x > 0$$

Let $y = \arctan x + \arctan(1/x)$. Then,

$$\begin{aligned} \tan y &= \frac{\tan(\arctan x) + \tan[\arctan(1/x)]}{1 - \tan(\arctan x) \tan[\arctan(1/x)]} \\ &= \frac{x + (1/x)}{1 - x(1/x)} \\ &= \frac{x + (1/x)}{0} \text{ (which is undefined).} \end{aligned}$$

Thus, $y = \pi/2$. Therefore, $\arctan x + \arctan(1/x) = \pi/2$.

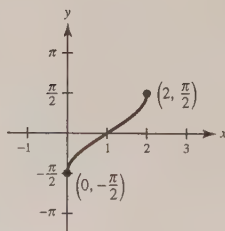
$$37. f(x) = \arcsin(x - 1)$$

$$x - 1 = \sin y$$

$$x = 1 + \sin y$$

$$\text{Domain: } [0, 2]$$

$$\text{Range: } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



$f(x)$ is the graph of $\arcsin x$ shifted 1 unit to the right.

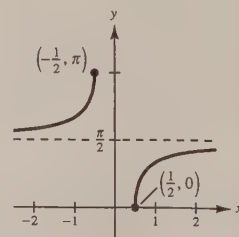
$$39. f(x) = \operatorname{arcsec} 2x$$

$$2x = \sec y$$

$$x = \frac{1}{2} \sec y$$

$$\text{Domain: } \left(-\infty, -\frac{1}{2}\right], \left[\frac{1}{2}, \infty\right)$$

$$\text{Range: } \left[0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right]$$



$$41. f(x) = 2 \arcsin(x - 1)$$

$$f'(x) = \frac{2}{\sqrt{1 - (x - 1)^2}} = \frac{2}{\sqrt{2x - x^2}}$$

$$45. f(x) = \arctan \frac{x}{a}$$

$$f'(x) = \frac{1/a}{1 + (x^2/a^2)} = \frac{a}{a^2 + x^2}$$

$$49. h(t) = \sin(\arccos t) = \sqrt{1 - t^2}$$

$$h'(t) = \frac{1}{2}(1 - t^2)^{-1/2}(-2t) = \frac{-t}{\sqrt{1 - t^2}}$$

$$53. y = \frac{1}{2} \left(\frac{1}{2} \ln \frac{x+1}{x-1} + \arctan x \right)$$

$$= \frac{1}{4} [\ln(x+1) - \ln(x-1)] + \frac{1}{2} \arctan x$$

$$\frac{dy}{dx} = \frac{1}{4} \left(\frac{1}{x+1} - \frac{1}{x-1} \right) + \frac{1/2}{1+x^2} = \frac{1}{1-x^4}$$

$$57. y = 8 \arcsin \frac{x}{4} - \frac{x\sqrt{16-x^2}}{2}$$

$$\begin{aligned} y' &= 2 \frac{1}{\sqrt{1 - (x/4)^2}} - \frac{\sqrt{16-x^2}}{2} - \frac{x}{4}(16-x^2)^{-1/2}(-2x) \\ &= \frac{8}{\sqrt{16-x^2}} - \frac{\sqrt{16-x^2}}{2} + \frac{x^2}{2\sqrt{16-x^2}} \\ &= \frac{16 - (16-x^2) + x^2}{2\sqrt{16-x^2}} = \frac{x^2}{\sqrt{16-x^2}} \end{aligned}$$

$$61. f(x) = \arcsin x, a = \frac{1}{2}$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f''(x) = \frac{x}{(1-x^2)^{3/2}}$$

$$P_1(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$$

$$P_2(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^2 = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right) + \frac{2\sqrt{3}}{9}\left(x - \frac{1}{2}\right)^2$$

$$43. g(x) = 3 \arccos \frac{x}{2}$$

$$g'(x) = \frac{-3(1/2)}{\sqrt{1 - (x^2/4)}} = \frac{-3}{\sqrt{4-x^2}}$$

$$47. g(x) = \frac{\arcsin 3x}{x}$$

$$\begin{aligned} g'(x) &= \frac{x(3/\sqrt{1-9x^2}) - \arcsin 3x}{x^2} \\ &= \frac{3x - \sqrt{1-9x^2} \arcsin 3x}{x^2 \sqrt{1-9x^2}} \end{aligned}$$

$$51. y = x \arccos x - \sqrt{1-x^2}$$

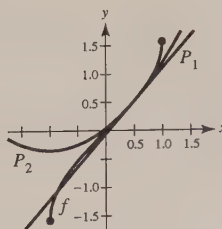
$$\begin{aligned} y' &= \arccos x - \frac{x}{\sqrt{1-x^2}} - \frac{1}{2}(1-x^2)^{-1/2}(-2x) \\ &= \arccos x \end{aligned}$$

$$55. y = x \arcsin x + \sqrt{1-x^2}$$

$$\frac{dy}{dx} = x \left(\frac{1}{\sqrt{1-x^2}} \right) + \arcsin x - \frac{x}{\sqrt{1-x^2}} = \arcsin x$$

$$59. y = \arctan x + \frac{x}{1+x^2}$$

$$\begin{aligned} y' &= \frac{1}{1+x^2} + \frac{(1+x^2) - x(2x)}{(1+x^2)^2} \\ &= \frac{(1+x^2) + (1-x^2)}{(1+x^2)^2} \\ &= \frac{2}{(1+x^2)^2} \end{aligned}$$



$$\begin{aligned}
 63. \quad f(x) &= \operatorname{arcsec} x - x \\
 f'(x) &= \frac{1}{|x|\sqrt{x^2 - 1}} - 1 \\
 &= 0 \text{ when } |x|\sqrt{x^2 - 1} = 1. \\
 x^2(x^2 - 1) &= 1 \\
 x^4 - x^2 - 1 &= 0 \text{ when } x^2 = \frac{1 + \sqrt{5}}{2} \text{ or} \\
 x &= \pm \sqrt{\frac{1 + \sqrt{5}}{2}} = \pm 1.272
 \end{aligned}$$

Relative maximum: (1.272, -0.606)

Relative minimum: (-1.272, 3.747)

67. The trigonometric functions are not one-to-one on $(-\infty, \infty)$, so their domains must be restricted to intervals on which they are one-to-one.

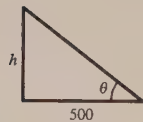
$$\begin{aligned}
 71. \quad (a) \quad \cot \theta &= \frac{x}{5} \\
 \theta &= \operatorname{arccot}\left(\frac{x}{5}\right)
 \end{aligned}$$

$$\begin{aligned}
 73. \quad (a) \quad h(t) &= -16t^2 + 256 \\
 -16t^2 + 256 &= 0 \text{ when } t = 4 \text{ sec.}
 \end{aligned}$$

$$(b) \quad \tan \theta = \frac{h}{500} = \frac{-16t^2 + 256}{500}$$

$$\theta = \arctan\left[\frac{16}{500}(-t^2 + 16)\right]$$

$$\frac{d\theta}{dt} = \frac{-8t/125}{1 + [(4/125)(-t^2 + 16)]^2} = \frac{-1000t}{15,625 + 16(16 - t^2)^2}$$

When $t = 1$, $d\theta/dt \approx -0.0520$ rad/sec.When $t = 2$, $d\theta/dt \approx -0.1116$ rad/sec.

$$\begin{aligned}
 65. \quad f(x) &= \arctan x - \arctan(x - 4) \\
 f'(x) &= \frac{1}{1 + x^2} - \frac{1}{1 + (x - 4)^2} = 0 \\
 1 + x^2 &= 1 + (x - 4)^2 \\
 0 &= -8x + 16 \\
 x &= 2
 \end{aligned}$$

By the First Derivative Test, (2, 2.214) is a relative maximum.

$$69. \quad y = \operatorname{arccot} x, \quad 0 < y < \pi$$

$$x = \cot y$$

$$\tan y = \frac{1}{x}$$

So, graph the function

$$y = \arctan\left(\frac{1}{x}\right) \text{ for } x > 0 \text{ and } y = \arctan\left(\frac{1}{x}\right) + \pi \text{ for } x < 0$$

$$(b) \quad \frac{d\theta}{dt} = \frac{-\frac{1}{5}}{1 + \left(\frac{x}{5}\right)^2} \frac{dx}{dt} = \frac{-5}{x^2 + 25} \frac{dx}{dt}$$

$$\text{If } \frac{dx}{dt} = -400 \text{ and } x = 10, \frac{d\theta}{dt} = 16 \text{ rad/hr.}$$

$$\text{If } \frac{dx}{dt} = -400 \text{ and } x = 3, \frac{d\theta}{dt} \approx 58.824 \text{ rad/hr.}$$

$$75. \tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x) \tan(\arctan y)} = \frac{x + y}{1 - xy}, xy \neq 1$$

Therefore,

$$\arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right), xy \neq 1.$$

Let $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan \frac{(1/2) + (1/3)}{1 - [(1/2) \cdot (1/3)]} = \arctan \frac{5/6}{1 - (1/6)} = \arctan \frac{5/6}{5/6} = \arctan 1 = \frac{\pi}{4}$$

$$77. f(x) = kx + \sin x$$

$$f'(x) = k + \cos x \geq 0 \text{ for } k \geq 1$$

$$f'(x) = k + \cos x \leq 0 \text{ for } k \leq -1$$

Therefore, $f(x) = kx + \sin x$ is strictly monotonic and has an inverse for $k \leq -1$ or $k \geq 1$.

79. True

$$\frac{d}{dx}[\arctan x] = \frac{1}{1 + x^2} > 0 \text{ for all } x.$$

81. True

$$\frac{d}{dx}[\arctan(\tan x)] = \frac{\sec^2 x}{1 + \tan^2 x} = \frac{\sec^2 x}{\sec^2 x} = 1$$

Section 5.9 Inverse Trigonometric Functions: Integration

$$1. \int \frac{5}{\sqrt{9 - x^2}} dx = 5 \arcsin\left(\frac{x}{3}\right) + C$$

$$3. \text{ Let } u = 3x, du = 3 dx.$$

$$\int_0^{1/6} \frac{1}{\sqrt{1 - 9x^2}} dx = \frac{1}{3} \int_0^{1/6} \frac{1}{\sqrt{1 - (3x)^2}} (3) dx = \left[\frac{1}{3} \arcsin(3x) \right]_0^{1/6} = \frac{\pi}{18}$$

$$5. \int \frac{7}{16 + x^2} dx = \frac{7}{4} \arctan\left(\frac{x}{4}\right) + C$$

$$7. \text{ Let } u = 2x, du = 2 dx.$$

$$\int_0^{\sqrt{3}/2} \frac{1}{1 + 4x^2} dx = \frac{1}{2} \int_0^{\sqrt{3}/2} \frac{2}{1 + (2x)^2} dx = \left[\frac{1}{2} \arctan(2x) \right]_0^{\sqrt{3}/2} = \frac{\pi}{6}$$

$$9. \int \frac{1}{x\sqrt{4x^2 - 1}} dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}} dx = \operatorname{arcsec}|2x| + C$$

$$11. \int \frac{x^3}{x^2 + 1} dx = \int \left[x - \frac{x}{x^2 + 1} \right] dx = \int x dx - \frac{1}{2} \int \frac{2x}{x^2 + 1} dx = \frac{1}{2} x^2 - \frac{1}{2} \ln(x^2 + 1) + C \quad (\text{Use long division.})$$

$$13. \int \frac{1}{\sqrt{1 - (x + 1)^2}} dx = \arcsin(x + 1) + C$$

$$15. \text{ Let } u = t^2, du = 2t dt.$$

$$\int \frac{t}{\sqrt{1 - t^4}} dt = \frac{1}{2} \int \frac{1}{\sqrt{1 - (t^2)^2}} (2t) dt = \frac{1}{2} \arcsin(t^2) + C$$

17. Let $u = \arcsin x$, $du = \frac{1}{\sqrt{1-x^2}} dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}} dx = \left[\frac{1}{2} \arcsin^2 x \right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$$

19. Let $u = 1 - x^2$, $du = -2x dx$.

$$\begin{aligned} \int_{-1/2}^0 \frac{x}{\sqrt{1-x^2}} dx &= -\frac{1}{2} \int_{-1/2}^0 (1-x^2)^{-1/2} (-2x) dx \\ &= \left[-\sqrt{1-x^2} \right]_{-1/2}^0 = \frac{\sqrt{3}-2}{2} \\ &\approx -0.134 \end{aligned}$$

21. Let $u = e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{e^{2x}}{4 + e^{4x}} dx = \frac{1}{2} \int \frac{2e^{2x}}{4 + (e^{2x})^2} dx = \frac{1}{4} \arctan \frac{e^{2x}}{2} + C$$

23. Let $u = \cos x$, $du = -\sin x dx$.

$$\begin{aligned} \int_{\pi/2}^{\pi} \frac{\sin x}{1 + \cos^2 x} dx &= -\int_{\pi/2}^{\pi} \frac{-\sin x}{1 + \cos^2 x} dx \\ &= \left[-\arctan(\cos x) \right]_{\pi/2}^{\pi} = \frac{\pi}{4} \end{aligned}$$

25. $\int \frac{1}{\sqrt{x}\sqrt{1-x}} dx$, $u = \sqrt{x}$, $x = u^2$, $dx = 2u du$

$$\begin{aligned} \int \frac{1}{u\sqrt{1-u^2}} (2u du) &= 2 \int \frac{du}{\sqrt{1-u^2}} = 2 \arcsin u + C \\ &= 2 \arcsin \sqrt{x} + C \end{aligned}$$

27. $\int \frac{x-3}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx - 3 \int \frac{1}{x^2+1} dx$

$$= \frac{1}{2} \ln(x^2+1) - 3 \arctan x + C$$

$$\begin{aligned} 29. \int \frac{x+5}{\sqrt{9-(x-3)^2}} dx &= \int \frac{(x-3)}{\sqrt{9-(x-3)^2}} dx + \int \frac{8}{\sqrt{9-(x-3)^2}} dx \\ &= -\sqrt{9-(x-3)^2} - 8 \arcsin\left(\frac{x-3}{3}\right) + C \\ &= -\sqrt{6x-x^2} + 8 \arcsin\left(\frac{x}{3} - 1\right) + C \end{aligned}$$

31. $\int_0^2 \frac{dx}{x^2-2x+2} = \int_0^2 \frac{1}{1+(x-1)^2} dx = \left[\arctan(x-1) \right]_0^2 = \frac{\pi}{2}$

$$\begin{aligned} 33. \int \frac{2x}{x^2+6x+13} dx &= \int \frac{2x+6}{x^2+6x+13} dx - 6 \int \frac{1}{x^2+6x+13} dx = \int \frac{2x+6}{x^2+6x+13} dx - 6 \int \frac{1}{4+(x+3)^2} dx \\ &= \ln|x^2+6x+13| - 3 \arctan\left(\frac{x+3}{2}\right) + C \end{aligned}$$

35. $\int \frac{1}{\sqrt{-x^2-4x}} dx = \int \frac{1}{\sqrt{4-(x+2)^2}} dx = \arcsin\left(\frac{x+2}{2}\right) + C$

37. Let $u = -x^2 - 4x$, $du = (-2x - 4) dx$.

$$\int \frac{x+2}{\sqrt{-x^2-4x}} dx = -\frac{1}{2} \int (-x^2-4x)^{-1/2} (-2x-4) dx = -\sqrt{-x^2-4x} + C$$

$$\begin{aligned} 39. \int_2^3 \frac{2x-3}{\sqrt{4x-x^2}} dx &= \int_2^3 \frac{2x-4}{\sqrt{4x-x^2}} dx + \int_2^3 \frac{1}{\sqrt{4x-x^2}} dx = -\int_2^3 (4x-x^2)^{-1/2} (4-2x) dx + \int_2^3 \frac{1}{\sqrt{4-(x-2)^2}} dx \\ &= \left[-2\sqrt{4x-x^2} + \arcsin\left(\frac{x-2}{2}\right) \right]_2^3 = 4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059 \end{aligned}$$

41. Let $u = x^2 + 1$, $du = 2x dx$.

$$\int \frac{x}{x^4 + 2x^2 + 2} dx = \frac{1}{2} \int \frac{2x}{(x^2 + 1)^2 + 1} dx = \frac{1}{2} \arctan(x^2 + 1) + C$$

43. Let $u = \sqrt{e^t - 3}$. Then $u^2 + 3 = e^t$, $2u du = e^t dt$, and $\frac{2u du}{u^2 + 3} = dt$.

$$\begin{aligned} \int \sqrt{e^t - 3} dt &= \int \frac{2u^2}{u^2 + 3} du = \int 2 du - \int 6 \frac{1}{u^2 + 3} du \\ &= 2u - 2\sqrt{3} \arctan \frac{u}{\sqrt{3}} + C = 2\sqrt{e^t - 3} - 2\sqrt{3} \arctan \sqrt{\frac{e^t - 3}{3}} + C \end{aligned}$$

45. A perfect square trinomial is an expression in x with three terms that factor as a perfect square.

Example: $x^2 + 6x + 9 = (x + 3)^2$

47. (a) $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$, $u = x$ (b) $\int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + C$, $u = 1-x^2$

(c) $\int \frac{1}{x\sqrt{1-x^2}} dx$ cannot be evaluated using the basic integration rules.

49. (a) $\int \sqrt{x-1} dx = \frac{2}{3}(x-1)^{3/2} + C$, $u = x-1$

(b) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u du$.

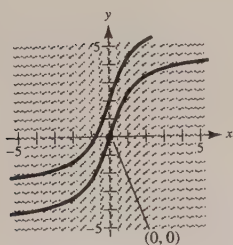
$$\begin{aligned} \int x\sqrt{x-1} dx &= \int (u^2 + 1)(u)(2u) du = 2 \int (u^4 + u^2) du = 2 \left(\frac{u^5}{5} + \frac{u^3}{3} \right) + C \\ &= \frac{2}{15} u^3(3u^2 + 5) + C = \frac{2}{15} (x-1)^{3/2} [3(x-1) + 5] + C = \frac{2}{15} (x-1)^{3/2} (3x+2) + C \end{aligned}$$

(c) Let $u = \sqrt{x-1}$. Then $x = u^2 + 1$ and $dx = 2u du$.

$$\int \frac{x}{\sqrt{x-1}} dx = \int \frac{u^2 + 1}{u} (2u) du = 2 \int (u^2 + 1) du = 2 \left(\frac{u^3}{3} + u \right) + C = \frac{2}{3} u(u^2 + 3) + C = \frac{2}{3} \sqrt{x-1} (x+2) + C$$

Note: In (b) and (c), substitution was necessary *before* the basic integration rules could be used.

51. (a)

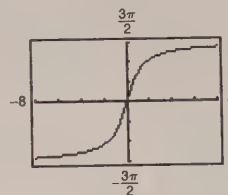


(b) $\frac{dy}{dx} = \frac{3}{1+x^2}$, $(0, 0)$

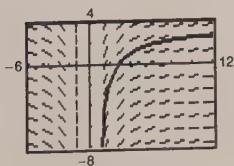
$$y = 3 \int \frac{dx}{1+x^2} = 3 \arctan x + C$$

$(0, 0): 0 = 3 \arctan(0) + C \Rightarrow C = 0$

$y = 3 \arctan x$



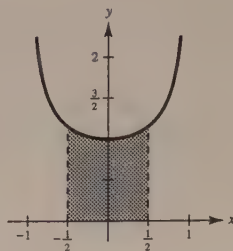
53. $\frac{dy}{dx} = \frac{10}{x\sqrt{x^2-1}}$, $y(3) = 0$



$$\begin{aligned} 55. A &= \int_1^3 \frac{1}{x^2 - 2x + 1 + 4} dx = \int_1^3 \frac{1}{(x-1)^2 + 2^2} dx \\ &= \left[\frac{1}{2} \arctan \left(\frac{x-1}{2} \right) \right]_1^3 = \frac{1}{2} \arctan(1) = \frac{\pi}{8} \approx 0.3927 \end{aligned}$$

57. Area $\approx (1)(1) = 1$

Matches (c)



$$59. (a) \int_0^1 \frac{4}{1+x^2} dx = \left[4 \arctan x \right]_0^1 = 4 \arctan 1 - 4 \arctan 0 = 4\left(\frac{\pi}{4}\right) - 4(0) = \pi$$

(b) Let $n = 6$.

$$4 \int_0^1 \frac{1}{1+x^2} dx \approx 4\left(\frac{1}{18}\right) \left[1 + \frac{4}{1+(1/36)} + \frac{2}{1+(1/9)} + \frac{4}{1+(1/4)} + \frac{2}{1+(4/9)} + \frac{4}{1+(25/36)} + \frac{1}{2} \right] \approx 3.1415918$$

(c) 3.1415927

$$61. (a) \frac{d}{dx} \left[\arcsin\left(\frac{u}{a}\right) + C \right] = \frac{1}{\sqrt{1-(u^2/a^2)}} \left(\frac{u'}{a}\right) = \frac{u'}{\sqrt{a^2-u^2}}$$

$$\text{Thus, } \int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C.$$

$$(b) \frac{d}{dx} \left[\frac{1}{a} \arctan \frac{u}{a} + C \right] = \frac{1}{a} \left[\frac{u'/a}{1+(u/a)^2} \right] = \frac{1}{a^2} \left[\frac{u'}{(a^2+u^2)/a^2} \right] = \frac{u'}{a^2+u^2}$$

$$\text{Thus, } \int \frac{du}{a^2+u^2} = \int \frac{u'}{a^2+u^2} dx = \frac{1}{a} \arctan \frac{u}{a} + C.$$

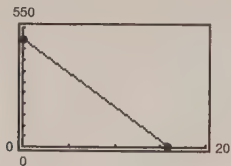
(c) Assume $u > 0$.

$$\frac{d}{dx} \left[\frac{1}{a} \operatorname{arccsc} \frac{u}{a} + C \right] = \frac{1}{a} \left[\frac{u'/a}{(u/a)\sqrt{(u/a)^2-1}} \right] = \frac{1}{a} \left[\frac{u'}{u\sqrt{(u^2-a^2)/a^2}} \right] = \frac{u'}{u\sqrt{u^2-a^2}}.$$

The case $u < 0$ is handled in a similar manner.

$$\text{Thus, } \int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{u'}{u\sqrt{u^2-a^2}} dx = \frac{1}{a} \operatorname{arccsc} \frac{|u|}{a} + C.$$

63. (a) $v(t) = -32t + 500$



$$(b) s(t) = \int v(t) dt = \int (-32t + 500) dt$$

$$= -16t^2 + 500t + C$$

$$s(0) = -16(0) + 500(0) + C = 0 \Rightarrow C = 0$$

$$s(t) = -16t^2 + 500t$$

When the object reaches its maximum height,
 $v(t) = 0$.

$$v(t) = -32t + 500 = 0$$

$$-32t = -500$$

$$t = 15.625$$

$$\begin{aligned} s(15.625) &= -16(15.625)^2 + 500(15.625) \\ &= 3906.25 \text{ ft (Maximum height)} \end{aligned}$$

63. —CONTINUED—

$$(c) \quad \int \frac{1}{32 + kv^2} dv = - \int dt$$

$$\frac{1}{\sqrt{32k}} \arctan\left(\sqrt{\frac{k}{32}} v\right) = -t + C_1$$

$$\arctan\left(\sqrt{\frac{k}{32}} v\right) = -\sqrt{32k} t + C$$

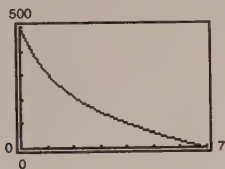
$$\sqrt{\frac{k}{32}} v = \tan(C - \sqrt{32k} t)$$

$$v = \sqrt{\frac{32}{k}} \tan(C - \sqrt{32k} t)$$

When $t = 0$, $v = 500$, $C = \arctan(500\sqrt{k/32})$, and we have

$$v(t) = \sqrt{\frac{32}{k}} \tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k} t\right]$$

$$(d) \text{ When } k = 0.001, v(t) = \sqrt{32,000} \tan[\arctan(500\sqrt{0.00003125}) - \sqrt{0.032} t].$$



$v(t) = 0$ when $t_0 \approx 6.86$ sec.

$$(e) h = \int_0^{6.86} \sqrt{32,000} \tan[\arctan(500\sqrt{0.00003125}) - \sqrt{0.032} t] dt$$

Simpson's Rule: $n = 10$; $h \approx 1088$ feet

(f) Air resistance lowers the maximum height.

Section 5.10 Hyperbolic Functions

$$1. (a) \sinh 3 = \frac{e^3 - e^{-3}}{2} \approx 10.018$$

$$(b) \tanh(-2) = \frac{\sinh(-2)}{\cosh(-2)} = \frac{e^{-2} - e^2}{e^{-2} + e^2} \approx -0.964$$

$$3. (a) \operatorname{csch}(\ln 2) = \frac{2}{e^{\ln 2} - e^{-\ln 2}} = \frac{2}{2 - (1/2)} = \frac{4}{3}$$

$$(b) \coth(\ln 5) = \frac{\cosh(\ln 5)}{\sinh(\ln 5)} = \frac{e^{\ln 5} + e^{-\ln 5}}{e^{\ln 5} - e^{-\ln 5}} = \frac{5 + (1/5)}{5 - (1/5)} = \frac{13}{12}$$

$$5. (a) \cosh^{-1}(2) = \ln(2 + \sqrt{3}) \approx 1.317$$

$$(b) \operatorname{sech}^{-1}\left(\frac{2}{3}\right) = \ln\left(\frac{1 + \sqrt{1 - (4/9)}}{2/3}\right) \approx 0.962$$

$$7. \tanh^2 x + \operatorname{sech}^2 x = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\frac{2}{e^x + e^{-x}}\right)^2 = \frac{e^{2x} - 2 + e^{-2x} + 4}{(e^x + e^{-x})^2} = \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1$$

$$\begin{aligned}
 9. \sinh x \cosh y + \cosh x \sinh y &= \left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^y + e^{-y}}{2} \right) + \left(\frac{e^x + e^{-x}}{2} \right) \left(\frac{e^y - e^{-y}}{2} \right) \\
 &= \frac{1}{4} [e^{x+y} - e^{-x+y} + e^{x-y} - e^{-(x+y)} + e^{x+y} + e^{-x+y} - e^{x-y} - e^{-(x+y)}] \\
 &= \frac{1}{4} [2(e^{x+y} - e^{-(x+y)})] = \frac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh(x+y)
 \end{aligned}$$

$$\begin{aligned}
 11. 3 \sinh x + 4 \sinh^3 x &= \sinh x (3 + 4 \sinh^2 x) = \left(\frac{e^x - e^{-x}}{2} \right) \left[3 + 4 \left(\frac{e^x - e^{-x}}{2} \right)^2 \right] \\
 &= \left(\frac{e^x - e^{-x}}{2} \right) [3 + e^{2x} - 2 + e^{-2x}] = \frac{1}{2} (e^x - e^{-x}) (e^{2x} + e^{-2x} + 1) \\
 &= \frac{1}{2} [e^{3x} + e^{-x} + e^x - e^x - e^{-3x} - e^{-x}] = \frac{e^{3x} - e^{-3x}}{2} = \sinh(3x)
 \end{aligned}$$

$$13. \quad \sinh x = \frac{3}{2}$$

$$\cosh^2 x - \left(\frac{3}{2} \right)^2 = 1 \Rightarrow \cosh^2 x = \frac{13}{4} \Rightarrow \cosh x = \frac{\sqrt{13}}{2}$$

$$\tanh x = \frac{3/2}{\sqrt{13}/2} = \frac{3\sqrt{13}}{13}$$

$$\operatorname{csch} x = \frac{1}{3/2} = \frac{2}{3}$$

$$\operatorname{sech} x = \frac{1}{\sqrt{13}/2} = \frac{2\sqrt{13}}{13}$$

$$\coth x = \frac{1}{3/\sqrt{13}} = \frac{\sqrt{13}}{3}$$

$$15. y = \sinh(1 - x^2)$$

$$y' = -2x \cosh(1 - x^2)$$

$$17. f(x) = \ln(\sinh x)$$

$$f'(x) = \frac{1}{\sinh x} (\cosh x) = \coth x$$

$$19. y = \ln \left(\tanh \frac{x}{2} \right)$$

$$\begin{aligned}
 y' &= \frac{1/2}{\tanh(x/2)} \operatorname{sech}^2 \left(\frac{x}{2} \right) = \frac{1}{2 \sinh(x/2) \cosh(x/2)} \\
 &= \frac{1}{\sinh x} = \operatorname{csch} x
 \end{aligned}$$

$$21. h(x) = \frac{1}{4} \sinh(2x) - \frac{x}{2}$$

$$h'(x) = \frac{1}{2} \cosh(2x) - \frac{1}{2} = \frac{\cosh(2x) - 1}{2} = \sinh^2 x$$

$$23. f(t) = \arctan(\sinh t)$$

$$\begin{aligned}
 f'(t) &= \frac{1}{1 + \sinh^2 t} (\cosh t) \\
 &= \frac{\cosh t}{\cosh^2 t} = \operatorname{sech} t
 \end{aligned}$$

$$25. \text{ Let } y = g(x).$$

$$y = x^{\cosh x}$$

$$\ln y = \cosh x \ln x$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{\cosh x}{x} + \sinh x \ln x$$

$$\frac{dy}{dx} = \frac{y}{x} [\cosh x + x(\sinh x) \ln x]$$

$$= \frac{x^{\cosh x}}{x} [\cosh x + x(\sinh x) \ln x]$$

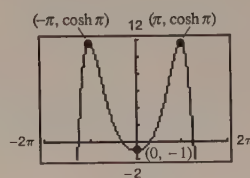
27. $y = (\cosh x - \sinh x)^2$

$$\begin{aligned} y' &= 2(\cosh x - \sinh x)(\sinh x - \cosh x) \\ &= -2(\cosh x - \sinh x)^2 = -2e^{-2x} \end{aligned}$$

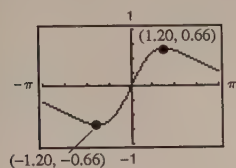
29. $f(x) = \sin x \sinh x - \cos x \cosh x, -4 \leq x \leq 4$

$$\begin{aligned} f'(x) &= \sin x \cosh x + \cos x \sinh x - \cos x \sinh x + \sin x \cosh x \\ &= 2 \sin x \cosh x = 0 \text{ when } x = 0, \pm\pi. \end{aligned}$$

 Relative maxima: $(\pm\pi, \cosh \pi)$

 Relative minimum: $(0, -1)$


31. $g(x) = x \operatorname{sech} x = \frac{x}{\cosh x}$


 Relative maximum: $(1.20, 0.66)$

 Relative minimum: $(-1.20, -0.66)$

33. $y = a \sinh x$

$y' = a \cosh x$

$y'' = a \sinh x$

$y''' = a \cosh x$

 Therefore, $y''' - y' = 0$.

35. $f(x) = \tanh x$

$f(1) = \tanh(1) \approx 0.7616$

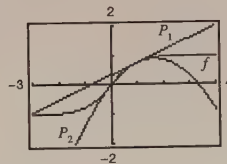
$f'(x) = \operatorname{sech}^2 x$

$f'(1) = \frac{1}{\cosh^2(1)} \approx 0.4200$

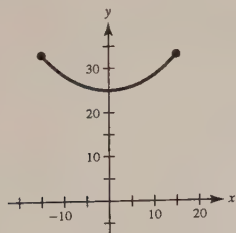
$f''(x) = -2 \operatorname{sech}^2 x \cdot \tanh x \quad f''(1) \approx -0.6397$

$P_1(x) = f(1) + f'(1)(x - 1) = 0.7616 + 0.42(x - 1)$

$P_2(x) = 0.7616 + 0.42(x - 1) - \frac{0.6397}{2}(x - 1)^2$



37. (a) $y = 10 + 15 \cosh \frac{x}{15}, -15 \leq x \leq 15$



(b) At $x = \pm 15, y = 10 + 15 \cosh(1) \approx 33.146$.

At $x = 0, y = 10 + 15 \cosh(0) = 25$.

(c) $y' = \sinh \frac{x}{15}$. At $x = 15, y' = \sinh(1) \approx 1.175$

39. Let $u = 1 - 2x, du = -2 dx$.

$$\begin{aligned} \int \sinh(1 - 2x) dx &= -\frac{1}{2} \int \sinh u (-2) du \\ &= -\frac{1}{2} \cosh(1 - 2x) + C \end{aligned}$$

41. Let $u = \cosh(x - 1), du = \sinh(x - 1) dx$.

$$\int \cosh^2(x - 1) \sinh(x - 1) dx = \frac{1}{3} \cosh^3(x - 1) + C$$

43. Let $u = \sinh x$, $du = \cosh x \, dx$.

$$\int \frac{\cosh x}{\sinh x} dx = \ln |\sinh x| + C$$

45. Let $u = \frac{x^2}{2}$, $du = x \, dx$.

$$\int x \operatorname{csch}^2 \frac{x^2}{2} dx = \int \left(\operatorname{csch}^2 \frac{x^2}{2} \right) x dx = -\coth \frac{x^2}{2} + C$$

47. Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2} dx$.

$$\int \frac{\operatorname{csch}(1/x) \coth(1/x)}{x^2} dx = -\int \operatorname{csch} \frac{1}{x} \coth \frac{1}{x} \left(-\frac{1}{x^2} \right) dx = \operatorname{csch} \frac{1}{x} + C$$

49. $\int_0^4 \frac{1}{25 - x^2} dx = \frac{1}{10} \int \frac{1}{5 - x} dx + \frac{1}{10} \int \frac{1}{5 + x} dx = \left[\frac{1}{10} \ln \left| \frac{5 + x}{5 - x} \right| \right]_0^4 = \frac{1}{10} \ln 9 = \frac{1}{5} \ln 3$

51. Let $u = 2x$, $du = 2 \, dx$.

$$\int_0^{\sqrt{2}/4} \frac{2}{\sqrt{1 - 4x^2}} dx = \int_0^{\sqrt{2}/4} \frac{1}{\sqrt{1 - (2x)^2}} (2) dx = \left[\arcsin(2x) \right]_0^{\sqrt{2}/4} = \frac{\pi}{4}$$

53. Let $u = x^2$, $du = 2x \, dx$.

$$\int \frac{x}{x^4 + 1} dx = \frac{1}{2} \int \frac{2x}{(x^2)^2 + 1} dx = \frac{1}{2} \arctan(x^2) + C$$

55. $y = \cosh^{-1}(3x)$

$$y' = \frac{3}{\sqrt{9x^2 - 1}}$$

57. $y = \sinh^{-1}(\tan x)$

$$y' = \frac{1}{\sqrt{\tan^2 x + 1}} (\sec^2 x) = |\sec x|$$

59. $y = \coth^{-1}(\sin 2x)$

$$y' = \frac{1}{1 - \sin^2 2x} (2 \cos 2x) = 2 \sec 2x$$

61. $y = 2x \sinh^{-1}(2x) - \sqrt{1 + 4x^2}$

$$y' = 2x \left(\frac{2}{\sqrt{1 + 4x^2}} \right) + 2 \sinh^{-1}(2x) - \frac{4x}{\sqrt{1 + 4x^2}} = 2 \sinh^{-1}(2x)$$

63. See page 395.

65. $y = a \operatorname{sech}^{-1} \frac{x}{a} - \sqrt{a^2 - x^2}$, $a > 0$

$$\frac{dy}{dx} = \frac{-1}{(x/a)\sqrt{1 - (x^2/a^2)}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{-a^2}{x\sqrt{a^2 - x^2}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{x^2 - a^2}{x\sqrt{a^2 - x^2}} = \frac{-\sqrt{a^2 - x^2}}{x}$$

67. $\int \frac{1}{\sqrt{1 + e^{2x}}} dx = \int \frac{e^x}{e^x \sqrt{1 + (e^x)^2}} dx = -\operatorname{csch}^{-1}(e^x) + C = -\ln \left(\frac{1 + \sqrt{1 + e^{2x}}}{e^x} \right) + C$

69. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$.

$$\int \frac{1}{\sqrt{x}\sqrt{1+x}} dx = 2 \int \frac{1}{\sqrt{1 + (\sqrt{x})^2}} \left(\frac{1}{2\sqrt{x}} \right) dx = 2 \sinh^{-1} \sqrt{x} + C = 2 \ln(\sqrt{x} + \sqrt{1+x}) + C$$

71. $\int \frac{-1}{4x - x^2} dx = \int \frac{1}{(x-2)^2 - 4} dx = \frac{1}{4} \ln \left| \frac{(x-2) - 2}{(x-2) + 2} \right| = \frac{1}{4} \ln \left| \frac{x-4}{x} \right| + C$

$$\begin{aligned}
 73. \int \frac{1}{1-4x-2x^2} dx &= \int \frac{1}{3-2(x+1)^2} dx = \frac{-1}{\sqrt{2}} \int \frac{\sqrt{2}}{[\sqrt{2}(x+1)]^2 - (\sqrt{3})^2} dx \\
 &= \frac{-1}{2\sqrt{6}} \ln \left| \frac{\sqrt{2}(x+1) - \sqrt{3}}{\sqrt{2}(x+1) + \sqrt{3}} \right| + C = \frac{1}{2\sqrt{6}} \ln \left| \frac{\sqrt{2}(x+1) + \sqrt{3}}{\sqrt{2}(x+1) - \sqrt{3}} \right| + C
 \end{aligned}$$

$$75. \text{ Let } u = 4x - 1, du = 4 dx.$$

$$y = \int \frac{1}{\sqrt{80+8x-16x^2}} dx = \frac{1}{4} \int \frac{4}{\sqrt{81-(4x-1)^2}} dx = \frac{1}{4} \arcsin\left(\frac{4x-1}{9}\right) + C$$

$$\begin{aligned}
 77. y &= \int \frac{x^3 - 21x}{5+4x-x^2} dx = \int \left(-x-4 + \frac{20}{5+4x-x^2}\right) dx = \int (-x-4) dx + 20 \int \frac{1}{3^2 - (x-2)^2} dx \\
 &= -\frac{x^2}{2} - 4x + \frac{20}{6} \ln \left| \frac{3+(x-2)}{3-(x-2)} \right| + C = -\frac{x^2}{2} - 4x + \frac{10}{3} \ln \left| \frac{1+x}{5-x} \right| + C = -\frac{x^2}{2} - 4x - \frac{10}{3} \ln \left| \frac{5-x}{x+1} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 79. A &= 2 \int_0^4 \operatorname{sech} \frac{x}{2} dx \\
 &= 2 \int_0^4 \frac{2}{e^{x/2} + e^{-x/2}} dx \\
 &= 4 \int_0^4 \frac{e^{x/2}}{(e^{x/2})^2 + 1} dx \\
 &= \left[8 \arctan(e^{x/2}) \right]_0^4 \\
 &= 8 \arctan(e^2) - 2\pi \approx 5.207
 \end{aligned}$$

$$\begin{aligned}
 81. A &= \int_0^2 \frac{5x}{\sqrt{x^4+1}} dx \\
 &= \frac{5}{2} \int_0^2 \frac{2x}{\sqrt{(x^2)^2+1}} dx \\
 &= \left[\frac{5}{2} \ln(x^2 + \sqrt{x^4+1}) \right]_0^2 \\
 &= \frac{5}{2} \ln(4 + \sqrt{17}) \approx 5.237
 \end{aligned}$$

$$\begin{aligned}
 83. \int \frac{3k}{16} dt &= \int \frac{1}{x^2 - 12x + 32} dx \\
 \frac{3kt}{16} &= \int \frac{1}{(x-6)^2 - 4} dx = \frac{1}{2(2)} \ln \left| \frac{(x-6)-2}{(x-6)+2} \right| + C = \frac{1}{4} \ln \left| \frac{x-8}{x-4} \right| + C
 \end{aligned}$$

$$\text{When } x = 0: \quad t = 0$$

$$C = -\frac{1}{4} \ln(2)$$

$$\text{When } x = 1: \quad t = 10$$

$$\frac{30k}{16} = \frac{1}{4} \ln \left| \frac{-7}{-3} \right| - \frac{1}{4} \ln(2) = \frac{1}{4} \ln\left(\frac{7}{6}\right)$$

$$k = \frac{2}{15} \ln\left(\frac{7}{6}\right)$$

$$\text{When } t = 20: \quad \left(\frac{3}{16}\right)\left(\frac{2}{15}\right) \ln\left(\frac{7}{6}\right)(20) = \frac{1}{4} \ln \frac{x-8}{2x-8}$$

$$\ln\left(\frac{7}{6}\right)^2 = \ln \frac{x-8}{2x-8}$$

$$\frac{49}{36} = \frac{x-8}{2x-8}$$

$$62x = 104$$

$$x = \frac{104}{62} = \frac{52}{31} \approx 1.677 \text{ kg}$$

85. As k increases, the time required for the object to reach the ground increases.

$$87. y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$89. y = \cosh^{-1} x$$

$$\cosh y = x$$

$$(\sinh y)(y') = 1$$

$$y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

$$91. y = \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

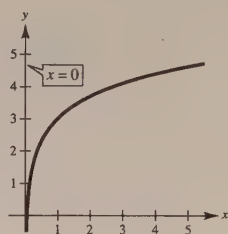
$$y' = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \left(\frac{-2}{e^x + e^{-x}}\right)\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$$

Review Exercises for Chapter 5

$$1. f(x) = \ln x + 3$$

Vertical shift 3 units upward

Vertical asymptote: $x = 0$



$$3. \ln \sqrt[5]{\frac{4x^2 - 1}{4x^2 + 1}} = \frac{1}{5} \ln \frac{(2x - 1)(2x + 1)}{4x^2 + 1} = \frac{1}{5} [\ln(2x - 1) + \ln(2x + 1) - \ln(4x^2 + 1)]$$

$$5. \ln 3 + \frac{1}{3} \ln(4 - x^2) - \ln x = \ln 3 + \ln \sqrt[3]{4 - x^2} - \ln x = \ln \left(\frac{3\sqrt[3]{4 - x^2}}{x} \right)$$

$$7. \ln \sqrt{x + 1} = 2$$

$$\sqrt{x + 1} = e^2$$

$$x + 1 = e^4$$

$$x = e^4 - 1 \approx 53.598$$

$$9. g(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$$

$$g'(x) = \frac{1}{2x}$$

$$11. f(x) = x\sqrt{\ln x}$$

$$f'(x) = \left(\frac{x}{2}\right)(\ln x)^{-1/2}\left(\frac{1}{x}\right) + \sqrt{\ln x}$$

$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1 + 2\ln x}{2\sqrt{\ln x}}$$

$$13. y = \frac{1}{b^2} \left[\ln(a + bx) + \frac{a}{a + bx} \right]$$

$$\frac{dy}{dx} = \frac{1}{b^2} \left[\frac{b}{a + bx} - \frac{ab}{(a + bx)^2} \right] = \frac{x}{(a + bx)^2}$$

$$15. y = -\frac{1}{a} \ln \left(\frac{a + bx}{x} \right) = -\frac{1}{a} [\ln(a + bx) - \ln x]$$

$$\frac{dy}{dx} = -\frac{1}{a} \left(\frac{b}{a + bx} - \frac{1}{x} \right) = \frac{1}{x(a + bx)}$$

$$17. u = 7x - 2, du = 7dx$$

$$\int \frac{1}{7x - 2} dx = \frac{1}{7} \int \frac{1}{7x - 2} (7) dx = \frac{1}{7} \ln |7x - 2| + C$$

$$19. \int \frac{\sin x}{1 + \cos x} dx = - \int \frac{-\sin x}{1 + \cos x} dx$$

$$= -\ln|1 + \cos x| + C$$

$$23. \int_0^{\pi/3} \sec \theta d\theta = \left[\ln|\sec \theta + \tan \theta| \right]_0^{\pi/3} = \ln(2 + \sqrt{3})$$

$$25. (a) \quad f(x) = \frac{1}{2}x - 3$$

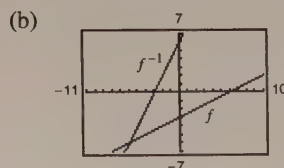
$$y = \frac{1}{2}x - 3$$

$$2(y + 3) = x$$

$$2(x + 3) = y$$

$$f^{-1}(x) = 2x + 6$$

$$21. \int_1^4 \frac{x+1}{x} dx = \int_1^4 \left(1 + \frac{1}{x}\right) dx = \left[x + \ln|x| \right]_1^4 = 3 + \ln 4$$



$$(c) \quad f^{-1}(f(x)) = f^{-1}\left(\frac{1}{2}x - 3\right) = 2\left(\frac{1}{2}x - 3\right) + 6 = x$$

$$f(f^{-1}(x)) = f(2x + 6) = \frac{1}{2}(2x + 6) - 3 = x$$

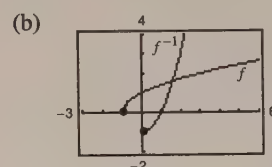
$$27. (a) \quad f(x) = \sqrt{x+1}$$

$$y = \sqrt{x+1}$$

$$y^2 - 1 = x$$

$$x^2 - 1 = y$$

$$f^{-1}(x) = x^2 - 1, x \geq 0$$



$$(c) \quad f^{-1}(f(x)) = f^{-1}(\sqrt{x+1}) = \sqrt{(x^2 - 1)^2} - 1 = x$$

$$f(f^{-1}(x)) = f(x^2 - 1) = \sqrt{(x^2 - 1) + 1}$$

$$= \sqrt{x^2} = x \text{ for } x \geq 0.$$

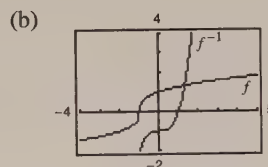
$$29. (a) \quad f(x) = \sqrt[3]{x+1}$$

$$y = \sqrt[3]{x+1}$$

$$y^3 - 1 = x$$

$$x^3 - 1 = y$$

$$f^{-1}(x) = x^3 - 1$$



$$(c) \quad f^{-1}(f(x)) = f^{-1}(\sqrt[3]{x+1}) = (\sqrt[3]{x+1})^3 - 1 = x$$

$$f(f^{-1}(x)) = f(x^3 - 1) = \sqrt[3]{(x^3 - 1) + 1} = x$$

$$31. \quad f(x) = x^3 + 2$$

$$f^{-1}(x) = (x - 2)^{1/3}$$

$$(f^{-1})'(x) = \frac{1}{3}(x - 2)^{-2/3}$$

$$(f^{-1})'(-1) = \frac{1}{3}(-1 - 2)^{-2/3} = \frac{1}{3(-3)^{2/3}}$$

$$= \frac{1}{3^{5/3}} \approx 0.160$$

$$33. \quad f(x) = \tan x$$

$$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$$

$$f'(x) = \sec^2 x$$

$$f'\left(\frac{\pi}{6}\right) = \frac{4}{3}$$

$$(f^{-1})'\left(\frac{\sqrt{3}}{3}\right) = \frac{1}{f'(\pi/6)} = \frac{3}{4}$$

35. (a) $f(x) = \ln \sqrt{x}$

$$y = \ln \sqrt{x}$$

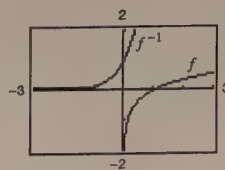
$$e^y = \sqrt{x}$$

$$e^{2y} = x$$

$$e^{2x} = y$$

$$f^{-1}(x) = e^{2x}$$

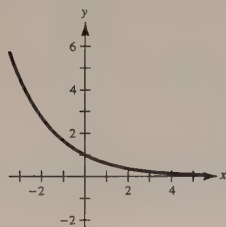
(b)



(c) $f^{-1}(f(x)) = f^{-1}(\ln \sqrt{x}) = e^{2 \ln \sqrt{x}} = e^{\ln x} = x$

$$f(f^{-1}(x)) = f(e^{2x}) = \ln \sqrt{e^{2x}} = \ln e^x = x$$

37. $y = e^{-x/2}$



39. $f(x) = \ln(e^{-x^2}) = -x^2$

$$f'(x) = -2x$$

41. $g(t) = t^2 e^t$

$$g'(t) = t^2 e^t + 2te^t = te^t(t + 2)$$

43. $y = \sqrt{e^{2x} + e^{-2x}}$

$$y' = \frac{1}{2}(e^{2x} + e^{-2x})^{-1/2}(2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

45. $g(x) = \frac{x^2}{e^x}$

$$g'(x) = \frac{e^x(2x) - x^2 e^x}{e^{2x}} = \frac{x(2-x)}{e^x}$$

47. $y(\ln x) + y^2 = 0$

$$y\left(\frac{1}{x}\right) + (\ln x)\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$$

$$(2y + \ln x)\frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{dx} = \frac{-y}{x(2y + \ln x)}$$

49. Let $u = -3x^2$, $du = -6x dx$.

$$\int x e^{-3x^2} dx = -\frac{1}{6} \int e^{-3x^2} (-6x) dx = -\frac{1}{6} e^{-3x^2} + C$$

51. $\int \frac{e^{4x} - e^{2x} + 1}{e^x} dx = \int (e^{3x} - e^x + e^{-x}) dx$

$$= \frac{1}{3} e^{3x} - e^x - e^{-x} + C$$

$$= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C$$

53. $\int x e^{1-x^2} dx = -\frac{1}{2} \int e^{1-x^2} (-2x) dx$

$$= -\frac{1}{2} e^{1-x^2} + C$$

55. Let $u = e^x - 1$, $du = e^x dx$.

$$\int \frac{e^x}{e^x - 1} dx = \ln|e^x - 1| + C$$

57.

$$y = e^x(a \cos 3x + b \sin 3x)$$

$$y' = e^x(-3a \sin 3x + 3b \cos 3x) + e^x(a \cos 3x + b \sin 3x)$$

$$= e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$

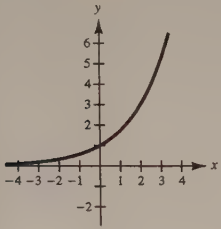
$$y'' = e^x[3(-3a + b) \cos 3x - 3(a + 3b) \sin 3x] + e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$$

$$= e^x[(-6a - 8b) \sin 3x + (-8a + 6b) \cos 3x]$$

$$y'' - 2y' + 10y = e^x[(-6a - 8b) - 2(-3a + b) + 10b] \sin 3x + [(-8a + 6b) - 2(a + 3b) + 10a] \cos 3x = 0$$

$$59. \text{Area} = \int_0^4 x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^4 = -\frac{1}{2}(e^{-16} - 1) \approx 0.500$$

$$61. y = 3^{x/2}$$



$$65. f(x) = 3^{x-1}$$

$$f'(x) = 3^{x-1} \ln 3$$

$$69. g(x) = \log_3 \sqrt{1-x} = \frac{1}{2} \log_3(1-x)$$

$$g'(x) = \left(\frac{1}{2} \right) \frac{-1}{(1-x) \ln 3} = \frac{1}{2(x-1) \ln 3}$$

$$73. (a) y = x^a$$

$$y' = ax^{a-1}$$

$$(b) y = a^x$$

$$y' = (\ln a)a^x$$

$$75. 10,000 = Pe^{(0.07)(15)}$$

$$P = \frac{10,000}{e^{1.05}} \approx \$3499.38$$

$$79. P = Ce^{0.015t}$$

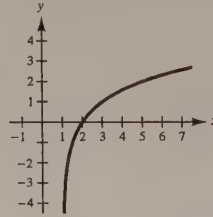
$$2C = Ce^{0.015t}$$

$$2 = e^{0.015t}$$

$$\ln 2 = 0.015t$$

$$t = \frac{\ln 2}{0.015} \approx 46.21 \text{ years}$$

$$63. y = \log_2(x-1)$$



$$67. y = x^{2x+1}$$

$$\ln y = (2x+1) \ln x$$

$$\frac{y'}{y} = \frac{2x+1}{x} + 2 \ln x$$

$$y' = y \left(\frac{2x+1}{x} + 2 \ln x \right) = x^{2x+1} \left(\frac{2x+1}{x} + 2 \ln x \right)$$

$$71. \int (x+1)5^{(x+1)^2} dx = \left(\frac{1}{2} \right) \frac{1}{\ln 5} 5^{(x+1)^2} + C$$

$$(c) y = x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y} y' = x \cdot \frac{1}{x} + (1) \ln x$$

$$y' = y(1 + \ln x)$$

$$y' = x^x(1 + \ln x)$$

$$(d) y = a^a$$

$$y' = 0$$

$$77. \frac{dP}{dh} = kP, P(0) = 30$$

$$P(h) = 30e^{kh}$$

$$P(18,000) = 30e^{18,000k} = 15$$

$$k = \frac{\ln(1/2)}{18,000} = \frac{-\ln 2}{18,000}$$

$$P(h) = 30e^{-(h \ln 2)/18,000}$$

$$P(35,000) = 30e^{-(35,000 \ln 2)/18,000} \approx 7.79 \text{ inches}$$

$$81. \frac{dy}{dx} = \frac{x^2 + 3}{x}$$

$$\int dy = \int \left(x + \frac{3}{x} \right) dx$$

$$y = \frac{x^2}{2} + 3 \ln|x| + C$$

83. $y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y} dy = \int 2x dx$$

$$\ln|y| = x^2 + C_1$$

$$e^{x^2 + C_1} = y$$

$$y = Ce^{x^2}$$

85. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ (homogeneous differential equation)

$$(x^2 + y^2) dx - 2xy dy = 0$$

$$\text{Let } y = vx, dy = x dv + v dx.$$

$$(x^2 + v^2x^2) dx - 2x(vx)(x dv + v dx) = 0$$

$$(x^2 + v^2x^2 - 2x^2v^2) dx - 2x^3v dv = 0$$

$$(x^2 - x^2v^2) dx = 2x^3v dv$$

$$(1 - v^2) dx = 2xv dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1 - v^2} dv$$

$$\ln|x| = -\ln|1 - v^2| + C_1 = -\ln|1 - v^2| + \ln C$$

$$x = \frac{C}{1 - v^2} = \frac{C}{1 - (y/x)^2} = \frac{Cx^2}{x^2 - y^2}$$

$$1 = \frac{Cx}{x^2 - y^2} \quad \text{or} \quad C_1 = \frac{x}{x^2 - y^2}$$

87. $y = C_1x + C_2x^3$

$$y' = C_1 + 3C_2x^2$$

$$y'' = 6C_2x$$

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + (C_1x + C_2x^3) \\ &= 6C_2x^3 - 3C_1x - 9C_2x^3 + 3C_1x + 3C_2x^3 = 0 \end{aligned}$$

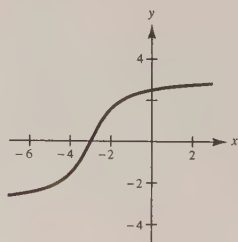
$$x = 2, y = 0: 0 = 2C_1 + 8C_2 \Rightarrow C_1 = -4C_2$$

$$x = 2, y' = 4: 4 = C_1 + 12C_2$$

$$4 = (-4C_2) + 12C_2 = 8C_2 \Rightarrow C_2 = \frac{1}{2}, C_1 = -2$$

$$y = -2x + \frac{1}{2}x^3$$

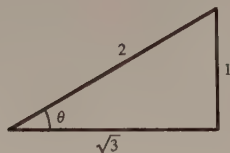
89. $f(x) = 2 \arctan(x + 3)$



91. (a) Let $\theta = \arcsin \frac{1}{2}$

$$\sin \theta = \frac{1}{2}$$

$$\sin\left(\arcsin \frac{1}{2}\right) = \sin \theta = \frac{1}{2}$$



(b) Let $\theta = \arcsin \frac{1}{2}$

$$\sin \theta = \frac{1}{2}$$

$$\cos\left(\arcsin \frac{1}{2}\right) = \cos \theta = \frac{\sqrt{3}}{2}$$

93. $y = \tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$

$$y' = \frac{(1-x^2)^{1/2} + x^2(1-x^2)^{-1/2}}{1-x^2} = (1-x^2)^{-3/2}$$

95. $y = x \operatorname{arcsec} x$

$$y' = \frac{x}{|x|\sqrt{x^2-1}} + \operatorname{arcsec} x$$

97. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2} \arcsin x$

$$y' = \frac{2x \arcsin x}{\sqrt{1-x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}} \arcsin x = (\arcsin x)^2$$

99. Let $u = e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{1}{e^{2x} + e^{-2x}} dx = \int \frac{e^{2x}}{1 + e^{4x}} dx = \frac{1}{2} \int \frac{1}{1 + (e^{2x})^2} (2e^{2x}) dx = \frac{1}{2} \arctan(e^{2x}) + C$$

101. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{1-x^4}} dx = \frac{1}{2} \int \frac{1}{\sqrt{1-(x^2)^2}} (2x) dx = \frac{1}{2} \arcsin x^2 + C$$

103. Let $u = 16 + x^2$, $du = 2x dx$.

$$\int \frac{x}{16+x^2} dx = \frac{1}{2} \int \frac{1}{16+x^2} (2x) dx = \frac{1}{2} \ln(16+x^2) + C$$

105. Let $u = \arctan\left(\frac{x}{2}\right)$, $du = \frac{2}{4+x^2} dx$.

$$\int \frac{\arctan(x/2)}{4+x^2} dx = \frac{1}{2} \int \left(\arctan \frac{x}{2}\right) \left(\frac{2}{4+x^2}\right) dx = \frac{1}{4} \left(\arctan \frac{x}{2}\right)^2 + C$$

107. $\int \frac{dy}{\sqrt{A^2 - y^2}} = \int \sqrt{\frac{k}{m}} dt$

$$\arcsin\left(\frac{y}{A}\right) = \sqrt{\frac{k}{m}} t + C$$

Since $y = 0$ when $t = 0$, you have $C = 0$. Thus,

$$\sin\left(\sqrt{\frac{k}{m}} t\right) = \frac{y}{A}$$

$$y = A \sin\left(\sqrt{\frac{k}{m}} t\right)$$

109. $y = 2x - \cosh \sqrt{x}$

$$y' = 2 - \frac{1}{2\sqrt{x}} (\sinh \sqrt{x}) = 2 - \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

111. Let $u = x^2$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{x^4-1}} dx = \frac{1}{2} \int \frac{1}{\sqrt{(x^2)^2-1}} (2x) dx = \frac{1}{2} \ln(x^2 + \sqrt{x^4-1}) + C$$

Problem Solving for Chapter 5

1. $\tan \theta_1 = \frac{3}{x}$

$$\tan \theta_2 = \frac{6}{10 - x}$$

Minimize $\theta_1 + \theta_2$:

$$f(x) = \theta_1 + \theta_2 = \arctan\left(\frac{3}{x}\right) + \arctan\left(\frac{6}{10 - x}\right)$$

$$f'(x) = \frac{1}{1 + \frac{9}{x^2}} \left(\frac{-3}{x^2}\right) + \frac{1}{1 + \frac{36}{(10 - x)^2}} \left(\frac{6}{(10 - x)^2}\right) = 0$$

$$\frac{3}{x^2 + 9} = \frac{6}{(10 - x)^2 + 36}$$

$$(10 - x)^2 + 36 = 2(x^2 + 9)$$

$$100 - 20x + x^2 + 36 = 2x^2 + 18$$

$$x^2 + 20x - 118 = 0$$

$$x = \frac{-20 \pm \sqrt{20^2 - 4(-118)}}{2} = -10 \pm \sqrt{218}$$

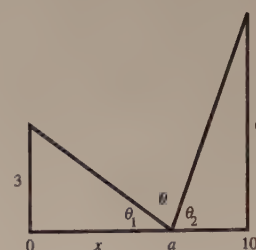
$$a = -10 + \sqrt{218} \approx 4.7648 \quad f(a) \approx 1.4153$$

$$\theta = \pi - (\theta_1 + \theta_2) \approx 1.7263 \quad \text{or} \quad 98.9^\circ$$

Endpoints: $a = 0$: $\theta \approx 1.0304$

$$a = 10$$
: $\theta \approx 1.2793$

Maximum is 1.7263 at $a = -10 + \sqrt{218} \approx 4.7648$.



3. $f(x) = \sin(\ln x)$

(a) Domain: $x > 0$ or $(0, \infty)$

(b) $f(x) = 1 = \sin(\ln x) \Rightarrow \ln x = \frac{\pi}{2} + 2k\pi$

Two values are $x = e^{\pi/2}, e^{(\pi/2) + 2\pi}$.

(c) $f(x) = -1 = \sin(\ln x) \Rightarrow \ln x = \frac{3\pi}{2} + 2k\pi$

Two values are $x = e^{-\pi/2}, e^{3\pi/2}$.

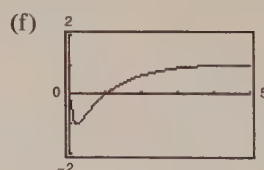
(d) Since the range of the sine function is $[-1, 1]$, parts (b) and (c) show that the range of f is $[-1, 1]$.

(e) $f'(x) = \frac{1}{x} \cos(\ln x)$

$$f'(x) = 0 \Rightarrow \cos(\ln x) = 0 \Rightarrow \ln x = \frac{\pi}{2} + k\pi \Rightarrow$$

$$x = e^{\pi/2} \text{ on } [1, 10]$$

$$\left. \begin{array}{l} f(e^{\pi/2}) = 1 \\ f(1) = 0 \\ f(10) \approx 0.7440 \end{array} \right\} \text{Maximum is 1 at } x = e^{\pi/2} \approx 4.8105$$



$\lim_{x \rightarrow 0^+} f(x)$ seems to be $-\frac{1}{2}$. (This is incorrect.)

(g) For the points $x = e^{\pi/2}, e^{-3\pi/2}, e^{-7\pi/2}, \dots$

we have $f(x) = 1$.

For the points $x = e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, \dots$

we have $f(x) = -1$.

That is, as $x \rightarrow 0^+$, there is an infinite number of points where $f(x) = 1$, and an infinite number where $f(x) = -1$. Thus $\lim_{x \rightarrow 0^+} \sin(\ln x)$ does not exist.

You can verify this by graphing $f(x)$ on small intervals close to the origin.

$$5. (a) \frac{\text{Area sector}}{\text{Area circle}} = \frac{t}{2\pi} \Rightarrow \text{Area sector} = \frac{t}{2\pi}(\pi) = \frac{t}{2}$$

$$(b) \text{Area } AOP = \frac{1}{2}(\text{base})(\text{height}) - \int_1^{\cosh t} \sqrt{x^2 - 1} dx$$

$$A(t) = \frac{1}{2} \cosh t \cdot \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1} dx$$

$$A'(t) = \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sqrt{\cosh^2 t - 1} \sinh t$$

$$= \frac{1}{2}[\cosh^2 t + \sinh^2 t] - \sinh^2 t$$

$$= \frac{1}{2}[\cosh^2 t - \sinh^2 t] = \frac{1}{2}$$

$$A(t) = \frac{1}{2}t + C. \text{ But, } A(0) = C = 0 \Rightarrow C = 0$$

$$\text{Thus, } A(t) = \frac{1}{2}t \text{ or } t = 2A(t).$$

$$7. y = \ln x$$

$$y' = \frac{1}{x}$$

$$y - b = \frac{1}{a}(x - a)$$

$$y = \frac{1}{a}x + b - 1 \text{ Tangent line}$$

$$\text{If } x = 0, c = b - 1. \text{ Thus, } b - c = b - (b - 1) = 1.$$

$$9. \text{ Let } u = 1 + \sqrt{x}, \sqrt{x} = u - 1, x = u^2 - 2u + 1,$$

$$dx = (2u - 2)du.$$

$$\text{Area} = \int_1^4 \frac{1}{\sqrt{x} + x} dx = \int_2^3 \frac{2u - 2}{(u - 1) + (u^2 - 2u + 1)} du$$

$$= \int_2^3 \frac{2(u - 1)}{u^2 - u} du$$

$$= \int_2^3 \frac{2}{u} du$$

$$= \left[2 \ln|u| \right]_2^3$$

$$= 2 \ln 3 - 2 \ln 2 = 2 \ln\left(\frac{3}{2}\right)$$

$$\approx 0.8109$$

$$11. (a) \frac{dy}{dt} = y^{1.01}$$

$$\int y^{-1.01} dy = \int dt$$

$$\frac{y^{-0.01}}{-0.01} = t + C_1$$

$$\frac{1}{y^{0.01}} = -0.01t + C$$

$$y^{0.01} = \frac{1}{C - 0.01t}$$

$$y = \frac{1}{(C - 0.01t)^{100}}$$

$$y(0) = 1: 1 = \frac{1}{C^{100}} \Rightarrow C = 1$$

$$\text{Hence, } y = \frac{1}{(1 - 0.01t)^{100}}$$

$$\text{For } T = 100, \lim_{t \rightarrow T^-} y = \infty.$$

$$(b) \int y^{-(1+\varepsilon)} dy = \int k dt$$

$$\frac{y^{-\varepsilon}}{-\varepsilon} = kt + C_1$$

$$y^{-\varepsilon} = -\varepsilon kt + C$$

$$y = \frac{1}{(C - \varepsilon kt)^{1/\varepsilon}}$$

$$y(0) = y_0 = \frac{1}{C^{1/\varepsilon}} \Rightarrow C^{1/\varepsilon} = \frac{1}{y_0} \Rightarrow C = \left(\frac{1}{y_0}\right)^\varepsilon$$

$$\text{Hence, } y = \frac{1}{\left(\frac{1}{y_0^\varepsilon} - \varepsilon kt\right)^{1/\varepsilon}}$$

$$\text{For } t \rightarrow \frac{1}{y_0^\varepsilon \varepsilon k}, y \rightarrow \infty.$$

13. Since $\frac{dy}{dt} = k(y - 20)$,

$$\int \frac{1}{y - 20} dy = \int k dt$$

$$\ln|y - 20| = kt + C$$

$$y = Ce^{kt} + 20.$$

When $t = 0$, $y = 72$. Therefore, $C = 52$.

When $t = 1$, $y = 48$. Therefore, $48 = 52e^k + 20$, $e^k = (28/52) = (7/13)$, and $k = \ln(7/13)$.

Thus, $y = 52e^{[\ln(7/13)]t} + 20$.

When $t = 5$, $y = 52e^{5 \ln(7/13)} + 20 \approx 22.35^\circ$.

15. (a) $\frac{dS}{dt} = k_1 S(L - S)$

$S = \frac{L}{1 + Ce^{-kt}}$ is a solution because

$$\frac{dS}{dt} = -L(1 + Ce^{-kt})^{-2}(-Cke^{-kt})$$

$$= \frac{LCke^{-kt}}{(1 + Ce^{-kt})^2}$$

$$= \left(\frac{k}{L}\right) \frac{L}{1 + Ce^{-kt}} \cdot \frac{CLe^{-kt}}{1 + Ce^{-kt}}$$

$$= \left(\frac{k}{L}\right) \frac{L}{1 + Ce^{-kt}} \cdot \left(L - \frac{L}{1 + Ce^{-kt}}\right)$$

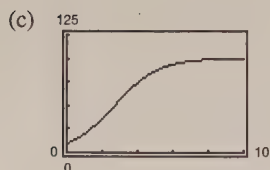
$$= k_1 S(L - S), \text{ where } k_1 = \frac{k}{L}.$$

$L = 100$. Also, $S = 10$ when $t = 0 \Rightarrow C = 9$. And,

$S = 20$ when $t = 1 \Rightarrow k = -\ln(4/9)$.

Particular Solution. $S = \frac{100}{1 + 9e^{\ln(4/9)t}}$

$$= \frac{100}{1 + 9e^{-0.8109t}}$$



(b) $\frac{dS}{dt} = k_1 S(100 - S)$

$$\frac{d^2S}{dt^2} = k_1 \left[S \left(-\frac{dS}{dt} \right) + (100 - S) \frac{dS}{dt} \right]$$

$$= k_1 (100 - 2S) \frac{dS}{dt}$$

$$= 0 \text{ when } S = 50 \text{ or } \frac{dS}{dt} = 0.$$

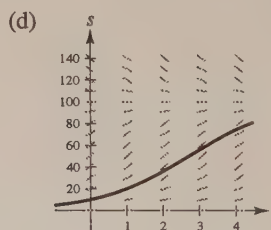
Choosing $S = 50$, we have:

$$50 = \frac{100}{1 + 9e^{\ln(4/9)t}}$$

$$2 = 1 + 9e^{\ln(4/9)t}$$

$$\frac{\ln(1/9)}{\ln(4/9)} = t$$

$t \approx 2.7$ months (This is the inflection point)



(e) Sales will decrease toward the line $S = L$.

CHAPTER 6

Applications of Integration

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CHAPTER 6

Applications of Integration

Section 6.1 Area of a Region Between Two Curves

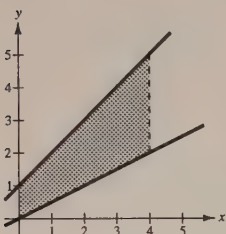
Solutions to Odd-Numbered Exercises

$$1. A = \int_0^6 [0 - (x^2 - 6x)] dx = - \int_0^6 (x^2 - 6x) dx$$

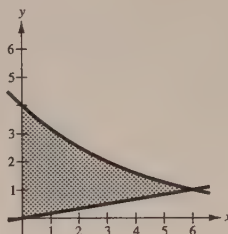
$$3. A = \int_0^3 [(-x^2 + 2x + 3) - (x^2 - 4x + 3)] dx = \int_0^3 (-2x^2 + 6x) dx$$

$$5. A = 2 \int_{-1}^0 3(x^3 - x) dx = 6 \int_{-1}^0 (x^3 - x) dx \quad \text{or} \quad -6 \int_0^1 (x^3 - x) dx$$

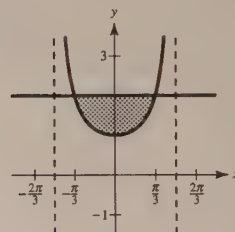
$$7. \int_0^4 \left[(x + 1) - \frac{x}{2} \right] dx$$



$$9. \int_0^6 \left[4(2^{-x/3}) - \frac{x}{6} \right] dx$$



$$11. \int_{-\pi/3}^{\pi/3} [2 - \sec x] dx$$

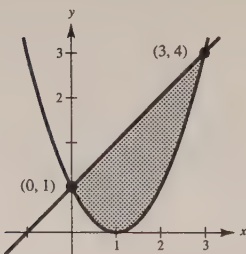


$$13. f(x) = x + 1$$

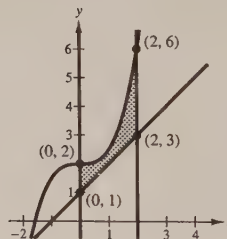
$$g(x) = (x - 1)^2$$

$$A \approx 4$$

Matches (d)



$$\begin{aligned} 15. A &= \int_0^2 \left[\left(\frac{1}{2}x^3 + 2 \right) - (x + 1) \right] dx \\ &= \int_0^2 \left(\frac{1}{2}x^3 - x + 1 \right) dx \\ &= \left[\frac{x^4}{8} - \frac{x^2}{2} + x \right]_0^2 \\ &= \left(\frac{16}{8} - \frac{4}{2} + 2 \right) - 0 = 2 \end{aligned}$$

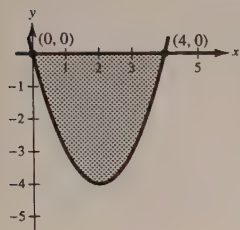


17. The points of intersection are given by:

$$x^2 - 4x = 0$$

$$x(x - 4) = 0 \text{ when } x = 0, 4$$

$$\begin{aligned} A &= \int_0^4 [g(x) - f(x)] dx \\ &= -\int_0^4 (x^2 - 4x) dx \\ &= -\left[\frac{x^3}{3} - 2x^2\right]_0^4 \\ &= \frac{32}{3} \end{aligned}$$

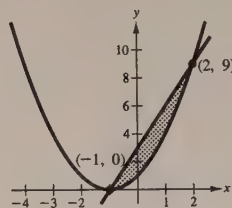


19. The points of intersection are given by:

$$x^2 + 2x + 1 = 3x + 3$$

$$(x - 2)(x + 1) = 0 \text{ when } x = -1, 2$$

$$\begin{aligned} A &= \int_{-1}^2 [g(x) - f(x)] dx \\ &= \int_{-1}^2 [(3x + 3) - (x^2 + 2x + 1)] dx \\ &= \int_{-1}^2 (2 + x - x^2) dx \\ &= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2 = \frac{9}{2} \end{aligned}$$



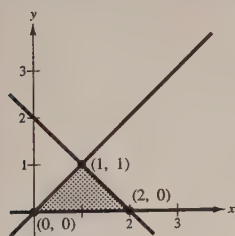
21. The points of intersection are given by:

$$x = 2 - x \text{ and } x = 0 \text{ and } 2 - x = 0$$

$$x = 1 \quad x = 0 \quad x = 2$$

$$A = \int_0^1 [(2 - y) - (y)] dy = \left[2y - y^2\right]_0^1 = 1$$

Note that if we integrate with respect to x , we need two integrals. Also, note that the region is a triangle.

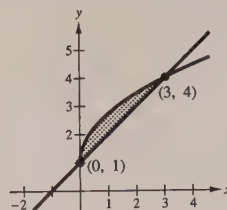


23. The points of intersection are given by:

$$\sqrt{3x} + 1 = x + 1$$

$$\sqrt{3x} = x \text{ when } x = 0, 3$$

$$\begin{aligned} A &= \int_0^3 [f(x) - g(x)] dx \\ &= \int_0^3 [(\sqrt{3x} + 1) - (x + 1)] dx \\ &= \int_0^3 [(3x)^{1/2} - x] dx \\ &= \left[\frac{2}{9}(3x)^{3/2} - \frac{x^2}{2}\right]_0^3 = \frac{3}{2} \end{aligned}$$

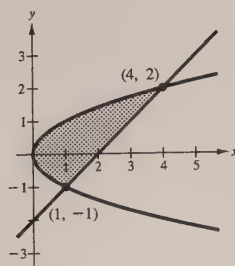


25. The points of intersection are given by:

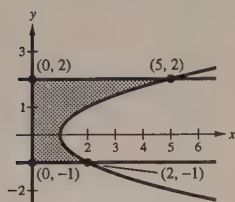
$$y^2 = y + 2$$

$$(y - 2)(y + 1) = 0 \text{ when } y = -1, 2$$

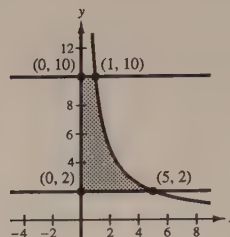
$$\begin{aligned} A &= \int_{-1}^2 [g(y) - f(y)] dy \\ &= \int_{-1}^2 [(y + 2) - y^2] dy \\ &= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^2 = \frac{9}{2} \end{aligned}$$



$$\begin{aligned}
 27. A &= \int_{-1}^2 [f(y) - g(y)] dy \\
 &= \int_{-1}^2 [(y^2 + 1) - 0] dy \\
 &= \left[\frac{y^3}{3} + y \right]_{-1}^2 = 6
 \end{aligned}$$

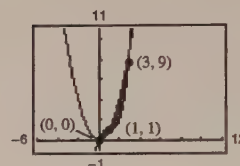


$$\begin{aligned}
 29. y &= \frac{10}{x} \Rightarrow x = \frac{10}{y} \\
 A &= \int_2^{10} \frac{10}{y} dy \\
 &= \left[10 \ln y \right]_2^{10} \\
 &= 10(\ln 10 - \ln 2) \\
 &= 10 \ln 5 \approx 16.0944
 \end{aligned}$$



31. The points of intersection are given by:

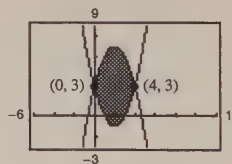
$$\begin{aligned}
 x^3 - 3x^2 + 3x &= x^2 \\
 x(x-1)(x-3) &= 0 \quad \text{when } x = 0, 1, 3 \\
 A &= \int_0^1 [f(x) - g(x)] dx + \int_1^3 [g(x) - f(x)] dx \\
 &= \int_0^1 [(x^3 - 3x^2 + 3x) - x^2] dx + \int_1^3 [x^2 - (x^3 - 3x^2 + 3x)] dx \\
 &= \int_0^1 (x^3 - 4x^2 + 3x) dx + \int_1^3 (-x^3 + 4x^2 - 3x) dx \\
 &= \left[\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2 \right]_0^1 + \left[-\frac{x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2 \right]_1^3 = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}
 \end{aligned}$$



Numerical Approximation: $0.417 + 2.667 \approx 3.083$

33. The points of intersection are given by:

$$\begin{aligned}
 x^2 - 4x + 3 &= 3 + 4x - x^2 \\
 2x(x-4) &= 0 \quad \text{when } x = 0, 4 \\
 A &= \int_0^4 [(3 + 4x - x^2) - (x^2 - 4x + 3)] dx \\
 &= \int_0^4 (-2x^2 + 8x) dx \\
 &= \left[-\frac{2x^3}{3} + 4x^2 \right]_0^4 = \frac{64}{3}
 \end{aligned}$$



Numerical Approximation: 21.333

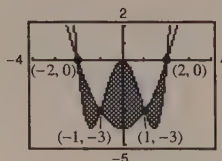
35. $f(x) = x^4 - 4x^2$, $g(x) = x^2 - 4$

The points of intersection are given by:

$$x^4 - 4x^2 = x^2 - 4$$

$$x^4 - 5x^2 + 4 = 0$$

$$(x^2 - 4)(x^2 - 1) = 0 \quad \text{when } x = \pm 2, \pm 1$$



By symmetry,

$$\begin{aligned} A &= 2 \int_0^1 [(x^4 - 4x^2) - (x^2 - 4)] dx + 2 \int_1^2 [(x^2 - 4) - (x^4 - 4x^2)] dx \\ &= 2 \int_0^1 (x^4 - 5x^2 + 4) dx + 2 \int_1^2 (-x^4 + 5x^2 - 4) dx \\ &= 2 \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x \right]_0^1 + 2 \left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x \right]_1^2 \\ &= 2 \left[\frac{1}{5} - \frac{5}{3} + 4 \right] + 2 \left[\left(-\frac{32}{5} + \frac{40}{3} - 8 \right) - \left(-\frac{1}{5} + \frac{5}{3} - 4 \right) \right] = 8. \end{aligned}$$

Numerical Approximation: $5.067 + 2.933 = 8.0$

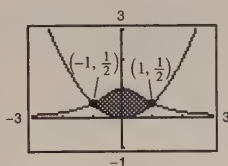
37. The points of intersection are given by:

$$\frac{1}{1+x^2} = \frac{x^2}{2}$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2 + 2)(x^2 - 1) = 0$$

$$x = \pm 1$$



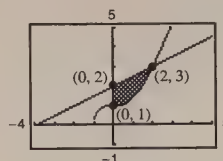
$$\begin{aligned} A &= 2 \int_0^1 [f(x) - g(x)] dx \\ &= 2 \int_0^1 \left[\frac{1}{1+x^2} - \frac{x^2}{2} \right] dx \\ &= 2 \left[\arctan x - \frac{x^3}{6} \right]_0^1 \\ &= 2 \left(\frac{\pi}{4} - \frac{1}{6} \right) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237 \end{aligned}$$

Numerical Approximation: 1.237

39. $\sqrt{1+x^3} \leq \frac{1}{2}x + 2$ on $[0, 2]$

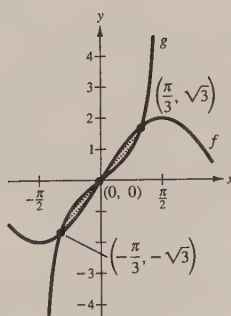
Numerical approximation: 1.759

$$A = \int_0^2 \left[\frac{1}{2}x + 2 - \sqrt{1+x^3} \right] dx \approx 1.759$$

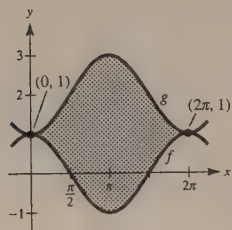


41. $A = 2 \int_0^{\pi/3} [f(x) - g(x)] dx$

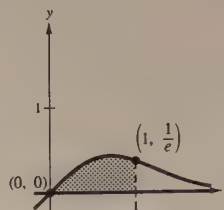
$$\begin{aligned} &= 2 \int_0^{\pi/3} (2 \sin x - \tan x) dx \\ &= 2 \left[-2 \cos x + \ln |\cos x| \right]_0^{\pi/3} \\ &= 2(1 - \ln 2) \approx 0.614 \end{aligned}$$



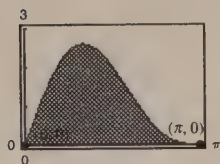
$$\begin{aligned}
 43. A &= \int_0^{2\pi} [(2 - \cos x) - \cos x] dx \\
 &= 2 \int_0^{2\pi} (1 - \cos x) dx \\
 &= 2 \left[x - \sin x \right]_0^{2\pi} = 4\pi \approx 12.566
 \end{aligned}$$



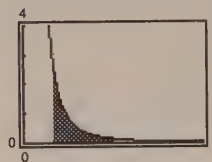
$$\begin{aligned}
 45. A &= \int_0^1 [xe^{-x^2} - 0] dx \\
 &= \left[-\frac{1}{2}e^{-x^2} \right]_0^1 = \frac{1}{2} \left(1 - \frac{1}{e} \right) \approx 0.316
 \end{aligned}$$



$$\begin{aligned}
 47. A &= \int_0^{\pi} [(2 \sin x + \sin 2x) - 0] dx \\
 &= \left[-2 \cos x - \frac{1}{2} \cos 2x \right]_0^{\pi} = 4.0
 \end{aligned}$$



$$\begin{aligned}
 49. A &= \int_1^3 \left[\frac{1}{x^2} e^{1/x} - 0 \right] dx \\
 &= \left[-e^{1/x} \right]_1^3 = e - e^{1/3} \approx 1.323
 \end{aligned}$$

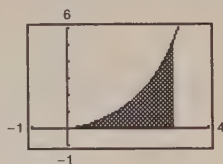


$$51. (a) y = \sqrt{\frac{x^3}{4-x}}, \quad y = 0, \quad x = 3$$

$$(b) A = \int_0^3 \sqrt{\frac{x^3}{4-x}} dx,$$

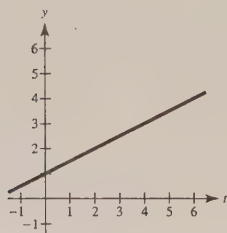
No, it cannot be evaluated by hand.

$$(c) 4.7721$$

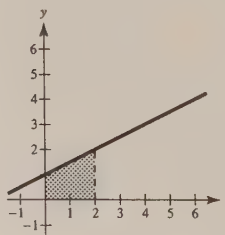


$$53. F(x) = \int_0^x \left(\frac{1}{2}t + 1 \right) dt = \left[\frac{t^2}{4} + t \right]_0^x = \frac{x^2}{4} + x$$

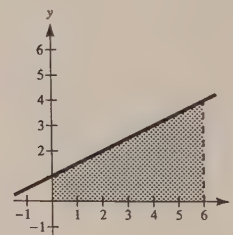
$$(a) F(0) = 0$$



$$(b) F(2) = \frac{2^2}{4} + 2 = 3$$



$$(c) F(6) = \frac{6^2}{4} + 6 = 15$$

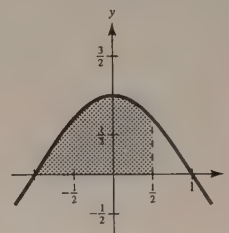
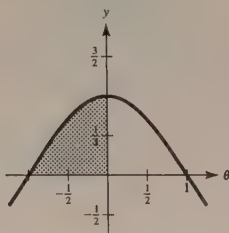
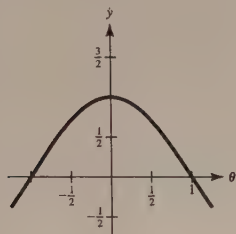


$$55. F(\alpha) = \int_{-1}^{\alpha} \cos \frac{\pi\theta}{2} d\theta = \left[\frac{2}{\pi} \sin \frac{\pi\theta}{2} \right]_{-1}^{\alpha} = \frac{2}{\pi} \sin \frac{\pi\alpha}{2} + \frac{2}{\pi}$$

$$(a) F(-1) = 0$$

$$(b) F(0) = \frac{2}{\pi} \approx 0.6366$$

$$(c) F\left(\frac{1}{2}\right) = \frac{2 + \sqrt{2}}{\pi} \approx 1.0868$$

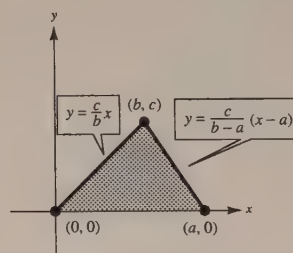


$$57. A = \int_0^c \left[\left(\frac{b-a}{c} y + a \right) - \frac{b}{c} y \right] dy$$

$$= \int_0^c \left(-\frac{a}{c} y + a \right) dy$$

$$= \left[-\frac{a}{2c} y^2 + ay \right]_0^c$$

$$= -\frac{ac}{2} + ac = \frac{ac}{2} \quad \left(= \frac{1}{2} (\text{base})(\text{height}) \right)$$



$$59. f(x) = x^3$$

$$f'(x) = 3x^2$$

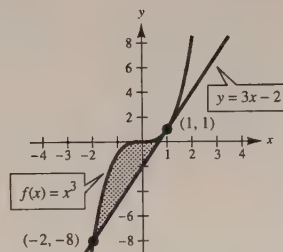
$$\text{At } (1, 1), f'(1) = 3.$$

Tangent line:

$$y - 1 = 3(x - 1) \text{ or } y = 3x - 2$$

The tangent line intersects $f(x) = x^3$ at $x = -2$.

$$A = \int_{-2}^1 [x^3 - (3x - 2)] dx = \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x \right]_{-2}^1 = \frac{27}{4}$$



61. The variable is y .

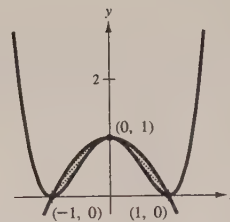
$$63. x^4 - 2x^2 + 1 \leq 1 - x^2 \text{ on } [-1, 1]$$

$$A = \int_{-1}^1 [(1 - x^2) - (x^4 - 2x^2 + 1)] dx$$

$$= \int_{-1}^1 (x^2 - x^4) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_{-1}^1 = \frac{4}{15}$$

You can use a single integral because $x^4 - 2x^2 + 1 \leq 1 - x^2$ on $[-1, 1]$.



65. Offer 2 is better because the accumulated salary (area under the curve) is larger.

$$67. \quad A = \int_{-3}^3 (9 - x^2) dx = 36$$

$$\int_{-\sqrt{9-b}}^{\sqrt{9-b}} [(9 - x^2) - b] dx = 18$$

$$\int_0^{\sqrt{9-b}} [(9 - b) - x^2] dx = 9$$

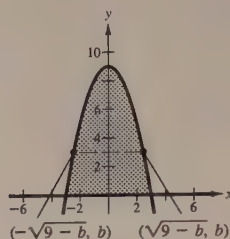
$$\left[(9 - b)x - \frac{x^3}{3} \right]_0^{\sqrt{9-b}} = 9$$

$$\frac{2}{3} (9 - b)^{3/2} = 9$$

$$(9 - b)^{3/2} = \frac{27}{2}$$

$$9 - b = \frac{9}{\sqrt[3]{4}}$$

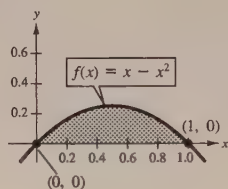
$$b = 9 - \frac{9}{\sqrt[3]{4}} \approx 3.330$$



$$69. \quad \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n (x_i - x_i^2) \Delta x$$

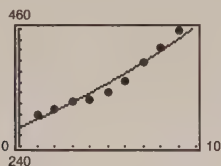
where $x_i = \frac{i}{n}$ and $\Delta x = \frac{1}{n}$ is the same as

$$\int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$

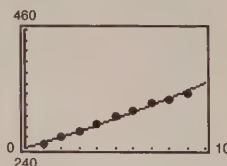


$$71. \quad \int_0^5 [(7.21 + 0.58t) - (7.21 + 0.45t)] dt = \int_0^5 0.13t dt = \left[\frac{0.13t^2}{2} \right]_0^5 = \$1.625 \text{ billion}$$

$$73. \quad (a) \quad y_1 = (275.0675)(1.0537)^t = (275.0675)e^{0.0523t}$$



$$(b) \quad y_2 = (239.9407)(1.0417)^t = (239.9407)e^{0.0408t}$$



$$(c) \quad \int_{10}^{15} (y_1 - y_2) dt \approx 649.5 \text{ billion dollars}$$

(d) No, model $y_1 > y_2$ forever because $1.0537 > 1.0417$.

No, these models are not accurate. According to news reports, $E > R$ eventually.

75. The total area is 8 times the area of the shaded region to the right. A point (x, y) is on the upper boundary of the region if

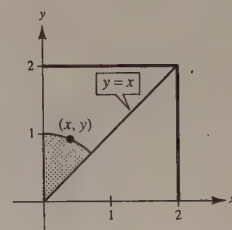
$$\sqrt{x^2 + y^2} = 2 - y$$

$$x^2 + y^2 = 4 - 4y + y^2$$

$$x^2 = 4 - 4y$$

$$4y = 4 - x^2$$

$$y = 1 - \frac{x^2}{4}$$



We now determine where this curve intersects the line $y = x$.

$$x = 1 - \frac{x^2}{4}$$

$$x^2 + 4x - 4 = 0$$

$$x = \frac{-4 \pm \sqrt{16 + 16}}{2} = -2 \pm 2\sqrt{2} \Rightarrow x = -2 + 2\sqrt{2}$$

$$\text{Total area} = 8 \int_0^{-2+2\sqrt{2}} \left(1 - \frac{x^2}{4} - x\right) dx$$

$$= 8 \left[x - \frac{x^3}{12} - \frac{x^2}{2} \right]_0^{-2+2\sqrt{2}} = \frac{16}{3}(4\sqrt{2} - 5) \approx 8(0.4379) = 3.503$$

$$77. (a) A = 2 \left[\int_0^5 \left(1 - \frac{1}{3}\sqrt{5-x}\right) dx + \int_5^{5.5} (1 - 0) dx \right]$$

$$= 2 \left[\left[x + \frac{2}{9}(5-x)^{3/2} \right]_0^5 + \left[x \right]_5^{5.5} \right] = 2 \left(5 - \frac{10\sqrt{5}}{9} + 5.5 - 5 \right) \approx 6.031 \text{ m}^2$$

$$(b) V = 2A \approx 2(6.031) \approx 12.062 \text{ m}^3$$

$$(c) 5000 V \approx 5000(12.062) = 60,310 \text{ pounds}$$

79. True

81. False. Let $f(x) = x$ and $g(x) = 2x - x^2$. f and g intersect at $(1, 1)$, the midpoint of $[0, 2]$. But

$$\int_a^b [f(x) - g(x)] dx = \int_0^2 [x - (2x - x^2)] dx = \frac{2}{3} \neq 0.$$

Section 6.2 Volume: The Disk Method

$$1. V = \pi \int_0^1 (-x + 1)^2 dx = \pi \int_0^1 (x^2 - 2x + 1) dx = \pi \left[\frac{x^3}{3} - x^2 + x \right]_0^1 = \frac{\pi}{3}$$

$$3. V = \pi \int_1^4 (\sqrt{x})^2 dx = \pi \int_1^4 x dx = \pi \left[\frac{x^2}{2} \right]_1^4 = \frac{15\pi}{2}$$

$$5. V = \pi \int_0^1 [(x^2)^2 - (x^3)^2] dx = \pi \int_0^1 (x^4 - x^6) dx = \pi \left[\frac{x^5}{5} - \frac{x^7}{7} \right]_0^1 = \frac{2\pi}{35}$$

$$7. y = x^2 \Rightarrow x = \sqrt{y}$$

$$9. y = x^{2/3} \Rightarrow x = y^{3/2}$$

$$V = \pi \int_0^4 (\sqrt{y})^2 dy = \pi \int_0^4 y dy$$

$$= \pi \left[\frac{y^2}{2} \right]_0^4 = 8\pi$$

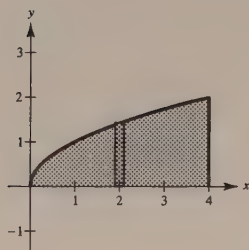
$$V = \pi \int_0^1 (y^{3/2})^2 dy = \pi \int_0^1 y^3 dy = \pi \left[\frac{y^4}{4} \right]_0^1 = \frac{\pi}{4}$$

11. $y = \sqrt{x}$, $y = 0$, $x = 4$

(a) $R(x) = \sqrt{x}$, $r(x) = 0$

$$V = \pi \int_0^4 (\sqrt{x})^2 dx$$

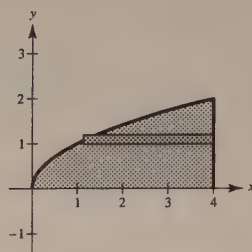
$$= \pi \int_0^4 x dx = \left[\frac{\pi}{2} x^2 \right]_0^4 = 8\pi$$



(b) $R(y) = 4$, $r(y) = y^2$

$$V = \pi \int_0^2 (16 - y^4) dy$$

$$= \pi \left[16y - \frac{1}{5} y^5 \right]_0^2 = \frac{128\pi}{5}$$

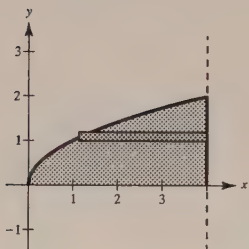


(c) $R(y) = 4 - y^2$, $r(y) = 0$

$$V = \pi \int_0^2 (4 - y^2)^2 dy$$

$$= \pi \int_0^2 (16 - 8y^2 + y^4) dy$$

$$= \pi \left[16y - \frac{8}{3} y^3 + \frac{1}{5} y^5 \right]_0^2 = \frac{256\pi}{15}$$

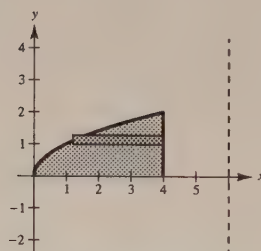


(d) $R(y) = 6 - y^2$, $r(y) = 2$

$$V = \pi \int_0^2 [(6 - y^2)^2 - 4] dy$$

$$= \pi \int_0^2 (32 - 12y^2 + y^4) dy$$

$$= \pi \left[32y - 4y^3 + \frac{1}{5} y^5 \right]_0^2 = \frac{192\pi}{5}$$



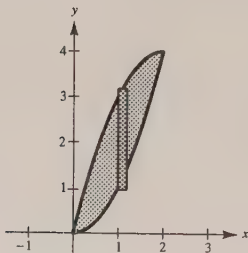
13. $y = x^2$, $y = 4x - x^2$ intersect at $(0, 0)$ and $(2, 4)$.

(a) $R(x) = 4x - x^2$, $r(x) = x^2$

$$V = \pi \int_0^2 [(4x - x^2)^2 - x^4] dx$$

$$= \pi \int_0^2 (16x^2 - 8x^3) dx$$

$$= \pi \left[\frac{16}{3} x^3 - 2x^4 \right]_0^2 = \frac{32\pi}{3}$$

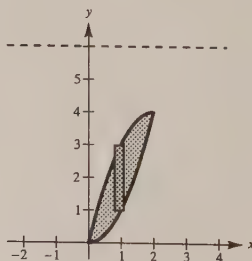


(b) $R(x) = 6 - x^2$, $r(x) = 6 - (4x - x^2)$

$$V = \pi \int_0^2 [(6 - x^2)^2 - (6 - 4x + x^2)^2] dx$$

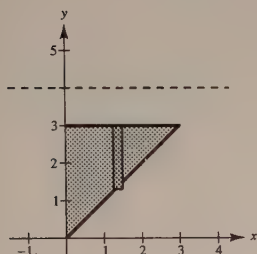
$$= 8\pi \int_0^2 (x^3 - 5x^2 + 6x) dx$$

$$= 8\pi \left[\frac{x^4}{4} - \frac{5}{3} x^3 + 3x^2 \right]_0^2 = \frac{64\pi}{3}$$



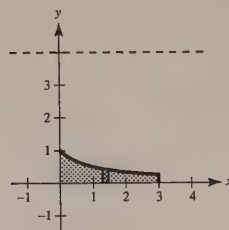
15. $R(x) = 4 - x$, $r(x) = 1$

$$\begin{aligned} V &= \pi \int_0^3 [(4 - x)^2 - (1)^2] dx \\ &= \pi \int_0^3 (x^2 - 8x + 15) dx \\ &= \pi \left[\frac{x^3}{3} - 4x^2 + 15x \right]_0^3 = 18\pi \end{aligned}$$



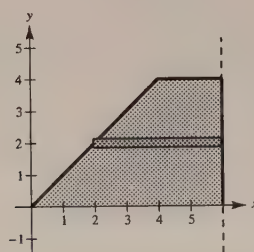
17. $R(x) = 4$, $r(x) = 4 - \frac{1}{1 + x}$

$$\begin{aligned} V &= \pi \int_0^3 \left[4^2 - \left(4 - \frac{1}{1 + x} \right)^2 \right] dx \\ &= \pi \int_0^3 \left[\frac{8}{1 + x} - \frac{1}{(1 + x)^2} \right] dx \\ &= \pi \left[8 \ln(1 + x) + \frac{1}{1 + x} \right]_0^3 \\ &= \pi \left[8 \ln 4 + \frac{1}{4} - 1 \right] \\ &= \left(8 \ln 4 - \frac{3}{4} \right) \pi \approx 32.485 \end{aligned}$$



19. $R(y) = 6 - y$, $r(y) = 0$

$$\begin{aligned} V &= \pi \int_0^4 (6 - y)^2 dy \\ &= \pi \int_0^4 (y^2 - 12y + 36) dy \\ &= \pi \left[\frac{y^3}{3} - 6y^2 + 36y \right]_0^4 \\ &= \frac{208\pi}{3} \end{aligned}$$

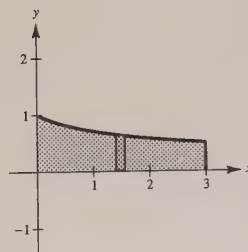


21. $R(y) = 6 - y^2$, $r(y) = 2$

$$\begin{aligned} V &= \pi \int_{-2}^2 [(6 - y^2)^2 - (2)^2] dy \\ &= 2\pi \int_0^2 (y^4 - 12y^2 + 32) dy \\ &= 2\pi \left[\frac{y^5}{5} - 4y^3 + 32y \right]_0^2 \\ &= \frac{384\pi}{5} \end{aligned}$$

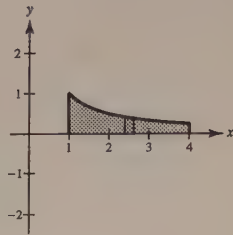
23. $R(x) = \frac{1}{\sqrt{x+1}}$, $r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^3 \left(\frac{1}{\sqrt{x+1}} \right)^2 dx \\ &= \pi \int_0^3 \frac{1}{x+1} dx \\ &= \left[\pi \ln|x+1| \right]_0^3 = \pi \ln 4 \end{aligned}$$



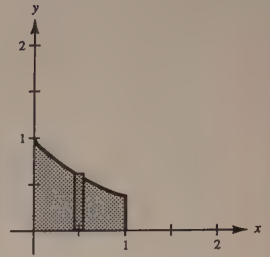
25. $R(x) = \frac{1}{x}$, $r(x) = 0$

$$\begin{aligned} V &= \pi \int_1^4 \left(\frac{1}{x}\right)^2 dx \\ &= \pi \left[-\frac{1}{x}\right]_1^4 \\ &= \frac{3\pi}{4} \end{aligned}$$



27. $R(x) = e^{-x}$, $r(x) = 0$

$$\begin{aligned} V &= \pi \int_0^1 (e^{-x})^2 dx \\ &= \pi \int_0^1 e^{-2x} dx \\ &= \left[-\frac{\pi}{2} e^{-2x}\right]_0^1 \\ &= \frac{\pi}{2}(1 - e^{-2}) \approx 1.358 \end{aligned}$$



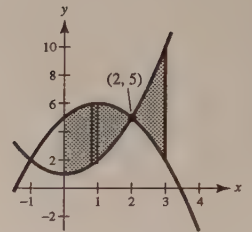
29. $x^2 + 1 = -x^2 + 2x + 5$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

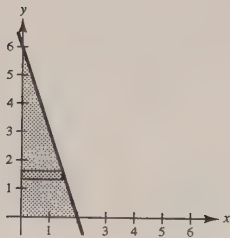
$(-1, 2)$, $(2, 5)$ are points of intersection.

$$\begin{aligned} V &= \pi \int_0^2 [(5 + 2x - x^2)^2 - (x^2 + 1)^2] dx + \pi \int_2^3 [(x^2 + 1)^2 - (5 + 2x - x^2)^2] dx \\ &= \pi \int_0^2 (-4x^3 - 8x^2 + 20x + 24) dx + \pi \int_2^3 (4x^3 + 8x^2 - 20x - 24) dx \\ &= \pi \left[-x^4 - \frac{8}{3}x^3 + 10x^2 + 24x\right]_0^2 + \pi \left[x^4 + \frac{8}{3}x^3 - 10x^2 - 24x\right]_2^3 \\ &= \pi \frac{152}{3} + \pi \frac{125}{3} = \frac{277\pi}{3} \end{aligned}$$

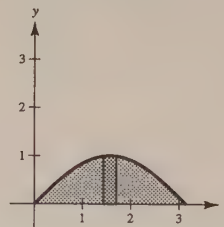


31. $y = 6 - 3x \Rightarrow x = \frac{1}{3}(6 - y)$

$$\begin{aligned} V &= \pi \int_0^6 \left[\frac{1}{3}(6 - y)\right]^2 dy \\ &= \frac{\pi}{9} \int_0^6 [36 - 12y + y^2] dy \\ &= \frac{\pi}{9} \left[36y - 6y^2 + \frac{y^3}{3}\right]_0^6 \\ &= \frac{\pi}{9} \left[216 - 216 + \frac{216}{3}\right] \\ &= 8\pi \left(= \frac{1}{3}\pi r^2 h \text{ volume of cone}\right) \end{aligned}$$



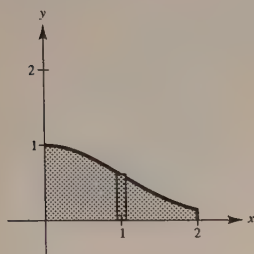
33. $V = \pi \int_0^\pi [\sin x]^2 dx \approx 4.9348$



$$35. V = \pi \int_0^2 [e^{-x^2}]^2 dx \approx 1.9686$$

$$39. A \approx 3$$

Matches (a)



$$37. V = \pi \int_{-1}^2 [e^{x/2} + e^{-x/2}]^2 dx \approx 49.0218$$

41. Disk Method:

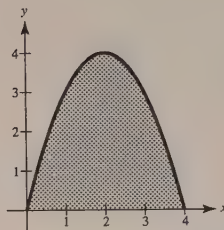
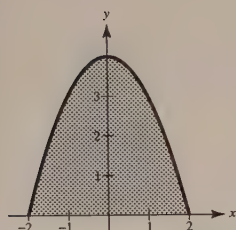
$$V = \pi \int_a^b [R(x)]^2 dx \quad \text{or} \quad V = \pi \int_c^d [R(y)]^2 dy$$

Washer Method:

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx \quad \text{or}$$

$$V = \pi \int_c^d ([R(y)]^2 - [r(y)]^2) dy$$

43.



The volumes are the same because the solid has been translated horizontally.

$$45. R(x) = \frac{1}{2}x, \quad r(x) = 0$$

$$V = \pi \int_0^6 \frac{1}{4}x^2 dx$$

$$= \left[\frac{\pi}{12}x^3 \right]_0^6 = 18\pi$$

Note: $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi(3^2)6$$

$$= 18\pi$$

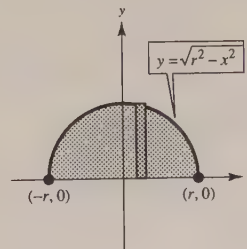
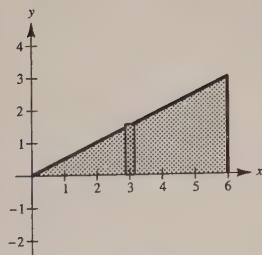
$$47. R(x) = \sqrt{r^2 - x^2}, \quad r(x) = 0$$

$$V = \pi \int_{-r}^r (r^2 - x^2) dx$$

$$= 2\pi \int_0^r (r^2 - x^2) dx$$

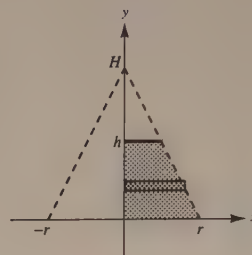
$$= 2\pi \left[r^2x - \frac{1}{3}x^3 \right]_0^r$$

$$= 2\pi \left(r^3 - \frac{1}{3}r^3 \right) = \frac{4}{3}\pi r^3$$



$$49. x = r - \frac{r}{H}y = r\left(1 - \frac{y}{H}\right), R(y) = r\left(1 - \frac{y}{H}\right), r(y) = 0$$

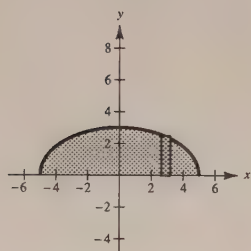
$$\begin{aligned} V &= \pi \int_0^h \left[r\left(1 - \frac{y}{H}\right) \right]^2 dy = \pi r^2 \int_0^h \left(1 - \frac{2}{H}y + \frac{1}{H^2}y^2 \right) dy \\ &= \pi r^2 \left[y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3 \right]_0^h \\ &= \pi r^2 \left(h - \frac{h^2}{H} + \frac{h^3}{3H^2} \right) \\ &= \pi r^2 h \left(1 - \frac{h}{H} + \frac{h^2}{3H^2} \right) \end{aligned}$$



$$51. V = \pi \int_0^2 \left(\frac{1}{8}x^2 \sqrt{2-x} \right)^2 dx = \frac{\pi}{64} \int_0^2 x^4(2-x) dx = \frac{\pi}{64} \left[\frac{2x^5}{5} - \frac{x^6}{6} \right]_0^2 = \frac{\pi}{30}$$

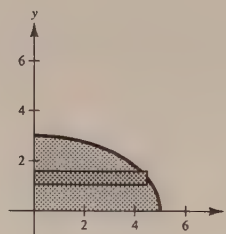
$$53. (a) R(x) = \frac{3}{5} \sqrt{25 - x^2}, r(x) = 0$$

$$\begin{aligned} V &= \frac{9\pi}{25} \int_{-5}^5 (25 - x^2) dx \\ &= \frac{18\pi}{25} \int_0^5 (25 - x^2) dx \\ &= \frac{18\pi}{25} \left[25x - \frac{x^3}{3} \right]_0^5 = 60\pi \end{aligned}$$



$$(b) R(y) = \frac{5}{3} \sqrt{9 - y^2}, r(y) = 0, x \geq 0$$

$$\begin{aligned} V &= \frac{25\pi}{9} \int_0^3 (9 - y^2) dy \\ &= \frac{25\pi}{9} \left[9y - \frac{y^3}{3} \right]_0^3 = 50\pi \end{aligned}$$



$$55. \text{Total volume: } V = \frac{4\pi(50)^3}{3} = \frac{500,000}{3} \text{ ft}^3$$

Volume of water in the tank:

$$\begin{aligned} \pi \int_{-50}^{y_0} (\sqrt{2500 - y^2})^2 dy &= \pi \int_{-50}^{y_0} (2500 - y^2) dy \\ &= \pi \left[2500y - \frac{y^3}{3} \right]_{-50}^{y_0} \\ &= \pi \left(2500y_0 - \frac{y_0^3}{3} + \frac{250,000}{3} \right) \end{aligned}$$

When the tank is one-fourth of its capacity:

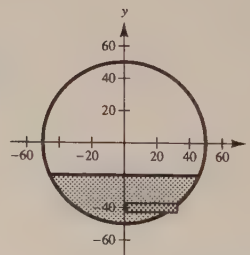
$$\begin{aligned} \frac{1}{4} \left(\frac{500,000\pi}{3} \right) &= \pi \left(2500y_0 - \frac{y_0^3}{3} + \frac{250,000}{3} \right) \\ 125,000 &= 7500y_0 - y_0^3 + 250,000 \end{aligned}$$

$$y_0^3 - 7500y_0 - 125,000 = 0$$

$$y_0 \approx -17.36$$

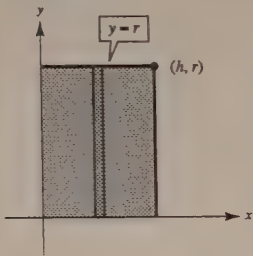
$$\text{Depth: } -17.36 - (-50) = 32.64 \text{ feet}$$

When the tank is three-fourths of its capacity the depth is $100 - 32.64 = 67.36$ feet.



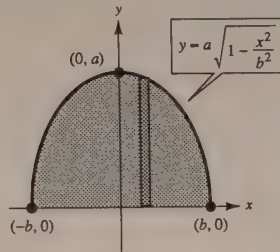
57. (a) $\pi \int_0^h r^2 dx$ (ii)

is the volume of a right circular cylinder with radius r and height h .



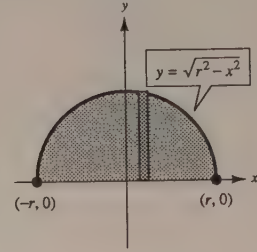
(b) $\pi \int_{-b}^b \left(a \sqrt{1 - \frac{x^2}{b^2}} \right)^2 dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



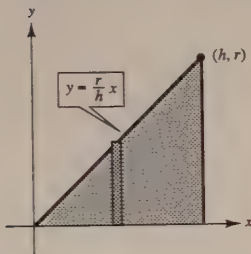
(c) $\pi \int_{-r}^r (\sqrt{r^2 - x^2})^2 dx$ (iii)

is the volume of a sphere with radius r .



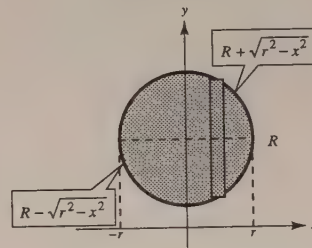
(d) $\pi \int_0^h \left(\frac{rx}{h} \right)^2 dx$ (i)

is the volume of a right circular cone with the radius of the base as r and height h .

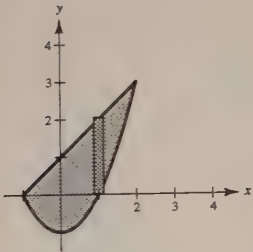


(e) $\pi \int_{-r}^r [(R + \sqrt{r^2 - x^2})^2 - (R - \sqrt{r^2 - x^2})^2] dx$ (v)

is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R .



59.



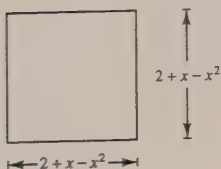
Base of Cross Section = $(x + 1) - (x^2 - 1) = 2 + x - x^2$

(a) $A(x) = b^2 = (2 + x - x^2)^2$

$$= 4 + 4x - 3x^2 - 2x^3 + x^4$$

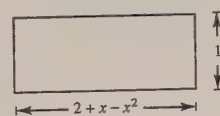
$$V = \int_{-1}^2 (4 + 4x - 3x^2 - 2x^3 + x^4) dx$$

$$= \left[4x + 2x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \right]_{-1}^2 = \frac{81}{10}$$

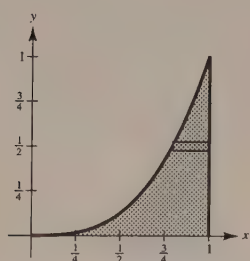


(b) $A(x) = bh = (2 + x - x^2)1$

$$V = \int_{-1}^2 (2 + x - x^2) dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^2 = \frac{9}{2}$$



61.

Base of Cross Section = $1 - \sqrt[3]{y}$

$$(b) A(y) = \frac{1}{2} \pi r^2 = \frac{1}{2} \pi \left(\frac{1 - \sqrt[3]{y}}{2} \right)^2 = \frac{1}{8} \pi (1 - \sqrt[3]{y})^2$$

$$V = \frac{1}{8} \pi \int_0^1 (1 - \sqrt[3]{y})^2 dy = \frac{\pi}{8} \left(\frac{1}{10} \right) = \frac{\pi}{80}$$

$$(c) A(y) = \frac{1}{2} bh = \frac{1}{2} (1 - \sqrt[3]{y}) \left(\frac{\sqrt{3}}{2} \right) (1 - \sqrt[3]{y})$$

$$= \frac{\sqrt{3}}{4} (1 - \sqrt[3]{y})^2$$

$$V = \frac{\sqrt{3}}{4} \int_0^1 (1 - \sqrt[3]{y})^2 dy = \frac{\sqrt{3}}{4} \left(\frac{1}{10} \right) = \frac{\sqrt{3}}{40}$$

$$(d) A(y) = \frac{1}{2} \pi ab = \frac{\pi}{2} (1 - \sqrt[3]{y}) \frac{1 - \sqrt[3]{y}}{2}$$

$$= \frac{\pi}{2} (1 - \sqrt[3]{y})^2$$

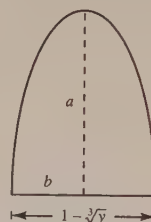
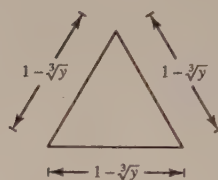
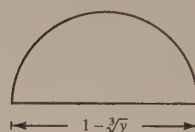
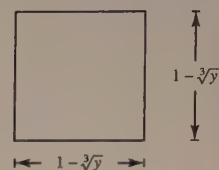
$$V = \frac{\pi}{2} \int_0^1 (1 - \sqrt[3]{y})^2 dy = \frac{\pi}{2} \left(\frac{1}{10} \right) = \frac{\pi}{20}$$

$$(a) A(y) = b^2 = (1 - \sqrt[3]{y})^2$$

$$V = \int_0^1 (1 - \sqrt[3]{y})^2 dy$$

$$= \int_0^1 (1 - 2y^{1/3} + y^{2/3}) dy$$

$$= \left[y - \frac{3}{2} y^{4/3} + \frac{3}{5} y^{5/3} \right]_0^1 = \frac{1}{10}$$



63. Let $A_1(x)$ and $A_2(x)$ equal the areas of the cross sections of the two solids for $a \leq x \leq b$. Since $A_1(x) = A_2(x)$, we have

$$V_1 = \int_a^b A_1(x) dx = \int_a^b A_2(x) dx = V_2$$

Thus, the volumes are the same.

$$65. \frac{4}{3} \pi (25 - r^2)^{3/2} = \frac{1}{2} \left(\frac{4}{3} \right) \pi (125)$$

$$(25 - r^2)^{3/2} = \frac{125}{2}$$

$$25 - r^2 = \left(\frac{125}{2} \right)^{2/3}$$

$$25 - \frac{25}{(2^{2/3})} = r^2$$

$$25(1 - 2^{-2/3}) = r^2$$

$$r = 5\sqrt{1 - 2^{-2/3}} \approx 3.0415$$

67. (a) Since the cross sections are isosceles right triangles:

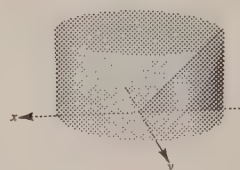
$$A(x) = \frac{1}{2} bh = \frac{1}{2} (\sqrt{r^2 - y^2}) (\sqrt{r^2 - y^2}) = \frac{1}{2} (r^2 - y^2)$$

$$V = \frac{1}{2} \int_{-r}^r (r^2 - y^2) dy = \int_0^r (r^2 - y^2) dy = \left[r^2 y - \frac{y^3}{3} \right]_0^r = \frac{2}{3} r^3$$

$$(b) A(x) = \frac{1}{2} bh = \frac{1}{2} \sqrt{r^2 - y^2} (\sqrt{r^2 - y^2} \tan \theta) = \frac{\tan \theta}{2} (r^2 - y^2)$$

$$V = \frac{\tan \theta}{2} \int_{-r}^r (r^2 - y^2) dy = \tan \theta \int_0^r (r^2 - y^2) dy = \tan \theta \left[r^2 y - \frac{y^3}{3} \right]_0^r = \frac{2}{3} r^3 \tan \theta$$

As $\theta \rightarrow 90^\circ$, $V \rightarrow \infty$.



Section 6.3 Volume: The Shell Method

1. $p(x) = x$

$h(x) = x$

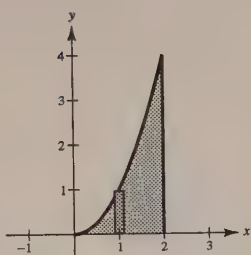
$$V = 2\pi \int_0^2 x(x) dx = \left[\frac{2\pi x^3}{3} \right]_0^2 = \frac{16\pi}{3}$$

5. $p(x) = x$

$h(x) = x^2$

$$V = 2\pi \int_0^2 x^3 dx$$

$$= \left[\frac{\pi}{2} x^4 \right]_0^2 = 8\pi$$



3. $p(x) = x$

$h(x) = \sqrt{x}$

$$V = 2\pi \int_0^4 x\sqrt{x} dx$$

$$= 2\pi \int_0^4 x^{3/2} dx$$

$$= \left[\frac{4\pi}{5} x^{5/2} \right]_0^4 = \frac{128\pi}{5}$$

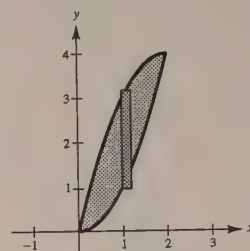
7. $p(x) = x$

$h(x) = (4x - x^2) - x^2 = 4x - 2x^2$

$$V = 2\pi \int_0^2 x(4x - 2x^2) dx$$

$$= 4\pi \int_0^2 (2x^2 - x^3) dx$$

$$= 4\pi \left[\frac{2}{3} x^3 - \frac{1}{4} x^4 \right]_0^2 = \frac{16\pi}{3}$$

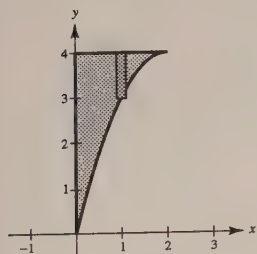


9. $p(x) = x$

$h(x) = 4 - (4x - x^2) = x^2 - 4x + 4$

$$V = 2\pi \int_0^2 (x^3 - 4x^2 + 4x) dx$$

$$= 2\pi \left[\frac{x^4}{4} - \frac{4}{3} x^3 + 2x^2 \right]_0^2 = \frac{8\pi}{3}$$



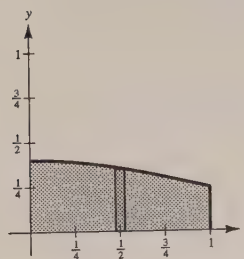
11. $p(x) = x$

$h(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$V = 2\pi \int_0^1 x \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right) dx$$

$$= \sqrt{2\pi} \int_0^1 e^{-x^2/2} x dx$$

$$= \left[-\sqrt{2\pi} e^{-x^2/2} \right]_0^1 = \sqrt{2\pi} \left(1 - \frac{1}{\sqrt{e}} \right) \approx 0.986$$



13. $p(y) = y$

$h(y) = 2 - y$

$$V = 2\pi \int_0^2 y(2 - y) dy$$

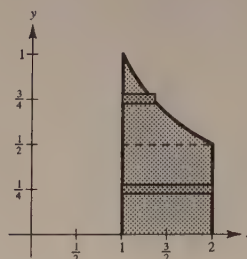
$$= 2\pi \int_0^2 (2y - y^2) dy$$

$$= 2\pi \left[y^2 - \frac{y^3}{3} \right]_0^2 = \frac{8\pi}{3}$$

15. $p(y) = y$ and $h(y) = 1$ if $0 \leq y < \frac{1}{2}$.

$p(y) = y$ and $h(y) = \frac{1}{y} - 1$ if $\frac{1}{2} \leq y \leq 1$.

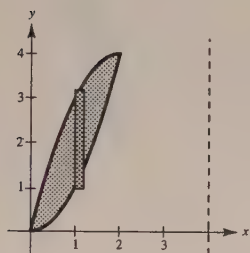
$$\begin{aligned} V &= 2\pi \int_0^{1/2} y \, dy + 2\pi \int_{1/2}^1 (1 - y) \, dy \\ &= 2\pi \left[\frac{y^2}{2} \right]_0^{1/2} + 2\pi \left[y - \frac{y^2}{2} \right]_{1/2}^1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \end{aligned}$$



17. $p(x) = 4 - x$

$h(x) = 4x - x^2 - x^2 = 4x - 2x^2$

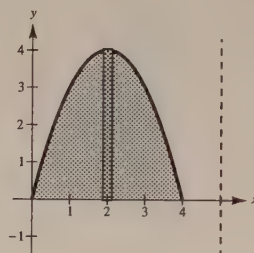
$$\begin{aligned} V &= 2\pi \int_0^2 (4 - x)(4x - 2x^2) \, dx \\ &= 2\pi(2) \int_0^2 (x^3 - 6x^2 + 8x) \, dx \\ &= 4\pi \left[\frac{x^4}{4} - 2x^3 + 4x^2 \right]_0^2 = 16\pi \end{aligned}$$



19. $p(x) = 5 - x$

$h(x) = 4x - x^2$

$$\begin{aligned} V &= 2\pi \int_0^4 (5 - x)(4x - x^2) \, dx \\ &= 2\pi \int_0^4 (x^3 - 9x^2 + 20x) \, dx \\ &= 2\pi \left[\frac{x^4}{4} - 3x^3 + 10x^2 \right]_0^4 = 64\pi \end{aligned}$$

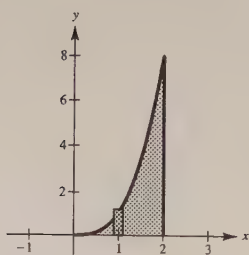


21. (a) Disk

$R(x) = x^3$

$r(x) = 0$

$$V = \pi \int_0^2 x^6 \, dx = \pi \left[\frac{x^7}{7} \right]_0^2 = \frac{128\pi}{7}$$

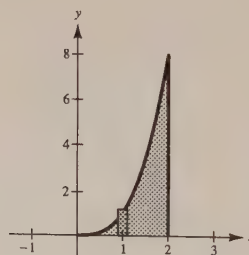


(b) Shell

$p(x) = x$

$h(x) = x^3$

$$V = 2\pi \int_0^2 x^4 \, dx = 2\pi \left[\frac{x^5}{5} \right]_0^2 = \frac{64\pi}{5}$$

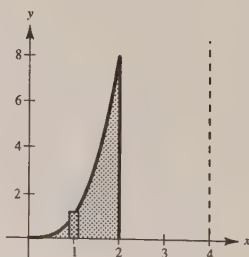


(c) Shell

$p(x) = 4 - x$

$h(x) = x^3$

$$\begin{aligned} V &= 2\pi \int_0^2 (4 - x)x^3 \, dx \\ &= 2\pi \int_0^2 (4x^3 - x^4) \, dx \\ &= 2\pi \left[x^4 - \frac{1}{5}x^5 \right]_0^2 = \frac{96\pi}{5} \end{aligned}$$

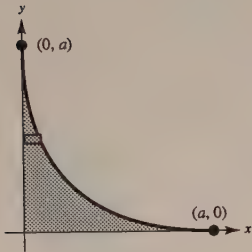


23. (a) Shell

$$p(y) = y$$

$$h(y) = (a^{1/2} - y^{1/2})^2$$

$$\begin{aligned} V &= 2\pi \int_0^a y(a - 2a^{1/2}y^{1/2} + y) dy \\ &= 2\pi \int_0^a (ay - 2a^{1/2}y^{3/2} + y^2) dy \\ &= 2\pi \left[\frac{a}{2}y^2 - \frac{4a^{1/2}}{5}y^{5/2} + \frac{y^3}{3} \right]_0^a \\ &= 2\pi \left[\frac{a^3}{2} - \frac{4a^3}{5} + \frac{a^3}{3} \right] = \frac{\pi a^3}{15} \end{aligned}$$

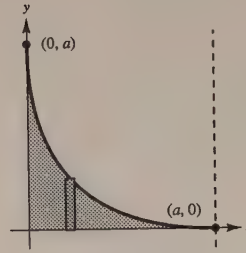


(c) Shell

$$p(x) = a - x$$

$$h(x) = (a^{1/2} - x^{1/2})^2$$

$$\begin{aligned} V &= 2\pi \int_0^a (a - x)(a^{1/2} - x^{1/2})^2 dx \\ &= 2\pi \int_0^a (a^2 - 2a^{3/2}x^{1/2} + 2a^{1/2}x^{3/2} - x^2) dx \\ &= 2\pi \left[a^2x - \frac{4}{3}a^{3/2}x^{3/2} + \frac{4}{5}a^{1/2}x^{5/2} - \frac{1}{3}x^3 \right]_0^a = \frac{4\pi a^3}{15} \end{aligned}$$



(b) Same as part (a) by symmetry

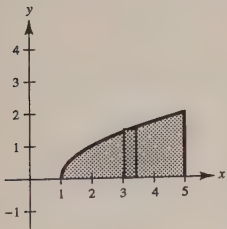
$$25. V = 2\pi \int_c^d p(y)h(y) dy \quad \text{or} \quad V = 2\pi \int_a^b p(x)h(x) dx$$

$$27. \pi \int_1^5 (x - 1) dx = \pi \int_1^5 (\sqrt{x - 1})^2 dx$$

This integral represents the volume of the solid generated by revolving the region bounded by $y = \sqrt{x - 1}$, $y = 0$, and $x = 5$ about the x -axis by using the Disk Method.

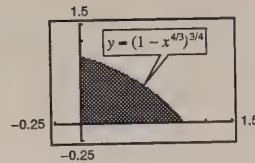
$$2\pi \int_0^2 y[5 - (y^2 + 1)] dy$$

represents this same volume by using the Shell Method.



Disk Method

29. (a)

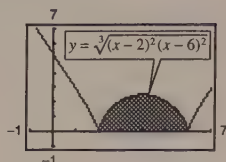


$$(b) x^{4/3} + y^{4/3} = 1, x = 0, y = 0$$

$$y = (1 - x^{4/3})^{3/4}$$

$$V = 2\pi \int_0^1 x(1 - x^{4/3})^{3/4} dx \approx 1.5056$$

31. (a)



$$(b) V = 2\pi \int_2^6 x \sqrt[3]{(x-2)^2(x-6)^2} dx \approx 187.249$$

35. $p(x) = x$

$$h(x) = 2 - \frac{1}{2}x^2$$

$$V = 2\pi \int_0^2 x \left(2 - \frac{1}{2}x^2\right) dx = 2\pi \int_0^2 \left(2x - \frac{1}{2}x^3\right) dx = 2\pi \left[x^2 - \frac{1}{8}x^4\right]_0^2 = 4\pi \text{ (total volume)}$$

Now find x_0 such that

$$\pi = 2\pi \int_0^{x_0} \left(2x - \frac{1}{2}x^3\right) dx$$

$$1 = 2 \left[x^2 - \frac{1}{8}x^4 \right]_0^{x_0}$$

$$1 = 2x_0^2 - \frac{1}{4}x_0^4$$

$$x_0^4 - 8x_0^2 + 4 = 0$$

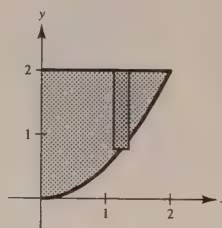
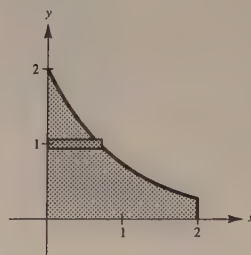
$$x_0^2 = 4 \pm 2\sqrt{3} \quad (\text{Quadratic Formula})$$

Take $x_0 = \sqrt{4 - 2\sqrt{3}}$ since the other root is too large.Diameter: $2\sqrt{4 - 2\sqrt{3}} \approx 1.464$

$$\begin{aligned} 37. V &= 4\pi \int_{-1}^1 (2-x)\sqrt{1-x^2} dx \\ &= 8\pi \int_{-1}^1 \sqrt{1-x^2} dx - 4\pi \int_{-1}^1 x\sqrt{1-x^2} dx \\ &= 8\pi \left(\frac{\pi}{2}\right) + 2\pi \int_{-1}^1 x(1-x^2)^{1/2}(-2) dx \\ &= 4\pi^2 + \left[2\pi \left(\frac{2}{3}\right)(1-x^2)^{3/2} \right]_{-1}^1 = 4\pi^2 \end{aligned}$$

33. $y = 2e^{-x}$, $y = 0$, $x = 0$, $x = 2$ Volume ≈ 7.5

Matches (d)

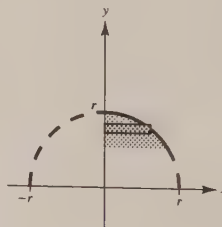


39. Disk Method

$$R(y) = \sqrt{r^2 - y^2}$$

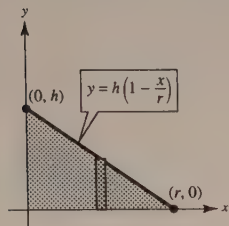
$$r(y) = 0$$

$$\begin{aligned} V &= \pi \int_{r-h}^r (r^2 - y^2) dy \\ &= \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^r = \frac{1}{3} \pi h^2 (3r - h) \end{aligned}$$



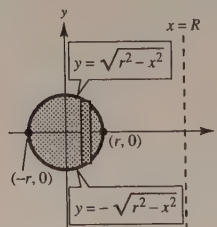
41. (a) $2\pi \int_0^r hx \left(1 - \frac{x}{r}\right) dx$ (ii)

is the volume of a right circular cone with the radius of the base as r and height h .

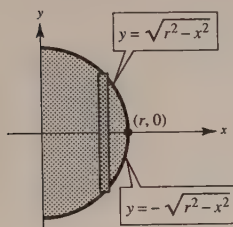


(b) $2\pi \int_{-r}^r (R - x)(2\sqrt{r^2 - x^2}) dx$ (v)

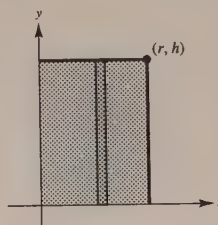
is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R .



(c) $2\pi \int_0^r 2x\sqrt{r^2 - x^2} dx$ (iii) is the volume of a sphere with radius r .

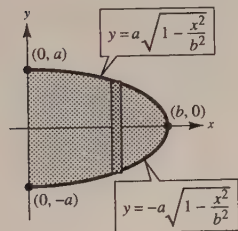


(d) $2\pi \int_0^r hx dx$ (i) is the volume of a right circular cylinder with a radius of r and a height of h .



(e) $2\pi \int_0^b 2ax\sqrt{1 - (x^2/b^2)} dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.

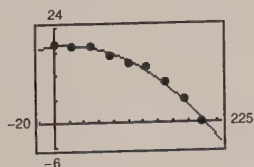


43. (a) $V = 2\pi \int_0^{200} xf(x) dx$

$$\approx \frac{2\pi(200)}{3(8)} [0 + 4(25)(19) + 2(50)(19) + 4(75)(17) + 2(100)15 + 4(125)(14) + 2(150)(10) + 4(175)(6) + 0]$$

$$\approx 1,366,593 \text{ cubic feet}$$

(b) $d = -0.000561x^2 + 0.0189x + 19.39$



(c) $V \approx 2\pi \int_0^{200} xd(x) dx \approx 2\pi(213,800) = 1,343,345 \text{ cubic feet}$

(d) Number gallons $\approx V(7.48) = 10,048,221 \text{ gallons}$

Section 6.4 Arc Length and Surfaces of Revolution

1. $(0, 0), (5, 12)$

(a) $d = \sqrt{(5-0)^2 + (12-0)^2} = 13$

(b) $y = \frac{12}{5}x$

$y' = \frac{12}{5}$

$s = \int_0^5 \sqrt{1 + \left(\frac{12}{5}\right)^2} dx = \left[\frac{13}{5}x\right]_0^5 = 13$

3. $y = \frac{2}{3}x^{3/2} + 1$

$y' = x^{1/2}, [0, 1]$

$$s = \int_0^1 \sqrt{1+x} dx$$

$$= \left[\frac{2}{3}(1+x)^{3/2}\right]_0^1$$

$$= \frac{2}{3}(\sqrt{8}-1) \approx 1.219$$

5. $y = \frac{3}{2}x^{2/3}$

$y' = \frac{1}{x^{1/3}}, [1, 8]$

$$s = \int_1^8 \sqrt{1 + \left(\frac{1}{x^{1/3}}\right)^2} dx$$

$$= \int_1^8 \sqrt{\frac{x^{2/3} + 1}{x^{2/3}}} dx$$

$$= \frac{3}{2} \int_1^8 \sqrt{x^{2/3} + 1} \left(\frac{2}{3x^{1/3}}\right) dx$$

$$= \frac{3}{2} \left[\frac{2}{3}(x^{2/3} + 1)^{3/2}\right]_1^8$$

$$= 5\sqrt{5} - 2\sqrt{2} \approx 8.352$$

7. $y = \frac{x^4}{8} + \frac{1}{4x^2}$

$y' = \frac{1}{2}x^3 - \frac{1}{2x^3}, [1, 2]$

$$1 + (y')^2 = \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)^2, [1, 2]$$

$$s = \int_a^b \sqrt{1 + (y')^2} dx$$

$$= \int_1^2 \left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right) dx$$

$$= \left[\frac{1}{8}x^4 - \frac{1}{4x^2}\right]_1^2 = \frac{33}{16} \approx 2.063$$

9. $y = \ln(\sin x), \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$y' = \frac{1}{\sin x} \cos x = \cot x$

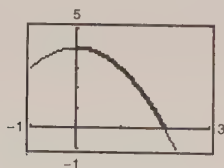
$1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$

$$s = \int_{\pi/4}^{3\pi/4} \csc x dx$$

$$= \left[\ln|\csc x - \cot x|\right]_{\pi/4}^{3\pi/4}$$

$$= \ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1) \approx 1.763$$

11. (a) $y = 4 - x^2, 0 \leq x \leq 2$



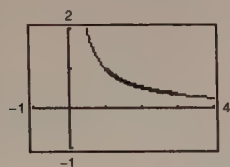
(b) $y' = -2x$

(c) $L \approx 4.647$

$1 + (y')^2 = 1 + 4x^2$

$L = \int_0^2 \sqrt{1 + 4x^2} dx$

13. (a) $y = \frac{1}{x}, 1 \leq x \leq 3$



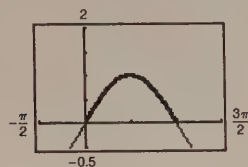
(b) $y' = -\frac{1}{x^2}$

(c) $L \approx 2.147$

$$1 + (y')^2 = 1 + \frac{1}{x^4}$$

$$L = \int_1^3 \sqrt{1 + \frac{1}{x^4}} dx$$

15. (a) $y = \sin x, 0 \leq x \leq \pi$



(b) $y' = \cos x$

(c) $L \approx 3.820$

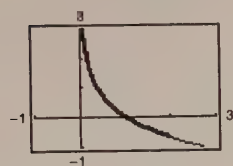
$$1 + (y')^2 = 1 + \cos^2 x$$

$$L = \int_0^\pi \sqrt{1 + \cos^2 x} dx$$

17. (a) $x = e^{-y}, 0 \leq y \leq 2$

$y = -\ln x$

$1 \geq x \geq e^{-2} \approx 0.135$



(b) $y' = -\frac{1}{x}$

(c) $L \approx 2.221$

$$1 + (y')^2 = 1 + \frac{1}{x^2}$$

$$L = \int_{e^{-2}}^1 \sqrt{1 + \frac{1}{x^2}} dx$$

Alternatively, you can do all the computations with respect to y .

(a) $x = e^{-y}, 0 \leq y \leq 2$

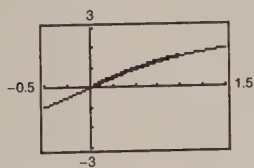
(b) $\frac{dx}{dy} = -e^{-y}$

(c) $L \approx 2.221$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + e^{-2y}$$

$$L = \int_0^2 \sqrt{1 + e^{-2y}} dy$$

19. (a) $y = 2 \arctan x, 0 \leq x \leq 1$



(b) $y' = \frac{2}{1+x^2}$

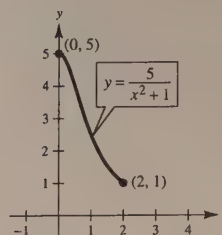
(c) $L \approx 1.871$

$$L = \int_0^1 \sqrt{1 + \frac{4}{(1+x^2)^2}} dx$$

$$21. \int_0^2 \sqrt{1 + \left[\frac{d}{dx} \left(\frac{5}{x^2 + 1} \right) \right]^2} dx$$

$$s \approx 5$$

Matches (b)



$$23. y = x^3, [0, 4]$$

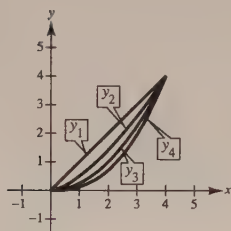
$$(a) d = \sqrt{(4-0)^2 + (64-0)^2} \approx 64.125$$

$$(b) d = \sqrt{(1-0)^2 + (1-0)^2} + \sqrt{(2-1)^2 + (8-1)^2} + \sqrt{(3-2)^2 + (27-8)^2} + \sqrt{(4-3)^2 + (64-27)^2} \approx 64.525$$

$$(c) s = \int_0^4 \sqrt{1 + (3x^2)^2} dx = \int_0^4 \sqrt{1 + 9x^4} dx \approx 64.666$$

$$(d) 64.672$$

$$25. (a)$$



$$(b) y_1, y_2, y_3, y_4$$

$$(c) y_1' = 1, L_1 = \int_0^4 \sqrt{2} dx \approx 5.657$$

$$y_2' = \frac{3}{4}x^{1/2}, L_2 = \int_0^4 \sqrt{1 + \frac{9x}{16}} dx \approx 5.759$$

$$y_3' = \frac{1}{2}x, L_3 = \int_0^4 \sqrt{1 + \frac{x^2}{4}} dx \approx 5.916$$

$$y_4' = \frac{5}{16}x^{3/2}, L_4 = \int_0^4 \sqrt{1 + \frac{25}{256}x^3} dx \approx 6.063$$

$$27. y = \frac{1}{3}[x^{3/2} - 3x^{1/2} + 2]$$

When $x = 0$, $y = \frac{2}{3}$. Thus, the fleeing object has traveled $\frac{2}{3}$ units when it is caught.

$$y' = \frac{1}{3} \left[\frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2} \right] = \left(\frac{1}{2} \right) \frac{x-1}{x^{1/2}}$$

$$1 + (y')^2 = 1 + \frac{(x-1)^2}{4x} = \frac{(x+1)^2}{4x}$$

$$s = \int_0^1 \frac{x+1}{2x^{1/2}} dx = \frac{1}{2} \int_0^1 (x^{1/2} + x^{-1/2}) dx = \frac{1}{2} \left[\frac{2}{3}x^{3/2} + 2x^{1/2} \right]_0^1 = \frac{4}{3} = 2\left(\frac{2}{3}\right)$$

The pursuer has traveled twice the distance that the fleeing object has traveled when it is caught.

$$29. y = 20 \cosh \frac{x}{20}, -20 \leq x \leq 20$$

$$y' = \sinh \frac{x}{20}$$

$$1 + (y')^2 = 1 + \sinh^2 \frac{x}{20} = \cosh^2 \frac{x}{20}$$

$$\begin{aligned} L &= \int_{-20}^{20} \cosh \frac{x}{20} dx = 2 \int_0^{20} \cosh \frac{x}{20} dx = 2(20) \sinh \frac{x}{20} \Big|_0^{20} \\ &= 40 \sinh(1) \approx 47.008 \text{ m.} \end{aligned}$$

$$\begin{aligned}
 31. \quad y &= \sqrt{9 - x^2} \\
 y' &= \frac{-x}{\sqrt{9 - x^2}} \\
 1 + (y')^2 &= \frac{9}{9 - x^2} \\
 s &= \int_0^2 \sqrt{\frac{9}{9 - x^2}} dx \\
 &= \int_0^2 \frac{3}{\sqrt{9 - x^2}} dx \\
 &= \left[3 \arcsin \frac{x}{3} \right]_0^2 \\
 &= 3 \left(\arcsin \frac{2}{3} - \arcsin 0 \right) \\
 &= 3 \arcsin \frac{2}{3} \approx 2.1892
 \end{aligned}$$

$$\begin{aligned}
 35. \quad y &= \frac{x^3}{6} + \frac{1}{2x} \\
 y' &= \frac{x^2}{2} - \frac{1}{2x^2} \\
 1 + (y')^2 &= \left(\frac{x^2}{2} + \frac{1}{2x^2} \right)^2, [1, 2] \\
 S &= 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x} \right) \left(\frac{x^2}{2} + \frac{1}{2x^2} \right) dx \\
 &= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3} \right) dx \\
 &= 2\pi \left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2} \right]_1^2 = \frac{47\pi}{16}
 \end{aligned}$$

$$\begin{aligned}
 39. \quad y &= \sin x \\
 y' &= \cos x, [0, \pi] \\
 S &= 2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x} dx \\
 &\approx 14.4236
 \end{aligned}$$

$$\begin{aligned}
 33. \quad y &= \frac{x^3}{3} \\
 y' &= x^2, [0, 3] \\
 S &= 2\pi \int_0^3 \frac{x^3}{3} \sqrt{1 + x^4} dx \\
 &= \frac{\pi}{6} \int_0^3 (1 + x^4)^{1/2} (4x^3) dx \\
 &= \left[\frac{\pi}{9} (1 + x^4)^{3/2} \right]_0^3 \\
 &= \frac{\pi}{9} (82\sqrt{82} - 1) \approx 258.85
 \end{aligned}$$

$$\begin{aligned}
 37. \quad y &= \sqrt[3]{x} + 2 \\
 y' &= \frac{1}{3x^{2/3}}, [1, 8] \\
 S &= 2\pi \int_1^8 x \sqrt{1 + \frac{1}{9x^{4/3}}} dx \\
 &= \frac{2\pi}{3} \int_1^8 x^{1/3} \sqrt{9x^{4/3} + 1} dx \\
 &= \frac{\pi}{18} \int_1^8 (9x^{4/3} + 1)^{1/2} (12x^{1/3}) dx \\
 &= \left[\frac{\pi}{27} (9x^{4/3} + 1)^{3/2} \right]_1^8 \\
 &= \frac{\pi}{27} (145\sqrt{145} - 10\sqrt{10}) \approx 199.48
 \end{aligned}$$

43. The precalculus formula is the surface area formula for the lateral surface of the frustum of a right circular cone. The representative element is

$$2\pi f(d_i) \sqrt{\Delta x_i^2 + \Delta y_i^2} = 2\pi f(d_i) \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i} \right)^2} \Delta x_i.$$

41. A rectifiable curve is one that has a finite arc length.

$$45. \quad y = \frac{hx}{r}$$

$$y' = \frac{h}{r}$$

$$1 + (y')^2 = \frac{r^2 + h^2}{r^2}$$

$$\begin{aligned} S &= 2\pi \int_0^r x \sqrt{\frac{r^2 + h^2}{r^2}} dx \\ &= \left[\frac{2\pi \sqrt{r^2 + h^2}}{r} \left(\frac{x^2}{2} \right) \right]_0^r = \pi r \sqrt{r^2 + h^2} \end{aligned}$$

$$47. \quad y = \sqrt{9 - x^2}$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \frac{3}{\sqrt{9 - x^2}}$$

$$\begin{aligned} S &= 2\pi \int_0^2 \frac{3x}{\sqrt{9 - x^2}} dx \\ &= -3\pi \int_0^2 \frac{-2x}{\sqrt{9 - x^2}} dx \\ &= \left[-6\pi \sqrt{9 - x^2} \right]_0^2 \\ &= 6\pi(3 - \sqrt{5}) \approx 14.40 \end{aligned}$$

See figure in Exercise 48.

$$49. \quad y = \frac{1}{3}x^{1/2} - x^{3/2}$$

$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{6}(x^{-1/2} - 9x^{1/2})$$

$$1 + (y')^2 = 1 + \frac{1}{36}(x^{-1} - 18 + 81x) = \frac{1}{36}(x^{-1/2} + 9x^{1/2})^2$$

$$\begin{aligned} S &= 2\pi \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2} \right) \sqrt{\frac{1}{36}(x^{-1/2} + 9x^{1/2})^2} dx = \frac{2\pi}{6} \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2} \right) (x^{-1/2} + 9x^{1/2}) dx \\ &= \frac{\pi}{3} \int_0^{1/3} \left(\frac{1}{3} + 2x - 9x^2 \right) dx = \frac{\pi}{3} \left[\frac{1}{3}x + x^2 - 3x^3 \right]_0^{1/3} = \frac{\pi}{27} \text{ ft}^2 \approx 0.1164 \text{ ft}^2 \approx 16.8 \text{ in}^2 \end{aligned}$$

$$\text{Amount of glass needed: } V = \frac{\pi}{27} \left(\frac{0.015}{12} \right) \approx 0.00015 \text{ ft}^3 \approx 0.25 \text{ in}^3$$

$$51. \text{ (a) } y = f(x) = 0.0000001953x^4 - 0.0001804x^3 + 0.0496x^2 - 4.8323x + 536.9270$$

$$\text{(b) Area} = \int_0^{400} f(x) dx \approx 131,734.5 \text{ square feet}$$

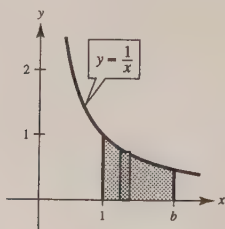
$$\approx 3.0 \text{ acres (1 acre} = 43,560 \text{ square feet)}$$

(Answers will vary.)

$$\text{(c) } L = \int_0^{400} \sqrt{1 + f'(x)^2} dx \approx 794.9 \text{ feet}$$

(Answers will vary.)

$$53. \text{ (a) } V = \pi \int_1^b \frac{1}{x^2} dx = \left[-\frac{\pi}{x} \right]_1^b = \pi \left(1 - \frac{1}{b} \right)$$



$$\begin{aligned} \text{(b) } S &= 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \left(-\frac{1}{x^2} \right)^2} dx \\ &= 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx \\ &= 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx \end{aligned}$$

53. —CONTINUED—

$$(c) \lim_{b \rightarrow \infty} V = \lim_{b \rightarrow \infty} \pi \left(1 - \frac{1}{b} \right) = \pi$$

(d) Since

$$\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x} > 0 \text{ on } [1, b]$$

we have

$$\int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx > \int_1^b \frac{1}{x} dx = [\ln x]_1^b = \ln b$$

and $\lim_{b \rightarrow \infty} \ln b \rightarrow \infty$. Thus,

$$\lim_{b \rightarrow \infty} 2\pi \int_1^b \frac{\sqrt{x^4 + 1}}{x^3} dx = \infty.$$

55. (a) Area of circle with radius L : $A = \pi L^2$ Area of sector with central angle θ (in radians)

$$S = \frac{\theta}{2\pi} A = \frac{\theta}{2\pi} (\pi L^2) = \frac{1}{2} L^2 \theta$$

(b) Let s be the arc length of the sector, which is the circumference of the base of the cone. Here, $s = L\theta = 2\pi r$, and you have

$$S = \frac{1}{2} L^2 \theta = \frac{1}{2} L^2 \left(\frac{s}{L} \right) = \frac{1}{2} L s = \frac{1}{2} L (2\pi r) = \pi r L$$

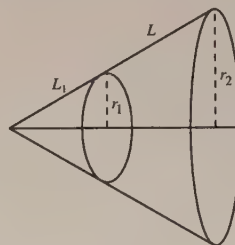
(c) The lateral surface area of the frustum is the difference of the large cone and the small one.

$$\begin{aligned} S &= \pi r_2(L + L_1) - \pi r_1 L_1 \\ &= \pi r_2 L + \pi L_1(r_2 - r_1) \end{aligned}$$

$$\text{By similar triangles, } \frac{L + L_1}{r_2} = \frac{L_1}{r_1} \Rightarrow L r_1 = L_1(r_2 - r_1)$$

Hence,

$$\begin{aligned} S &= \pi r_2 L + \pi L_1(r_2 - r_1) = \pi r_2 L + \pi L r_1 \\ &= \pi L(r_1 + r_2). \end{aligned}$$



Section 6.5 Work

$$1. W = Fd = (100)(10) = 1000 \text{ ft} \cdot \text{lb}$$

$$5. \text{ Work equals force times distance, } W = FD.$$

$$9. F(x) = kx$$

$$5 = k(4)$$

$$k = \frac{5}{4}$$

$$W = \int_0^7 \frac{5}{4} x dx = \left[\frac{5}{8} x^2 \right]_0^7$$

$$= \frac{245}{8} \text{ in} \cdot \text{lb}$$

$$= 30.625 \text{ in} \cdot \text{lb} \approx 2.55 \text{ ft} \cdot \text{lb}$$

$$3. W = Fd = (112)(4) = 448 \text{ joules (newton-meters)}$$

7. Since the work equals the area under the force function, you have (c) < (d) < (a) < (b).

$$11. F(x) = kx$$

$$250 = k(30) \Rightarrow k = \frac{25}{3}$$

$$W = \int_{20}^{50} F(x) dx = \int_{20}^{50} \frac{25}{3} x dx = \left[\frac{25x^2}{6} \right]_{20}^{50}$$

$$= 8750 \text{ n} \cdot \text{cm} = 87.5 \text{ joules or Nm}$$

13. $F(x) = kx$

$20 = k(9)$

$k = \frac{20}{9}$

$$W = \int_0^{12} \frac{20}{9}x \, dx = \left[\frac{10}{9}x^2 \right]_0^{12} = 160 \text{ in} \cdot \text{lb} = \frac{40}{3} \text{ ft} \cdot \text{lb}$$

15. $W = 18 = \int_0^{1/3} kx \, dx = \left[\frac{kx^2}{2} \right]_0^{1/3} = \frac{k}{18} \Rightarrow k = 324$

$$W = \int_{1/3}^{7/12} 324x \, dx = 162x^2 \Big|_{1/3}^{7/12} = 37.125 \text{ ft} \cdot \text{lbs}$$

[Note: 4 inches = $\frac{1}{3}$ foot]

17. Assume that Earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

$5 = \frac{k}{(4000)^2}$

$k = 80,000,000$

$F(x) = \frac{80,000,000}{x^2}$

(a)
$$W = \int_{4000}^{4100} \frac{80,000,000}{x^2} \, dx = \left[\frac{-80,000,000}{x} \right]_{4000}^{4100} \approx 487.8 \text{ mi} \cdot \text{tons}$$

 $\approx 5.15 \times 10^9 \text{ ft} \cdot \text{lb}$

(b)
$$W = \int_{4000}^{4300} \frac{80,000,000}{x^2} \, dx \approx 1395.3 \text{ mi} \cdot \text{ton}$$

 $\approx 1.47 \times 10^{10} \text{ ft} \cdot \text{lb}$

19. Assume that the earth has a radius of 4000 miles.

$F(x) = \frac{k}{x^2}$

$10 = \frac{k}{(4000)^2}$

$k = 160,000,000$

$F(x) = \frac{160,000,000}{x^2}$

(a)
$$W = \int_{4000}^{15,000} \frac{160,000,000}{x^2} \, dx = \left[-\frac{160,000,000}{x} \right]_{4000}^{15,000} \approx -10,666.667 + 40,000$$

 $= 29,333.333 \text{ mi} \cdot \text{ton}$
 $\approx 2.93 \times 10^4 \text{ mi} \cdot \text{ton}$
 $\approx 3.10 \times 10^{11} \text{ ft} \cdot \text{lb}$

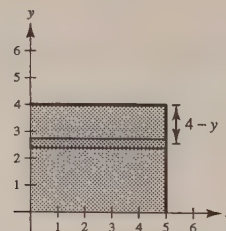
(b)
$$W = \int_{4000}^{26,000} \frac{160,000,000}{x^2} \, dx = \left[-\frac{160,000,000}{x} \right]_{4000}^{26,000} \approx -6,153.846 + 40,000$$

 $= 33,846.154 \text{ mi} \cdot \text{ton}$
 $\approx 3.38 \times 10^4 \text{ mi} \cdot \text{ton}$
 $\approx 3.57 \times 10^{11} \text{ ft} \cdot \text{lb}$

21. Weight of each layer: $62.4(20) \Delta y$ Distance: $4 - y$

(a)
$$W = \int_2^4 62.4(20)(4 - y) \, dy = \left[4992y - 624y^2 \right]_2^4 = 2496 \text{ ft} \cdot \text{lb}$$

(b)
$$W = \int_0^4 62.4(20)(4 - y) \, dy = \left[4992y - 624y^2 \right]_0^4 = 9984 \text{ ft} \cdot \text{lb}$$

23. Volume of disk: $\pi(2)^2 \Delta y = 4\pi \Delta y$ Weight of disk of water: $9800(4\pi) \Delta y$ Distance the disk of water is moved: $5 - y$

$$W = \int_0^4 (5 - y)(9800)4\pi \, dy = 39,200\pi \int_0^4 (5 - y) \, dy$$

$$= 39,200\pi \left[5y - \frac{y^2}{2} \right]_0^4$$

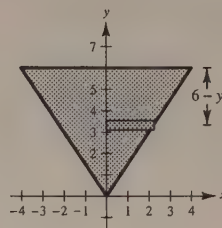
$$= 39,200\pi(12) = 470,400\pi \text{ newton-meters}$$

25. Volume of disk: $\pi \left(\frac{2}{3}y\right)^2 \Delta y$

Weight of disk: $62.4\pi \left(\frac{2}{3}y\right)^2 \Delta y$

Distance: $6 - y$

$$W = \frac{4(62.4)\pi}{9} \int_0^6 (6 - y)y^2 dy = \frac{4}{9}(62.4)\pi \left[2y^3 - \frac{1}{4}y^4 \right]_0^6 = 2995.2\pi \text{ ft} \cdot \text{lb}$$

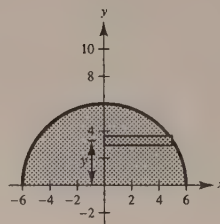


27. Volume of disk: $\pi(\sqrt{36 - y^2})^2 \Delta y$

Weight of disk: $62.4\pi(36 - y^2) \Delta y$

Distance: y

$$\begin{aligned} W &= 62.4\pi \int_0^6 y(36 - y^2) dy \\ &= 62.4\pi \int_0^6 (36y - y^3) dy = 62.4\pi \left[18y^2 - \frac{1}{4}y^4 \right]_0^6 \\ &= 20,217.6\pi \text{ ft} \cdot \text{lb} \end{aligned}$$



29. Volume of layer: $V = lwh = 4(2)\sqrt{(9/4) - y^2} \Delta y$

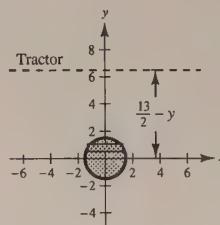
Weight of layer: $W = 42(8)\sqrt{(9/4) - y^2} \Delta y$

Distance: $\frac{13}{2} - y$

$$\begin{aligned} W &= \int_{-1.5}^{1.5} 42(8)\sqrt{(9/4) - y^2} \left(\frac{13}{2} - y \right) dy \\ &= 336 \left[\frac{13}{2} \int_{-1.5}^{1.5} \sqrt{(9/4) - y^2} dy - \int_{-1.5}^{1.5} \sqrt{(9/4) - y^2} y dy \right] \end{aligned}$$

The second integral is zero since the integrand is odd and the limits of integration are symmetric to the origin. The first integral represents the area of a semicircle of radius $\frac{3}{2}$. Thus, the work is

$$W = 336 \left(\frac{13}{2} \right) \pi \left(\frac{3}{2} \right)^2 \left(\frac{1}{2} \right) = 2457\pi \text{ ft} \cdot \text{lb}$$



31. Weight of section of chain: $3 \Delta y$

Distance: $15 - y$

$$\begin{aligned} W &= 3 \int_0^{15} (15 - y) dy \\ &= \left[-\frac{3}{2}(15 - y)^2 \right]_0^{15} \\ &= 337.5 \text{ ft} \cdot \text{lb} \end{aligned}$$

33. The lower 5 feet of chain are raised 10 feet with a constant force.

$$W_1 = 3(5)(10) = 150 \text{ ft} \cdot \text{lb}$$

The top 10 feet of chain are raised with a variable force.

Weight per section: $3 \Delta y$

Distance: $10 - y$

$$\begin{aligned} W_2 &= 3 \int_0^{10} (10 - y) dy = \left[-\frac{3}{2}(10 - y)^2 \right]_0^{10} \\ &= 150 \text{ ft} \cdot \text{lb} \end{aligned}$$

$$W = W_1 + W_2 = 300 \text{ ft} \cdot \text{lb}$$

35. Weight of section of chain: $3 \Delta y$ Distance: $15 - 2y$

$$W = 3 \int_0^{7.5} (15 - 2y) dy = \left[-\frac{3}{4}(15 - 2y)^2 \right]_0^{7.5} \\ = \frac{3}{4}(15)^2 = 168.75 \text{ ft} \cdot \text{lb}$$

39. $p = \frac{k}{V}$

$$1000 = \frac{k}{2}$$

$$k = 2000$$

$$W = \int_2^3 \frac{2000}{V} dV = \left[2000 \ln |V| \right]_2^3 \\ = 2000 \ln \left(\frac{3}{2} \right) \approx 810.93 \text{ ft} \cdot \text{lb}$$

43. $W = \int_0^5 1000[1.8 - \ln(x + 1)] dx \approx 3249.44 \text{ ft} \cdot \text{lb}$

37. Work to pull up the ball: $W_1 = 500(15) = 7500 \text{ ft} \cdot \text{lb}$

Work to wind up the top 15 feet of cable: force is variable

Weight per section: $1 \Delta y$ Distance: $15 - x$

$$W_2 = \int_0^{15} (15 - x) dx = \left[-\frac{1}{2}(15 - x)^2 \right]_0^{15} \\ = 112.5 \text{ ft} \cdot \text{lb}$$

Work to lift the lower 25 feet of cable with a constant force:

$$W_3 = (1)(25)(15) = 375 \text{ ft} \cdot \text{lb}$$

$$W = W_1 + W_2 + W_3 = 7500 + 112.5 + 375 \\ = 7987.5 \text{ ft} \cdot \text{lb}$$

41. $F(x) = \frac{k}{(2 - x)^2}$

$$W = \int_{-2}^1 \frac{k}{(2 - x)^2} dx = \left[\frac{k}{2 - x} \right]_{-2}^1 = k \left(1 - \frac{1}{4} \right) \\ = \frac{3k}{4} (\text{units of work})$$

45. $W = \int_0^5 100x\sqrt{125 - x^3} dx \approx 10,330.3 \text{ ft} \cdot \text{lb}$

Section 6.6 Moments, Centers of Mass, and Centroids

1. $\bar{x} = \frac{6(-5) + 3(1) + 5(3)}{6 + 3 + 5} = -\frac{6}{7}$

3. $\bar{x} = \frac{1(7) + 1(8) + 1(12) + 1(15) + 1(18)}{1 + 1 + 1 + 1 + 1} = 12$

5. (a) $\bar{x} = \frac{(7 + 5) + (8 + 5) + (12 + 5) + (15 + 5) + (18 + 5)}{5} = 17 = 12 + 5$

(b) $\bar{x} = \frac{12(-6 - 3) + 1(-4 - 3) + 6(-2 - 3) + 3(0 - 3) + 11(8 - 3)}{12 + 1 + 6 + 3 + 11} = \frac{-99}{33} = -3$

7. $50x = 75(L - x) = 75(10 - x)$

$$50x = 750 - 75x$$

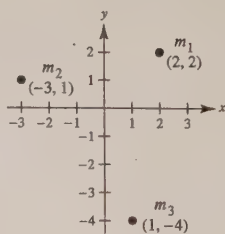
$$125x = 750$$

$$x = 6 \text{ feet}$$

9. $\bar{x} = \frac{5(2) + 1(-3) + 3(1)}{5 + 1 + 3} = \frac{10}{9}$

$$\bar{y} = \frac{5(2) + 1(1) + 3(-4)}{5 + 1 + 3} = -\frac{1}{9}$$

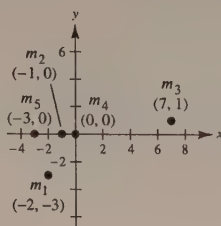
$$(\bar{x}, \bar{y}) = \left(\frac{10}{9}, -\frac{1}{9} \right)$$



$$11. \quad \bar{x} = \frac{3(-2) + 4(-1) + 2(7) + 1(0) + 6(-3)}{3 + 4 + 2 + 1 + 6} = -\frac{7}{8}$$

$$\bar{y} = \frac{3(-3) + 4(0) + 2(1) + 1(0) + 6(0)}{3 + 4 + 2 + 1 + 6} = -\frac{7}{16}$$

$$(\bar{x}, \bar{y}) = \left(-\frac{7}{8}, -\frac{7}{16}\right)$$



$$13. \quad m = \rho \int_0^4 \sqrt{x} \, dx = \left[\frac{2\rho}{3} x^{3/2} \right]_0^4 = \frac{16\rho}{3}$$

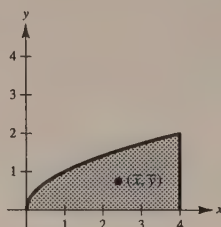
$$M_x = \rho \int_0^4 \frac{\sqrt{x}}{2} (\sqrt{x}) \, dx = \left[\rho \frac{x^2}{4} \right]_0^4 = 4\rho$$

$$\bar{y} = \frac{M_x}{m} = 4\rho \left(\frac{3}{16\rho} \right) = \frac{3}{4}$$

$$M_y = \rho \int_0^4 x \sqrt{x} \, dx = \left[\rho \frac{2}{5} x^{5/2} \right]_0^4 = \frac{64\rho}{5}$$

$$\bar{x} = \frac{M_y}{m} = \frac{64\rho}{5} \left(\frac{3}{16\rho} \right) = \frac{12}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{12}{5}, \frac{3}{4} \right)$$



$$15. \quad m = \rho \int_0^1 (x^2 - x^3) \, dx = \rho \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\rho}{12}$$

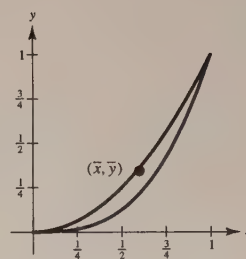
$$M_x = \rho \int_0^1 \frac{(x^2 + x^3)}{2} (x^2 - x^3) \, dx = \frac{\rho}{2} \int_0^1 (x^4 - x^6) \, dx = \frac{\rho}{2} \left[\frac{x^5}{5} - \frac{x^7}{7} \right]_0^1 = \frac{\rho}{35}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho}{35} \left(\frac{12}{\rho} \right) = \frac{12}{35}$$

$$M_y = \rho \int_0^1 x(x^2 - x^3) \, dx = \rho \int_0^1 (x^3 - x^4) \, dx = \rho \left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^1 = \frac{\rho}{20}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\rho}{20} \left(\frac{12}{\rho} \right) = \frac{3}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{5}, \frac{12}{35} \right)$$



$$17. \quad m = \rho \int_0^3 [(-x^2 + 4x + 2) - (x + 2)] \, dx = -\rho \left[\frac{x^3}{3} + \frac{3x^2}{2} \right]_0^3 = \frac{9\rho}{2}$$

$$M_x = \rho \int_0^3 \left[\frac{(-x^2 + 4x + 2) + (x + 2)}{2} \right] [(-x^2 + 4x + 2) - (x + 2)] \, dx$$

$$= \frac{\rho}{2} \int_0^3 (-x^2 + 5x + 4)(-x^2 + 3x) \, dx = \frac{\rho}{2} \int_0^3 (x^4 - 8x^3 + 11x^2 + 12x) \, dx$$

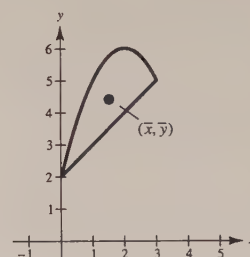
$$= \frac{\rho}{2} \left[\frac{x^5}{5} - 2x^4 + \frac{11x^3}{3} + 6x^2 \right]_0^3 = \frac{99\rho}{5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{99\rho}{5} \left(\frac{2}{9\rho} \right) = \frac{22}{5}$$

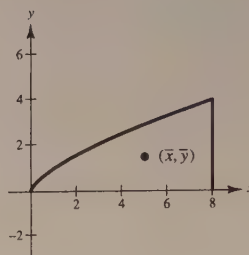
$$M_y = \rho \int_0^3 x[(-x^2 + 4x + 2) - (x + 2)] \, dx = \rho \int_0^3 (-x^3 + 3x^2) \, dx = \rho \left[-\frac{x^4}{4} + x^3 \right]_0^3 = \frac{27\rho}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{27\rho}{4} \left(\frac{2}{9\rho} \right) = \frac{3}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{2}, \frac{22}{5} \right)$$



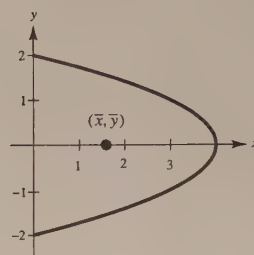
$$\begin{aligned}
 19. \quad m &= \rho \int_0^8 x^{2/3} dx = \rho \left[\frac{3}{5} x^{5/3} \right]_0^8 = \frac{96\rho}{5} \\
 M_x &= \rho \int_0^8 \frac{x^{2/3}}{2} (x^{2/3}) dx = \frac{\rho}{2} \left[\frac{3}{7} x^{7/3} \right]_0^8 = \frac{192\rho}{7} \\
 \bar{y} &= \frac{M_x}{m} = \frac{192\rho}{7} \left(\frac{5}{96\rho} \right) = \frac{10}{7} \\
 M_y &= \rho \int_0^8 x(x^{2/3}) dx = \rho \left[\frac{3}{8} x^{8/3} \right]_0^8 = 96\rho \\
 \bar{x} &= \frac{M_y}{m} = 96\rho \left(\frac{5}{96\rho} \right) = 5 \\
 (\bar{x}, \bar{y}) &= \left(5, \frac{10}{7} \right)
 \end{aligned}$$



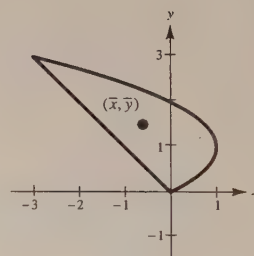
$$\begin{aligned}
 21. \quad m &= 2\rho \int_0^2 (4 - y^2) dy = 2\rho \left[4y - \frac{y^3}{3} \right]_0^2 = \frac{32\rho}{3} \\
 M_y &= 2\rho \int_0^2 \left(\frac{4 - y^2}{2} \right) (4 - y^2) dy = \rho \left[16y - \frac{8}{3}y^3 + \frac{y^5}{5} \right]_0^2 = \frac{256\rho}{15} \\
 \bar{x} &= \frac{M_y}{m} = \frac{256\rho}{15} \left(\frac{3}{32\rho} \right) = \frac{8}{5} \\
 \bar{y} &= 0 \quad (\text{by symmetry})
 \end{aligned}$$

By symmetry, M_x and $\bar{y} = 0$.

$$(\bar{x}, \bar{y}) = \left(\frac{8}{5}, 0 \right)$$



$$\begin{aligned}
 23. \quad m &= \rho \int_0^3 [(2y - y^2) - (-y)] dy = \rho \left[\frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3 = \frac{9\rho}{2} \\
 M_y &= \rho \int_0^3 \frac{[(2y - y^2) + (-y)]}{2} [(2y - y^2) - (-y)] dy = \frac{\rho}{2} \int_0^3 (y - y^2)(3y - y^2) dy \\
 &= \frac{\rho}{2} \int_0^3 (y^4 - 4y^3 + 3y^2) dy = \frac{\rho}{2} \left[\frac{y^5}{5} - y^4 + y^3 \right]_0^3 = -\frac{27\rho}{10} \\
 \bar{x} &= \frac{M_y}{m} = -\frac{27\rho}{10} \left(\frac{2}{9\rho} \right) = -\frac{3}{5} \\
 M_x &= \rho \int_0^3 y[(2y - y^2) - (-y)] dy = \rho \int_0^3 (3y^2 - y^3) dy = \rho \left[y^3 - \frac{y^4}{4} \right]_0^3 = \frac{27\rho}{4} \\
 \bar{y} &= \frac{M_x}{m} = \frac{27\rho}{4} \left(\frac{2}{9\rho} \right) = \frac{3}{2} \\
 (\bar{x}, \bar{y}) &= \left(-\frac{3}{5}, \frac{3}{2} \right)
 \end{aligned}$$



$$\begin{aligned}
 25. \quad A &= \int_0^1 (x - x^2) dx = \left[\frac{1}{2}x^2 - \frac{x^3}{3} \right]_0^1 = \frac{1}{6} \\
 M_x &= \frac{1}{2} \int_0^1 (x^2 - x^4) dx = \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{1}{15} \\
 M_y &= \int_0^1 (x^2 - x^3) dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{1}{12}
 \end{aligned}$$

$$27. A = \int_0^3 (2x + 4) dx = \left[x^2 + 4x \right]_0^3 = 9 + 12 = 21$$

$$M_x = \frac{1}{2} \int_0^3 (2x + 4)^2 dx = \int_0^3 (2x^2 + 8x + 8) dx = \left[\frac{2x^3}{3} + 4x^2 + 8x \right]_0^3 = 18 + 36 + 24 = 78$$

$$M_y = \int_0^3 (2x^2 + 4x) dx = \left[\frac{2x^3}{3} + 2x^2 \right]_0^3 = 18 + 18 = 36$$

$$29. m = \rho \int_0^5 10x\sqrt{125 - x^3} dx \approx 1033.0\rho$$

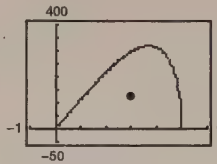
$$M_x = \rho \int_0^5 \left(\frac{10x\sqrt{125 - x^3}}{2} \right) (10x\sqrt{125 - x^3}) dx = 50\rho \int_0^5 x^2(125 - x^3) dx = \frac{3,124,375\rho}{24} \approx 130,208\rho$$

$$M_y = \rho \int_0^5 10x^2\sqrt{125 - x^3} dx = -\frac{10\rho}{3} \int_0^5 \sqrt{125 - x^3}(-3x^2) dx = \frac{12,500\sqrt{5}\rho}{9} \approx 3105.6\rho$$

$$\bar{x} = \frac{M_y}{m} \approx 3.0$$

$$\bar{y} = \frac{M_x}{m} \approx 126.0$$

Therefore, the centroid is (3.0, 126.0).

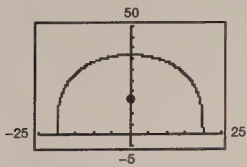


$$31. m = \rho \int_{-20}^{20} 5\sqrt[3]{400 - x^2} dx \approx 1239.76\rho$$

$$M_x = \rho \int_{-20}^{20} \frac{5\sqrt[3]{400 - x^2}}{2} (5\sqrt[3]{400 - x^2}) dx \\ = \frac{25\rho}{2} \int_{-20}^{20} (400 - x^2)^{2/3} dx \approx 20064.27$$

$$\bar{y} = \frac{M_x}{m} \approx 16.18$$

$\bar{x} = 0$ by symmetry. Therefore, the centroid is (0, 16.2).



$$33. A = \frac{1}{2}(2a)c = ac$$

$$\frac{1}{A} = \frac{1}{ac}$$

$$\bar{x} = \left(\frac{1}{ac} \right) \frac{1}{2} \int_0^c \left[\left(\frac{b-a}{c}y + a \right)^2 - \left(\frac{b+a}{c}y - a \right)^2 \right] dy$$

$$= \frac{1}{2ac} \int_0^c \left[\frac{4ab}{c}y - \frac{4ab}{c^2}y^2 \right] dy$$

$$= \frac{1}{2ac} \left[\frac{2ab}{c}y^2 - \frac{4ab}{3c^2}y^3 \right]_0^c = \frac{1}{2ac} \left(\frac{2}{3}abc \right) = \frac{b}{3}$$

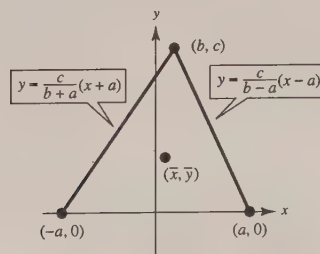
$$\bar{y} = \frac{1}{ac} \int_0^c y \left[\left(\frac{b-a}{c}y + a \right) - \left(\frac{b+a}{c}y - a \right) \right] dy$$

$$= \frac{1}{ac} \int_0^c y \left(-\frac{2a}{c}y + 2a \right) dy = \frac{2}{c} \int_0^c \left(y - \frac{y^2}{c} \right) dy$$

$$= \frac{2}{c} \left[\frac{y^2}{2} - \frac{y^3}{3c} \right]_0^c = \frac{c}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3} \right)$$

In Exercise 73 of Section P.2, you found that $(b/3, c/3)$ is the point of intersection of the medians.



$$35. A = \frac{c}{2}(a+b)$$

$$\frac{1}{A} = \frac{2}{c(a+b)}$$

$$\bar{x} = \frac{2}{c(a+b)} \int_0^c x \left(\frac{b-a}{c}x + a \right) dx = \frac{2}{c(a+b)} \int_0^c \left(\frac{b-a}{c}x^2 + ax \right) dx = \frac{2}{c(a+b)} \left[\frac{b-a}{c} \frac{x^3}{3} + \frac{ax^2}{2} \right]_0^c$$

$$= \frac{2}{c(a+b)} \left[\frac{(b-a)c^2}{3} + \frac{ac^2}{2} \right] = \frac{2}{c(a+b)} \left[\frac{2bc^2 - 2ac^2 + 3ac^2}{6} \right] = \frac{c(2b+a)}{3(a+b)} = \frac{(a+2b)c}{3(a+b)}$$

$$\bar{y} = \frac{2}{c(a+b)} \frac{1}{2} \int_0^c \left(\frac{b-a}{c}x + a \right)^2 dx = \frac{1}{c(a+b)} \int_0^c \left[\left(\frac{b-a}{c} \right)^2 x^2 + \frac{2a(b-a)}{c}x + a^2 \right] dx$$

$$= \frac{1}{c(a+b)} \left[\left(\frac{b-a}{c} \right)^2 \frac{x^3}{3} + \frac{2a(b-a)}{c} \frac{x^2}{2} + a^2x \right]_0^c = \frac{1}{c(a+b)} \left[\frac{(b-a)^2c}{3} + ac(b-a) + a^2c \right]$$

$$= \frac{1}{3c(a+b)} [(b^2 - 2ab + a^2)c + 3ac(b-a) + 3a^2c]$$

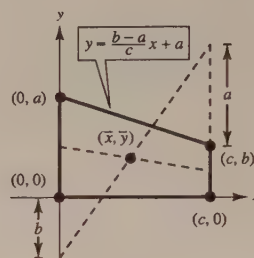
$$= \frac{1}{3(a+b)} [b^2 - 2ab + a^2 + 3ab - 3a^2 + 3a^2] = \frac{a^2 + ab + b^2}{3(a+b)}$$

$$\text{Thus, } (\bar{x}, \bar{y}) = \left(\frac{(a+2b)c}{3(a+b)}, \frac{a^2 + ab + b^2}{3(a+b)} \right).$$

The one line passes through $(0, a/2)$ and $(c, b/2)$. Its equation is $y = \frac{b-a}{2c}x + \frac{a}{2}$.

The other line passes through $(0, -b)$ and $(c, a+b)$. Its equation is $y = \frac{a+2b}{c}x - b$.

(\bar{x}, \bar{y}) is the point of intersection of these two lines.



37. $\bar{x} = 0$ by symmetry

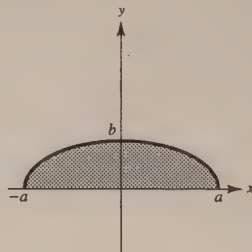
$$A = \frac{1}{2}\pi ab$$

$$\frac{1}{A} = \frac{2}{\pi ab}$$

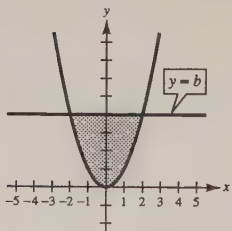
$$\bar{y} = \frac{2}{\pi ab} \frac{1}{2} \int_{-a}^a \left(\frac{b}{a} \sqrt{a^2 - x^2} \right)^2 dx$$

$$= \frac{1}{\pi ab} \left(\frac{b^2}{a^2} \right) \left[a^2x - \frac{x^3}{3} \right]_{-a}^a = \frac{b}{\pi} \left[\frac{4a^3}{3} \right] = \frac{4b}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4b}{3\pi} \right)$$



39. (a)



(b) $\bar{x} = 0$ by symmetry

$$(c) M_y = \int_{-\sqrt{b}}^{\sqrt{b}} x(b - x^2) dx = 0 \text{ because } bx - x^3 \text{ is odd}$$

(d) $\bar{y} > \frac{b}{2}$ since there is more area above $y = \frac{b}{2}$ than below

$$\begin{aligned} (e) M_x &= \int_{-\sqrt{b}}^{\sqrt{b}} \frac{(b+x^2)(b-x^2)}{2} dx \\ &= \int_{-\sqrt{b}}^{\sqrt{b}} \frac{b^2 - x^4}{2} dx = \frac{1}{2} \left[b^2x - \frac{x^5}{5} \right]_{-\sqrt{b}}^{\sqrt{b}} \\ &= b^2\sqrt{b} - \frac{b^2\sqrt{b}}{5} = \frac{4b^2\sqrt{b}}{5} \end{aligned}$$

$$\begin{aligned} A &= \int_{-\sqrt{b}}^{\sqrt{b}} (b - x^2) dx = \left[bx - \frac{x^3}{3} \right]_{-\sqrt{b}}^{\sqrt{b}} \\ &= \left(b\sqrt{b} - \frac{b\sqrt{b}}{3} \right) 2 = \frac{4b\sqrt{b}}{3} \end{aligned}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4b^2\sqrt{b}/5}{4b\sqrt{b}/3} = \frac{3}{5}b.$$

41. (a) $\bar{x} = 0$ by symmetry

$$A = 2 \int_0^{40} f(x) dx = \frac{2(40)}{3(4)} [30 + 4(29) + 2(26) + 4(20) + 0] = \frac{20}{3} (278) = \frac{5560}{3}$$

$$M_x = \int_{-40}^{40} \frac{f(x)^2}{2} dx = \frac{40}{3(4)} [30^2 + 4(29)^2 + 2(26)^2 + 4(20)^2 + 0] = \frac{10}{3} (7216) = \frac{72160}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{72160/3}{5560/3} = \frac{72160}{5560} \approx 12.98$$

$$(\bar{x}, \bar{y}) = (0, 12.98)$$

(b) $y = (-1.02 \times 10^{-5})x^4 - 0.0019x^2 + 29.28$

(c) $\bar{y} = \frac{M_x}{A} \approx \frac{23697.68}{1843.54} \approx 12.85$

$$(\bar{x}, \bar{y}) = (0, 12.85)$$

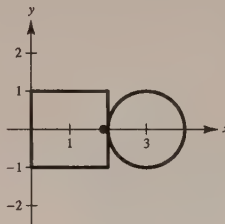
43. Centroids of the given regions: (1, 0) and (3, 0)

Area: $A = 4 + \pi$

$$\bar{x} = \frac{4(1) + \pi(3)}{4 + \pi} = \frac{4 + 3\pi}{4 + \pi}$$

$$\bar{y} = \frac{4(0) + \pi(0)}{4 + \pi} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{4 + 3\pi}{4 + \pi}, 0 \right) \approx (1.88, 0)$$



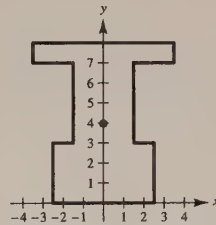
45. Centroids of the given regions: $\left(0, \frac{3}{2}\right)$, (0, 5), and $\left(0, \frac{15}{2}\right)$

Area: $A = 15 + 12 + 7 = 34$

$$\bar{x} = \frac{15(0) + 12(0) + 7(0)}{34} = 0$$

$$\bar{y} = \frac{15(3/2) + 12(5) + 7(15/2)}{34} = \frac{135}{34}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{135}{34} \right)$$



47. Centroids of the given regions: (1, 0) and (3, 0)

Mass: $4 + 2\pi$

$$\bar{x} = \frac{4(1) + 2\pi(3)}{4 + 2\pi} = \frac{2 + 3\pi}{2 + \pi}$$

$$\bar{y} = 0$$

$$(\bar{x}, \bar{y}) = \left(\frac{2 + 3\pi}{2 + \pi}, 0 \right) \approx (2.22, 0)$$

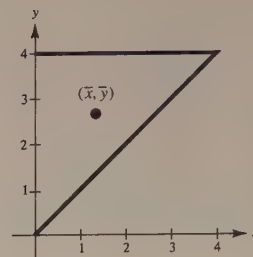
49. $V = 2\pi rA = 2\pi(5)(16\pi) = 160\pi^2 \approx 1579.14$

$$51. A = \frac{1}{2}(4)(4) = 8$$

$$\bar{y} = \left(\frac{1}{8}\right) \frac{1}{2} \int_0^4 (4+x)(4-x) dx = \frac{1}{16} \left[16x - \frac{x^3}{3} \right]_0^4 = \frac{8}{3}$$

$$r = \bar{y} = \frac{8}{3}$$

$$V = 2\pi r A = 2\pi \left(\frac{8}{3}\right)(8) = \frac{128\pi}{3} \approx 134.04$$



$$53. m = m_1 + \cdots + m_n$$

$$M_y = m_1 x_1 + \cdots + m_n x_n$$

$$M_x = m_1 y_1 + \cdots + m_n y_n$$

$$\bar{x} = \frac{M_y}{m}, \bar{y} = \frac{M_x}{m}$$

$$55. (a) \text{ Yes. } (\bar{x}, \bar{y}) = \left(\frac{5}{6}, \frac{5}{18} + 2\right) = \left(\frac{5}{6}, \frac{41}{18}\right)$$

$$(b) \text{ Yes. } (\bar{x}, \bar{y}) = \left(\frac{5}{6} + 2, \frac{5}{18}\right) = \left(\frac{17}{6}, \frac{5}{18}\right)$$

$$(c) \text{ Yes. } (\bar{x}, \bar{y}) = \left(\frac{5}{6}, -\frac{5}{18}\right)$$

$$(d) \text{ No.}$$

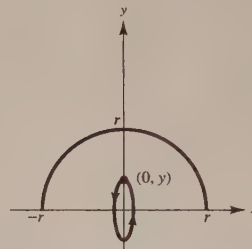
57. The surface area of the sphere is $S = 4\pi r^2$. The arc length of C is $s = \pi r$. The distance traveled by the centroid is

$$d = \frac{S}{s} = \frac{4\pi r^2}{\pi r} = 4r.$$

This distance is also the circumference of the circle of radius y .

$$d = 2\pi y$$

Thus, $2\pi y = 4r$ and we have $y = 2r/\pi$. Therefore, the centroid of the semicircle $y = \sqrt{r^2 - x^2}$ is $(0, 2r/\pi)$.



$$59. A = \int_0^1 x^n dx = \left[\frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1}$$

$$m = \rho A = \frac{\rho}{n+1}$$

$$M_x = \frac{\rho}{2} \int_0^1 (x^n)^2 dx = \left[\frac{\rho}{2} \cdot \frac{x^{2n+1}}{2n+1} \right]_0^1 = \frac{\rho}{2(2n+1)}$$

$$M_y = \rho \int_0^1 x(x^n) dx = \left[\rho \cdot \frac{x^{n+2}}{n+2} \right]_0^1 = \frac{\rho}{n+2}$$

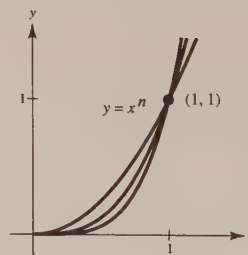
$$\bar{x} = \frac{M_y}{m} = \frac{n+1}{n+2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{n+1}{2(2n+1)} = \frac{n+1}{4n+2}$$

$$\text{Centroid: } \left(\frac{n+1}{n+2}, \frac{n+1}{4n+2} \right)$$

$$\text{As } n \rightarrow \infty, (\bar{x}, \bar{y}) \rightarrow \left(1, \frac{1}{4}\right).$$

The graph approaches the x -axis and the line $x = 1$ as $n \rightarrow \infty$.



Section 6.7 Fluid Pressure and Fluid Force

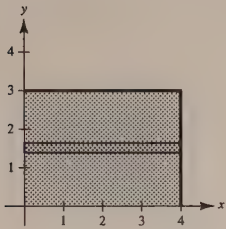
$$1. F = PA = [62.4(5)](3) = 936 \text{ lb}$$

$$\begin{aligned} 3. F &= 62.4(h+2)(6) - (62.4)(h)(6) \\ &= 62.4(2)(6) = 748.8 \text{ lb} \end{aligned}$$

$$5. h(y) = 3 - y$$

$$L(y) = 4$$

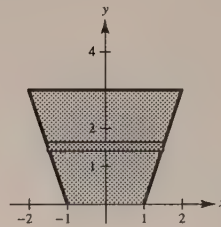
$$\begin{aligned} F &= 62.4 \int_0^3 (3-y)(4) dy \\ &= 249.6 \int_0^3 (3-y) dy \\ &= 249.6 \left[3y - \frac{y^2}{2} \right]_0^3 = 1123.2 \text{ lb} \end{aligned}$$



$$7. h(y) = 3 - y$$

$$L(y) = 2\left(\frac{y}{3} + 1\right)$$

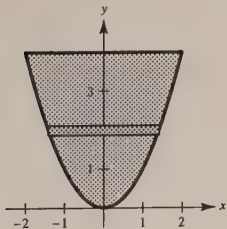
$$\begin{aligned} F &= 2(62.4) \int_0^3 (3-y)\left(\frac{y}{3} + 1\right) dy \\ &= 124.8 \int_0^3 \left(3 - \frac{y^2}{3}\right) dy \\ &= 124.8 \left[3y - \frac{y^3}{9} \right]_0^3 = 748.8 \text{ lb} \end{aligned}$$



$$9. h(y) = 4 - y$$

$$L(y) = 2\sqrt{y}$$

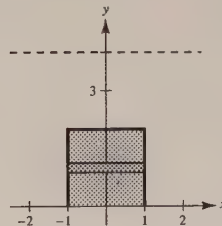
$$\begin{aligned} F &= 2(62.4) \int_0^4 (4-y)\sqrt{y} dy \\ &= 124.8 \int_0^4 (4y^{1/2} - y^{3/2}) dy \\ &= 124.8 \left[\frac{8y^{3/2}}{3} - \frac{2y^{5/2}}{5} \right]_0^4 = 1064.96 \text{ lb} \end{aligned}$$



$$11. h(y) = 4 - y$$

$$L(y) = 2$$

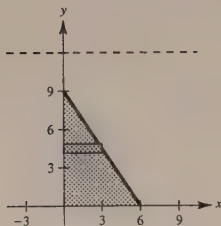
$$\begin{aligned} F &= 9800 \int_0^2 2(4-y) dy \\ &= 9800 \left[8y - y^2 \right]_0^2 = 117,600 \text{ Newtons} \end{aligned}$$



13. $h(y) = 12 - y$

$$L(y) = 6 - \frac{2y}{3}$$

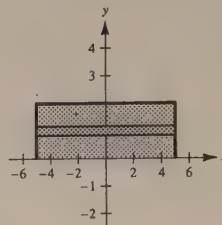
$$\begin{aligned}
 F &= 9800 \int_0^9 (12 - y) \left(6 - \frac{2y}{3} \right) dy \\
 &= 9800 \left[72y - 7y^2 + \frac{2y^3}{9} \right]_0^9 = 2,381,400 \text{ Newtons}
 \end{aligned}$$



15. $h(y) = 2 - y$

$$L(y) = 10$$

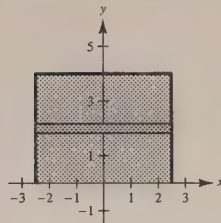
$$\begin{aligned}
 F &= 140.7 \int_0^2 (2 - y)(10) dy \\
 &= 1407 \int_0^2 (2 - y) dy \\
 &= 1407 \left[2y - \frac{y^2}{2} \right]_0^2 = 2814 \text{ lb}
 \end{aligned}$$



17. $h(y) = 4 - y$

$$L(y) = 6$$

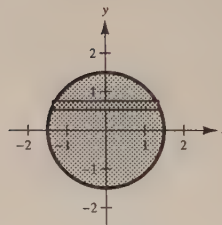
$$\begin{aligned}
 F &= 140.7 \int_0^4 (4 - y)(6) dy \\
 &= 844.2 \int_0^4 (4 - y) dy \\
 &= 844.2 \left[4y - \frac{y^2}{2} \right]_0^4 = 6753.6 \text{ lb}
 \end{aligned}$$



19. $h(y) = -y$

$$L(y) = 2 \left(\frac{1}{2} \right) \sqrt{9 - 4y^2}$$

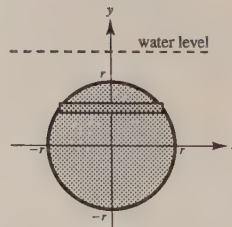
$$\begin{aligned}
 F &= 42 \int_{-3/2}^0 (-y) \sqrt{9 - 4y^2} dy \\
 &= \frac{42}{8} \int_{-3/2}^0 (9 - 4y^2)^{1/2} (-8y) dy \\
 &= \left[\left(\frac{21}{4} \right) \left(\frac{2}{3} \right) (9 - 4y^2)^{3/2} \right]_{-3/2}^0 = 94.5 \text{ lb}
 \end{aligned}$$



21. $h(y) = k - y$

$$L(y) = 2\sqrt{r^2 - y^2}$$

$$\begin{aligned}
 F &= w \int_{-r}^r (k - y) \sqrt{r^2 - y^2} (2) dy \\
 &= w \left[2k \int_{-r}^r \sqrt{r^2 - y^2} dy + \int_{-r}^r \sqrt{r^2 - y^2} (-2y) dy \right]
 \end{aligned}$$



The second integral is zero since its integrand is odd and the limits of integration are symmetric to the origin. The first integral is the area of a semicircle with radius r .

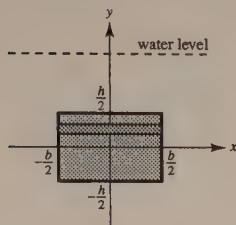
$$F = w \left[(2k) \frac{\pi r^2}{2} + 0 \right] = w k \pi r^2$$

23. $h(y) = k - y$

$$L(y) = b$$

$$F = w \int_{-h/2}^{h/2} (k - y)b \, dy$$

$$= wb \left[ky - \frac{y^2}{2} \right]_{-h/2}^{h/2} = wb(hk) = wkhb$$



27. $h(y) = 4 - y$

$$F = 62.4 \int_0^4 (4 - y)L(y) \, dy$$

Using Simpson's Rule with $n = 8$ we have:

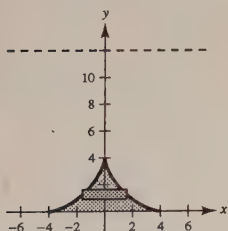
$$\begin{aligned} F &\approx 62.4 \left(\frac{4 - 0}{3(8)} \right) [0 + 4(3.5)(3) + 2(3)(5) + 4(2.5)(8) + 2(2)(9) + 4(1.5)(10) + 2(1)(10.25) + 4(0.5)(10.5) + 0] \\ &= 3010.8 \text{ lb} \end{aligned}$$

29. $h(y) = 12 - y$

$$L(y) = 2(4^{2/3} - y^{2/3})^{3/2}$$

$$F = 62.4 \int_0^4 2(12 - y)(4^{2/3} - y^{2/3})^{3/2} \, dy$$

$$\approx 6448.73 \text{ lb}$$



25. From Exercise 23:

$$F = 64(15)(1)(1) = 960 \text{ lb}$$

 31. (a) If the fluid force is one half of 1123.2 lb, and the height of the water is b , then

$$h(y) = b - y$$

$$L(y) = 4$$

$$F = 62.4 \int_0^b (b - y)(4) \, dy = \frac{1}{2}(1123.2)$$

$$\int_0^b (b - y) \, dy = 2.25$$

$$\left[by - \frac{y^2}{2} \right]_0^b = 2.25$$

$$b^2 - \frac{b^2}{2} = 2.25$$

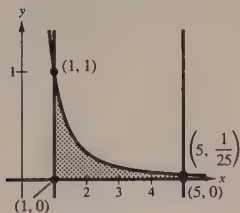
$$b^2 = 4.5 \Rightarrow b \approx 2.12 \text{ ft.}$$

(b) The pressure increases with increasing depth.

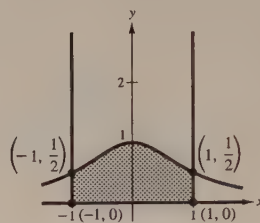
33. $F = F_w = w \int_c^d h(y)L(y) \, dy$, see page 471.

Review Exercises for Chapter 6

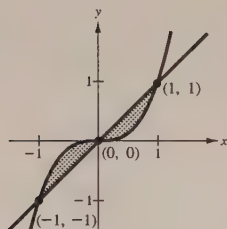
$$1. A = \int_1^5 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^5 = \frac{4}{5}$$



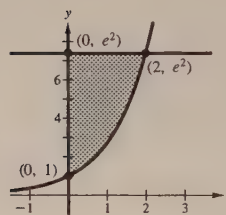
$$\begin{aligned} 3. A &= \int_{-1}^1 \frac{1}{x^2 + 1} dx \\ &= \left[\arctan x \right]_{-1}^1 \\ &= \frac{\pi}{4} - \left(-\frac{\pi}{4} \right) = \frac{\pi}{2} \end{aligned}$$



$$\begin{aligned} 5. A &= 2 \int_0^1 (x - x^3) dx \\ &= 2 \left[\frac{1}{2}x^2 - \frac{1}{4}x^4 \right]_0^1 \\ &= \frac{1}{2} \end{aligned}$$

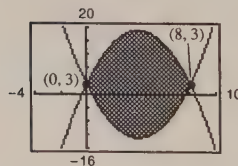


$$\begin{aligned} 7. A &= \int_0^2 (e^2 - e^x) dx \\ &= \left[xe^2 - e^x \right]_0^2 \\ &= e^2 + 1 \end{aligned}$$



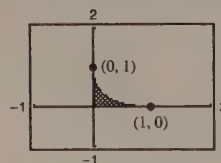
$$\begin{aligned} 9. A &= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx \\ &= \left[-\cos x - \sin x \right]_{\pi/4}^{5\pi/4} \\ &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \\ &= \frac{4}{\sqrt{2}} = 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} 11. A &= \int_0^8 [(3 + 8x - x^2) - (x^2 - 8x + 3)] dx \\ &= \int_0^8 (16x - 2x^2) dx \\ &= \left[8x^2 - \frac{2}{3}x^3 \right]_0^8 = \frac{512}{3} \approx 170.667 \end{aligned}$$



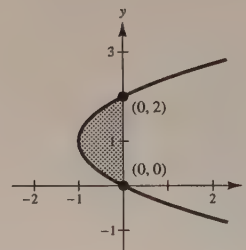
13. $y = (1 - \sqrt{x})^2$

$$\begin{aligned} A &= \int_0^1 (1 - \sqrt{x})^2 dx \\ &= \int_0^1 (1 - 2x^{1/2} + x) dx \\ &= \left[x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2 \right]_0^1 = \frac{1}{6} \approx 0.1667 \end{aligned}$$



15. $x = y^2 - 2y \Rightarrow x + 1 = (y - 1)^2 \Rightarrow y = 1 \pm \sqrt{x + 1}$

$$\begin{aligned} A &= \int_{-1}^0 [(1 + \sqrt{x + 1}) - (1 - \sqrt{x + 1})] dx = \int_{-1}^0 2\sqrt{x + 1} dx \\ A &= \int_0^2 [0 - (y^2 - 2y)] dy = \int_0^2 (2y - y^2) dy = \left[y^2 - \frac{1}{3}y^3 \right]_0^2 = \frac{4}{3} \end{aligned}$$



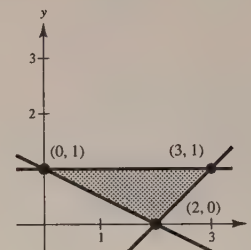
17. $A = \int_0^2 \left[1 - \left(1 - \frac{x}{2} \right) \right] dx + \int_2^3 [1 - (x - 2)] dx$

$$= \int_0^2 \frac{x}{2} dx + \int_2^3 (3 - x) dx$$

$$y = 1 - \frac{x}{2} \Rightarrow x = 2 - 2y$$

$$y = x - 2 \Rightarrow x = y + 2, y = 1$$

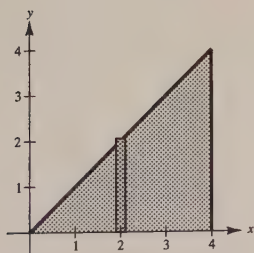
$$\begin{aligned} A &= \int_0^1 [(y + 2) - (2 - 2y)] dy \\ &= \int_0^1 3y dy = \left[\frac{3}{2}y^2 \right]_0^1 = \frac{3}{2} \end{aligned}$$



19. Job 1 is better. The salary for Job 1 is greater than the salary for Job 2 for all the years except the first and 10th years.

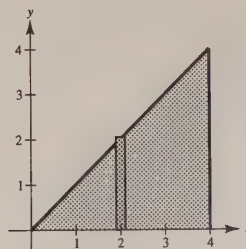
21. (a) Disk

$$V = \pi \int_0^4 x^2 dx = \left[\frac{\pi x^3}{3} \right]_0^4 = \frac{64\pi}{3}$$



(b) Shell

$$V = 2\pi \int_0^4 x^2 dx = \left[\frac{2\pi x^3}{3} \right]_0^4 = \frac{128\pi}{3}$$

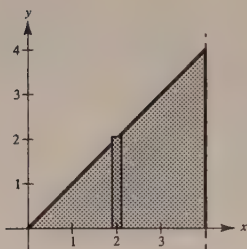


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21. —CONTINUED—

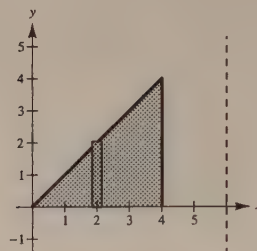
(c) Shell

$$\begin{aligned}
 V &= 2\pi \int_0^4 (4-x)x \, dx \\
 &= 2\pi \int_0^4 (4x - x^2) \, dx \\
 &= 2\pi \left[2x^2 - \frac{x^3}{3} \right]_0^4 = \frac{64\pi}{3}
 \end{aligned}$$



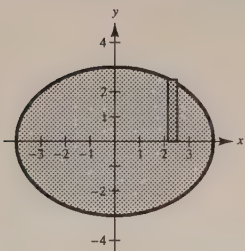
(d) Shell

$$\begin{aligned}
 V &= 2\pi \int_0^4 (6-x)x \, dx \\
 &= 2\pi \int_0^4 (6x - x^2) \, dx \\
 &= 2\pi \left[3x^2 - \frac{1}{3}x^3 \right]_0^4 = \frac{160\pi}{3}
 \end{aligned}$$



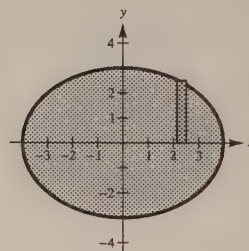
23. (a) Shell

$$\begin{aligned}
 V &= 4\pi \int_0^4 x \left(\frac{3}{4} \right) \sqrt{16-x^2} \, dx \\
 &= \left[3\pi \left(-\frac{1}{2} \right) \left(\frac{2}{3} \right) (16-x^2)^{3/2} \right]_0^4 = 64\pi
 \end{aligned}$$



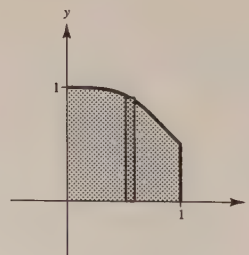
(b) Disk

$$\begin{aligned}
 V &= 2\pi \int_0^4 \left[\frac{3}{4} \sqrt{16-x^2} \right]^2 dx \\
 &= \frac{9\pi}{8} \left[16x - \frac{x^3}{3} \right]_0^4 = 48\pi
 \end{aligned}$$



25. Shell

$$\begin{aligned}
 V &= 2\pi \int_0^1 \frac{x}{x^4 + 1} \, dx \\
 &= \pi \int_0^1 \frac{(2x)}{(x^2)^2 + 1} \, dx \\
 &= \left[\pi \arctan(x^2) \right]_0^1 \\
 &= \pi \left[\frac{\pi}{4} - 0 \right] = \frac{\pi^2}{4}
 \end{aligned}$$



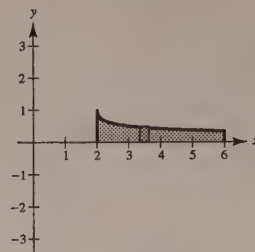
27. Shell

$$u = \sqrt{x-2}$$

$$x = u^2 + 2$$

$$dx = 2u \, du$$

$$\begin{aligned} V &= 2\pi \int_2^6 \frac{x}{1 + \sqrt{x-2}} dx = 4\pi \int_0^2 \frac{(u^2 + 2)u}{1 + u} du \\ &= 4\pi \int_0^2 \frac{u^3 + 2u}{1 + u} du = 4\pi \int_0^2 \left(u^2 - u + 3 - \frac{3}{1+u} \right) du \\ &= 4\pi \left[\frac{1}{3}u^3 - \frac{1}{2}u^2 + 3u - 3 \ln(1+u) \right]_0^2 = \frac{4\pi}{3}(20 - 9 \ln 3) \approx 42.359 \end{aligned}$$



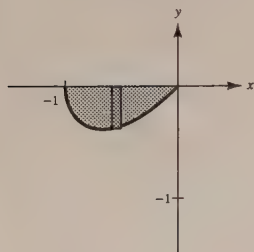
29. Since $y \leq 0$, $A = -\int_{-1}^0 x\sqrt{x+1} \, dx$.

$$u = x + 1$$

$$x = u - 1$$

$$dx = du$$

$$\begin{aligned} A &= -\int_0^1 (u-1)\sqrt{u} \, du = -\int_0^1 (u^{3/2} - u^{1/2}) \, du \\ &= -\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_0^1 = \frac{4}{15} \end{aligned}$$



33. $f(x) = \frac{4}{5}x^{5/4}$

$$f'(x) = x^{1/4}$$

$$1 + [f'(x)]^2 = 1 + \sqrt{x}$$

$$u = 1 + \sqrt{x}$$

$$x = (u-1)^2$$

$$dx = 2(u-1) \, du$$

$$\begin{aligned} s &= \int_0^4 \sqrt{1 + \sqrt{x}} \, dx = 2 \int_1^3 \sqrt{u}(u-1) \, du \\ &= 2 \int_1^3 (u^{3/2} - u^{1/2}) \, du \\ &= 2 \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_1^3 = \frac{4}{15} \left[u^{3/2}(3u-5) \right]_1^3 \\ &= \frac{8}{15}(1 + 6\sqrt{3}) \approx 6.076 \end{aligned}$$

31. From Exercise 23(a) we have: $V = 64\pi \text{ ft}^3$

$$\frac{1}{4}V = 16\pi$$

Disk: $\pi \int_{-3}^{y_0} \frac{16}{9}(9 - y^2) \, dy = 16\pi$

$$\frac{1}{9} \int_{-3}^{y_0} (9 - y^2) \, dy = 1$$

$$\left[9y - \frac{1}{3}y^3 \right]_{-3}^{y_0} = 9$$

$$\left(9y_0 - \frac{1}{3}y_0^3 \right) - (-27 + 9) = 9$$

$$y_0^3 - 27y_0 - 27 = 0$$

By Newton's Method, $y_0 \approx -1.042$ and the depth of the gasoline is $3 - 1.042 = 1.958 \text{ ft}$.

35. $y = 300 \cosh\left(\frac{x}{2000}\right) - 280, -2000 \leq x \leq 2000$

$$y' = \frac{3}{20} \sinh\left(\frac{x}{2000}\right)$$

$$\begin{aligned} s &= \int_{-2000}^{2000} \sqrt{1 + \left[\frac{3}{20} \sinh\left(\frac{x}{2000}\right) \right]^2} \, dx \\ &= \frac{1}{20} \int_{-2000}^{2000} \sqrt{400 + 9 \sinh^2\left(\frac{x}{2000}\right)} \, dx \end{aligned}$$

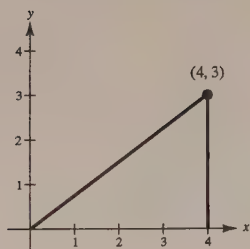
$$\approx 4018.2 \text{ ft (by Simpson's Rule or graphing utility)}$$

$$37. y = \frac{3}{4}x$$

$$y' = \frac{3}{4}$$

$$1 + (y')^2 = \frac{25}{16}$$

$$S = 2\pi \int_0^4 \left(\frac{3}{4}x\right) \sqrt{\frac{25}{16}} dx = \left[\left(\frac{15\pi}{8}\right)\frac{x^2}{2}\right]_0^4 = 15\pi$$



$$39. F = kx$$

$$4 = k(1)$$

$$F = 4x$$

$$W = \int_0^5 4x dx = \left[2x^2\right]_0^5 = 50 \text{ in} \cdot \text{lb} \approx 4.167 \text{ ft} \cdot \text{lb}$$

$$41. \text{ Volume of disk: } \pi\left(\frac{1}{3}\right)^2 \Delta y$$

$$\text{Weight of disk: } 62.4\pi\left(\frac{1}{3}\right)^2 \Delta y$$

$$\text{Distance: } 175 - y$$

$$W = \frac{62.4\pi}{9} \int_0^{150} (175 - y) dy = \frac{62.4\pi}{9} \left[175y - \frac{y^2}{2}\right]_0^{150}$$

$$= 104,000\pi \text{ ft} \cdot \text{lb} \approx 163.4 \text{ ft} \cdot \text{ton}$$

$$43. \text{ Weight of section of chain: } 5 \Delta x$$

$$\text{Distance moved: } 10 - x$$

$$W = 5 \int_0^{10} (10 - x) dx = \left[-\frac{5}{2}(10 - x)^2\right]_0^{10} = 250 \text{ ft} \cdot \text{lb}$$

$$45. W = \int_a^b F(x) dx$$

$$80 = \int_0^4 ax^2 dx = \left[\frac{ax^3}{3}\right]_0^4 = \frac{64}{3}a$$

$$a = \frac{3(80)}{64} = \frac{15}{4} = 3.75$$

$$47. A = \int_0^a (\sqrt{a} - \sqrt{x})^2 dx = \int_0^a (a - 2\sqrt{a}x^{1/2} + x) dx = \left[ax - \frac{4}{3}\sqrt{a}x^{3/2} + \frac{1}{2}x^2\right]_0^a = \frac{a^2}{6}$$

$$\frac{1}{A} = \frac{6}{a^2}$$

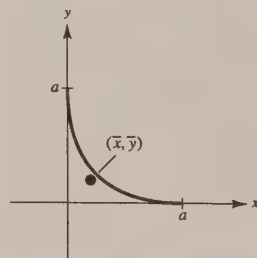
$$\bar{x} = \frac{6}{a^2} \int_0^a x(\sqrt{a} - \sqrt{x})^2 dx = \frac{6}{a^2} \int_0^a (ax - 2\sqrt{a}x^{3/2} + x^2) dx = \frac{a}{5}$$

$$\bar{y} = \left(\frac{6}{a^2}\right) \frac{1}{2} \int_0^a (\sqrt{a} - \sqrt{x})^4 dx$$

$$= \frac{3}{a^2} \int_0^a (a^2 - 4a^{3/2}x^{1/2} + 6ax - 4a^{1/2}x^{3/2} + x^2) dx$$

$$= \frac{3}{a^2} \left[a^2x - \frac{8}{3}a^{3/2}x^{3/2} + 3ax^2 - \frac{8}{5}a^{1/2}x^{5/2} + \frac{1}{3}x^3\right]_0^a = \frac{a}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{a}{5}, \frac{a}{5}\right)$$



49. By symmetry, $\bar{x} = 0$.

$$A = 2 \int_0^1 (a^2 - x^2) dx = 2 \left[a^2x - \frac{x^3}{3} \right]_0^1 = \frac{4a^3}{3}$$

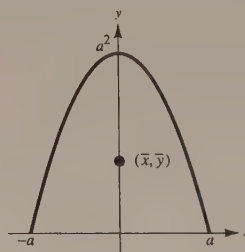
$$\frac{1}{A} = \frac{3}{4a^3}$$

$$\begin{aligned} \bar{y} &= \left(\frac{3}{4a^3} \right) \frac{1}{2} \int_{-a}^a (a^2 - x^2)^2 dx \\ &= \frac{6}{8a^3} \int_0^a (a^4 - 2a^2x^2 + x^4) dx \end{aligned}$$

$$= \frac{6}{8a^3} \left[a^4x - \frac{2a^2}{3}x^3 + \frac{1}{5}x^5 \right]_0^a$$

$$= \frac{6}{8a^3} \left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5 \right) = \frac{2a^2}{5}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{2a^2}{5} \right)$$

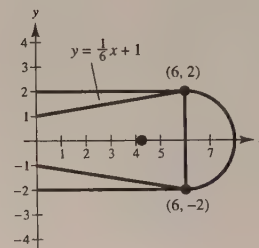


51. $\bar{y} = 0$ by symmetry

For the trapezoid:

$$m = [(4)(6) - (1)(6)]\rho = 18\rho$$

$$\begin{aligned} M_y &= \rho \int_0^6 x \left[\left(\frac{1}{6}x + 1 \right) - \left(-\frac{1}{6}x - 1 \right) \right] dx \\ &= \rho \int_0^6 \left(\frac{1}{3}x^2 + 2x \right) dx = \rho \left[\frac{x^3}{9} + x^2 \right]_0^6 = 60\rho \end{aligned}$$



For the semicircle:

$$m = \left(\frac{1}{2} \right) (\pi)(2)^2 \rho = 2\pi\rho$$

$$M_y = \rho \int_6^8 x \left[\sqrt{4 - (x-6)^2} - (-\sqrt{4 - (x-6)^2}) \right] dx = 2\rho \int_6^8 x \sqrt{4 - (x-6)^2} dx$$

Let $u = x - 6$, then $x = u + 6$ and $dx = du$. When $x = 6$, $u = 0$. When $x = 8$, $u = 2$.

$$\begin{aligned} M_y &= 2\rho \int_0^2 (u + 6) \sqrt{4 - u^2} du = 2\rho \int_0^2 u \sqrt{4 - u^2} du + 12\rho \int_0^2 \sqrt{4 - u^2} du \\ &= 2\rho \left[\left(-\frac{1}{2} \right) \left(\frac{2}{3} \right) (4 - u^2)^{3/2} \right]_0^2 + 12\rho \left[\frac{\pi(2)^2}{4} \right] = \frac{16\rho}{3} + 12\pi\rho = \frac{4\rho(4 + 9\pi)}{3} \end{aligned}$$

Thus, we have:

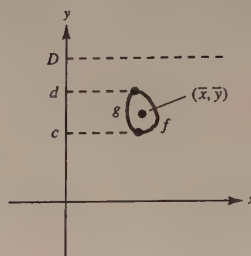
$$\bar{x}(18\rho + 2\pi\rho) = 60\rho + \frac{4\rho(4 + 9\pi)}{3}$$

$$\bar{x} = \frac{180\rho + 4\rho(4 + 9\pi)}{3} \cdot \frac{1}{2\rho(9 + \pi)} = \frac{2(9\pi + 49)}{3(\pi + 9)}$$

The centroid of the blade is $\left(\frac{2(9\pi + 49)}{3(\pi + 9)}, 0 \right)$.

53. Let D = surface of liquid; ρ = weight per cubic volume.

$$\begin{aligned}
 F &= \rho \int_c^d (D - y)[f(y) - g(y)] dy \\
 &= \rho \left[\int_c^d D[f(y) - g(y)] dy - \int_c^d y[f(y) - g(y)] dy \right] \\
 &= \rho \left[\int_c^d [f(y) - g(y)] dy \right] \left[D - \frac{\int_c^d y[f(y) - g(y)] dy}{\int_c^d [f(y) - g(y)] dy} \right] \\
 &= \rho(\text{Area})(D - \bar{y}) \\
 &= \rho(\text{Area})(\text{depth of centroid})
 \end{aligned}$$



Problem Solving for Chapter 6

1. $T = \frac{1}{2}c(c^2) = \frac{1}{2}c^3$

$$R = \int_0^c (cx - x^2) dx = \left[\frac{cx^2}{2} - \frac{x^3}{3} \right]_0^c = \frac{c^3}{2} - \frac{c^3}{3} = \frac{c^3}{6}$$

$$\lim_{c \rightarrow 0^+} \frac{T}{R} = \lim_{c \rightarrow 0^+} \frac{\frac{1}{2}c^3}{\frac{1}{6}c^3} = 3$$

3. (a) $\frac{1}{2}V = \int_0^1 [\pi(2 + \sqrt{1 - y^2})^2 - \pi(2 - \sqrt{1 - y^2})^2] dy$

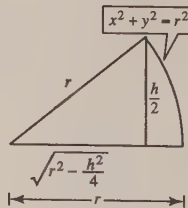
$$\begin{aligned}
 &= \pi \int_0^1 [(4 + 4\sqrt{1 - y^2} + (1 - y^2)) - (4 - 4\sqrt{1 - y^2} + (1 - y^2))] dy \\
 &= 8\pi \int_0^1 \sqrt{1 - y^2} dy \quad (\text{Integral represents } 1/4 \text{ (area of circle)}) \\
 &= 8\pi \left(\frac{\pi}{4} \right) = 2\pi^2 \Rightarrow V = 4\pi^2.
 \end{aligned}$$

(b) $(x - R)^2 + y^2 = r^2 \Rightarrow x = R \pm \sqrt{r^2 - y^2}$

$$\begin{aligned}
 \frac{1}{2}V &= \int_0^r [\pi(R + \sqrt{r^2 - y^2})^2 - \pi(R - \sqrt{r^2 - y^2})^2] dy \\
 &= \pi \int_0^r 4R\sqrt{r^2 - y^2} dy \\
 &= \pi(4R) \frac{1}{4} \pi r^2 = \pi^2 r^2 R \\
 V &= 2\pi^2 r^2 R
 \end{aligned}$$

5. $V = 2(2\pi) \int_{\sqrt{r^2 - (h^2/4)}}^r x\sqrt{r^2 - x^2} dx$

$$\begin{aligned}
 &= -2\pi \left[\frac{2}{3}(r^2 - x^2)^{3/2} \right]_{\sqrt{r^2 - (h^2/4)}}^r \\
 &= \frac{-4\pi}{3} \left[-\frac{h^3}{8} \right] = \frac{\pi h^3}{6} \text{ which does not depend on } r!
 \end{aligned}$$



7. (a) Tangent at A: $y = x^3, y' = 3x^2$

$$y - 1 = 3(x - 1)$$

$$y = 3x - 2$$

 To find point B: $x^3 = 3x - 2$

$$x^3 - 3x + 2 = 0$$

$$(x - 1)^2(x + 2) = 0 \Rightarrow B = (-2, -8)$$

 Tangent at B: $y = x^3, y' = 3x^2$

$$y + 8 = 12(x + 2)$$

$$y = 12x + 16$$

 To find point C: $x^3 = 12x + 16$

$$x^3 - 12x - 16 = 0$$

$$(x + 2)^2(x - 4) = 0 \Rightarrow C = (4, 64)$$

$$\text{Area of } R = \int_{-2}^1 (x^3 - 3x + 2) dx = \frac{27}{4}$$

$$\text{Area of } S = \int_{-2}^4 (12x + 16 - x^3) dx = 108$$

$$\text{Area of } S = 16(\text{area of } R) \quad \left[\frac{\text{area } S}{\text{area } R} = 16 \right]$$

 (b) Tangent at A(a, a^3): $y - a^3 = 3a^2(x - a)$

$$y = 3a^2x - 2a^3$$

 To find point B: $x^3 - 3a^2x + 2a^3 = 0$

$$(x - a)^2(x + 2a) = 0 \Rightarrow$$

$$B = (-2a, -8a^3)$$

 Tangent at B: $y + 8a^3 = 12a^2(x + 2a)$

$$y = 12a^2x + 16a^3$$

 To find point C: $x^3 - 12a^2x - 16a^3 = 0$

$$(x + 2a)^2(x - 4a) = 0 \Rightarrow$$

$$C = (4a, 64a^3)$$

$$\text{Area of } R = \int_{-2a}^a [x^3 - 3a^2x + 2a^3] dx = \frac{27}{4}a^4$$

$$\text{Area of } S = \int_{-2a}^{4a} [12a^2x + 16a^3 - x^3] dx = 108a^4$$

$$\text{Area of } S = 16(\text{area of } R)$$

$$9. s(x) = \int_{\alpha}^x \sqrt{1 + f'(t)^2} dt$$

$$(a) s'(x) = \frac{ds}{dx} = \sqrt{1 + f'(x)^2}$$

$$(b) ds = \sqrt{1 + f'(x)^2} dx$$

$$(ds)^2 = [1 + f'(x)^2](dx)^2 = \left[1 + \left(\frac{dy}{dx} \right)^2 \right] (dx)^2 = (dx)^2 + (dy)^2$$

$$(c) s(x) = \int_1^x \sqrt{1 + \left(\frac{3}{2}t^{1/2} \right)^2} dt = \int_1^x \sqrt{1 + \frac{9}{4}t} dt$$

$$(d) s(2) = \int_1^2 \sqrt{1 + \frac{9}{4}t} dt = \left[\frac{8}{27} \left(1 + \frac{9}{4}t \right)^{3/2} \right]_1^2 = \frac{22}{27}\sqrt{22} - \frac{13}{27}\sqrt{13} \approx 2.0858$$

 This is the length of the curve $y = x^{3/2}$ from $x = 1$ to $x = 2$.

 11. (a) $\bar{y} = 0$ by symmetry

$$M_y = \int_1^6 x \left(\frac{1}{x^3} - \left(-\frac{1}{x^3} \right) \right) dx = \int_1^6 \frac{2}{x^2} dx = \left[-2\frac{1}{x} \right]_1^6 = \frac{5}{3}$$

$$m = 2 \int_1^6 \frac{1}{x^3} dx = \left[-\frac{1}{x^2} \right]_1^6 = \frac{35}{36}$$

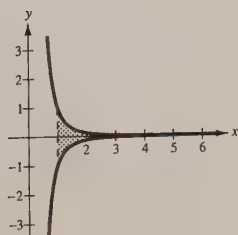
$$\bar{x} = \frac{5/3}{35/36} = \frac{12}{7} \quad (\bar{x}, \bar{y}) = \left(\frac{12}{7}, 0 \right)$$

$$(b) m = 2 \int_1^b \frac{1}{x^3} dx = \frac{b^2 - 1}{b^2}$$

$$M_y = 2 \int_1^b \frac{1}{x^2} dx = \frac{2(b - 1)}{b}$$

$$\bar{x} = \frac{2(b - 1)/b}{(b^2 - 1)/b^2} = \frac{2b}{b + 1} \quad (\bar{x}, \bar{y}) = \left(\frac{2b}{b + 1}, 0 \right)$$

$$(c) \lim_{b \rightarrow \infty} \bar{x} = \lim_{b \rightarrow \infty} \frac{2b}{b + 1} = 2 \quad (\bar{x}, \bar{y}) = (2, 0)$$



13. (a) $W = \text{area} = 2 + 4 + 6 = 12$

(b) $W = \text{area} = 3 + (1 + 1) + 2 + \frac{1}{2} = 7\frac{1}{2}$

17. We use Exercise 23 in Section 6.7:

(a) Wall at shallow end

From Exercise 23: $F = 62.4(2)(4)(20) = 9984 \text{ lb}$

(b) Wall at deep end

From Exercise 23: $F = 62.4(4)(8)(20) = 39,936 \text{ lb}$

(c) Side wall

From Exercise 23: $F_1 = 62.4(2)(4)(40) = 19,968 \text{ lb}$

$$\begin{aligned} F_2 &= 62.4 \int_0^4 (8 - y)(10y) dy \\ &= 624 \int_0^4 (8y - y^2) dy = 624 \left[4y^2 - \frac{y^3}{3} \right]_0^4 \\ &= 26,624 \text{ lb} \end{aligned}$$

Total force: $F_1 + F_2 = 46,592 \text{ lb}$

15. Point of equilibrium: $50 - 0.5x = 0.125x$

$$x = 80, p = 10$$

$$(P_0, x_0) = (10, 80)$$

$$\text{Consumer surplus} = \int_0^{80} [(50 - 0.5x) - 10] dx = 1600$$

$$\text{Producer surplus} = \int_0^{80} [10 - 0.125x] dx = 400$$

C H A P T E R 7

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

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CHAPTER 7

Integration Techniques, L'Hôpital's Rule, and Improper Integrals

Section 7.1 Basic Integration Rules

Solutions to Odd-Numbered Exercises

$$1. (a) \frac{d}{dx}[2\sqrt{x^2+1} + C] = 2\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x) = \frac{2x}{\sqrt{x^2+1}}$$

$$(b) \frac{d}{dx}[\sqrt{x^2+1} + C] = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$$

$$(c) \frac{d}{dx}\left[\frac{1}{2}\sqrt{x^2+1} + C\right] = \frac{1}{2}\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x) = \frac{x}{2\sqrt{x^2+1}}$$

$$(d) \frac{d}{dx}[\ln(x^2+1) + C] = \frac{2x}{x^2+1}$$

$$\int \frac{x}{\sqrt{x^2+1}} dx \text{ matches (b).}$$

$$3. (a) \frac{d}{dx}[\ln\sqrt{x^2+1} + C] = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) = \frac{x}{x^2+1}$$

$$(b) \frac{d}{dx}\left[\frac{2x}{(x^2+1)^2} + C\right] = \frac{(x^2+1)^2(2) - (2x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(1-3x^2)}{(x^2+1)^3}$$

$$(c) \frac{d}{dx}[\arctan x + C] = \frac{1}{1+x^2}$$

$$(d) \frac{d}{dx}[\ln(x^2+1) + C] = \frac{2x}{x^2+1}$$

$$\int \frac{1}{x^2+1} dx \text{ matches (c).}$$

$$5. \int (3x-2)^4 dx$$

$$u = 3x-2, du = 3 dx, n = 4$$

$$\text{Use } \int u^n du.$$

$$7. \int \frac{1}{\sqrt{x}(1-2\sqrt{x})} dx$$

$$u = 1-2\sqrt{x}, du = -\frac{1}{\sqrt{x}} dx$$

$$\text{Use } \int \frac{du}{u}.$$

$$9. \int \frac{3}{\sqrt{1-t^2}} dt$$

$$u = t, du = dt, a = 1$$

$$\text{Use } \int \frac{du}{\sqrt{a^2-u^2}}$$

$$11. \int t \sin t^2 dt$$

$$u = t^2, du = 2t dt$$

$$\text{Use } \int \sin u du.$$

$$13. \int (\cos x)e^{\sin x} dx$$

$$u = \sin x, du = \cos x dx$$

$$\text{Use } \int e^u du.$$

15. Let $u = -2x + 5$, $du = -2 dx$.

$$\begin{aligned}\int (-2x + 5)^{3/2} dx &= -\frac{1}{2} \int (-2x + 5)^{3/2} (-2) dx \\ &= -\frac{1}{5} (-2x + 5)^{5/2} + C\end{aligned}$$

17. Let $u = z - 4$, $du = dz$

$$\begin{aligned}\int \frac{5}{(z-4)^5} dz &= 5 \int (z-4)^{-5} dz = 5 \frac{(z-4)^{-4}}{-4} + C \\ &= \frac{-5}{4(z-4)^4} + C\end{aligned}$$

19. Let $u = t^3 - 1$, $du = 3t^2 dt$.

$$\begin{aligned}\int t^2 \sqrt[3]{t^3 - 1} dt &= \frac{1}{3} \int (t^3 - 1)^{1/3} (3t^2) dt \\ &= \frac{1}{3} \frac{(t^3 - 1)^{4/3}}{4/3} + C \\ &= \frac{(t^3 - 1)^{4/3}}{4} + C\end{aligned}$$

21. $\int \left[v + \frac{1}{(3v-1)^3} \right] dv = \int v dv + \frac{1}{3} \int (3v-1)^{-3} (3) dv$
 $= \frac{1}{2} v^2 - \frac{1}{6(3v-1)^2} + C$

23. Let $u = -t^3 + 9t + 1$, $du = (-3t^2 + 9) dt = -3(t^2 - 3) dt$.

$$\int \frac{t^2 - 3}{-t^3 + 9t + 1} dt = -\frac{1}{3} \int \frac{-3(t^2 - 3)}{-t^3 + 9t + 1} dt = -\frac{1}{3} \ln |-t^3 + 9t + 1| + C$$

25. $\int \frac{x^2}{x-1} dx = \int (x+1) dx + \int \frac{1}{x-1} dx$
 $= \frac{1}{2} x^2 + x + \ln|x-1| + C$

27. Let $u = 1 + e^x$, $du = e^x dx$.

$$\int \frac{e^x}{1 + e^x} dx = \ln(1 + e^x) + C$$

29. $\int (1 + 2x^2)^2 dx = \int (4x^4 + 4x^2 + 1) dx = \frac{4}{5} x^5 + \frac{4}{3} x^3 + x + C = \frac{x}{15} (12x^4 + 20x^2 + 15) + C$

31. Let $u = 2\pi x^2$, $du = 4\pi x dx$.

$$\begin{aligned}\int x(\cos 2\pi x^2) dx &= \frac{1}{4\pi} \int (\cos 2\pi x^2) (4\pi x) dx \\ &= \frac{1}{4\pi} \sin 2\pi x^2 + C\end{aligned}$$

33. Let $u = \pi x$, $du = \pi dx$.

$$\int \csc(\pi x) \cot(\pi x) dx = \frac{1}{\pi} \int \csc(\pi x) \cot(\pi x) \pi dx = -\frac{1}{\pi} \csc(\pi x) + C$$

35. Let $u = 5x$, $du = 5 dx$.

$$\int e^{5x} dx = \frac{1}{5} \int e^{5x} (5) dx = \frac{1}{5} e^{5x} + C$$

37. Let $u = 1 + e^x$, $du = e^x dx$.

$$\begin{aligned}\int \frac{2}{e^{-x} + 1} dx &= 2 \int \left(\frac{1}{e^{-x} + 1} \right) \left(\frac{e^x}{e^x} \right) dx \\ &= 2 \int \frac{e^x}{1 + e^x} dx = 2 \ln(1 + e^x) + C\end{aligned}$$

$$39. \int \frac{\ln x^2}{x} dx = 2 \int (\ln x) \frac{1}{x} dx = 2 \frac{(\ln x)^2}{2} + C = (\ln x)^2 + C$$

$$41. \int \frac{1 + \sin x}{\cos x} dx = \int (\sec x + \tan x) dx = \ln|\sec x + \tan x| + \ln|\sec x| + C = \ln|\sec x(\sec x + \tan x)| + C$$

$$43. \frac{1}{\cos \theta - 1} = \frac{1}{\cos \theta - 1} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} = \frac{\cos \theta + 1}{\cos^2 \theta - 1} = \frac{\cos \theta + 1}{-\sin^2 \theta}$$

$$= -\csc \theta \cdot \cot \theta - \csc^2 \theta$$

$$\int \frac{1}{\cos \theta - 1} d\theta = \int (-\csc \theta \cot \theta - \csc^2 \theta) d\theta$$

$$= \csc \theta + \cot \theta + C$$

$$= \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} + C$$

$$= \frac{1 + \cos \theta}{\sin \theta} + C$$

$$45. \int \frac{3z + 2}{z^2 + 9} dz = \frac{3}{2} \int \frac{2z}{z^2 + 9} dz + 2 \int \frac{dz}{z^2 + 9}$$

$$= \frac{3}{2} \ln(z^2 + 9) + \frac{2}{3} \arctan\left(\frac{z}{3}\right) + C$$

$$47. \text{ Let } u = 2t - 1, du = 2 dt.$$

$$\int \frac{-1}{\sqrt{1 - (2t - 1)^2}} dt = -\frac{1}{2} \int \frac{2}{\sqrt{1 - (2t - 1)^2}} dt$$

$$= -\frac{1}{2} \arcsin(2t - 1) + C$$

$$49. \text{ Let } u = \cos\left(\frac{2}{t}\right), du = \frac{2 \sin(2/t)}{t^2} dt.$$

$$\int \frac{\tan(2/t)}{t^2} dt = \frac{1}{2} \int \frac{1}{\cos(2/t)} \left[\frac{2 \sin(2/t)}{t^2} \right] dt$$

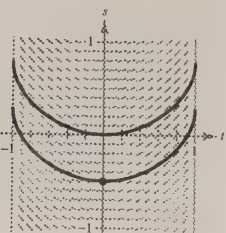
$$= \frac{1}{2} \ln \left| \cos\left(\frac{2}{t}\right) \right| + C$$

$$51. \int \frac{3}{\sqrt{6x - x^2}} dx = 3 \int \frac{1}{\sqrt{9 - (x - 3)^2}} dx = 3 \arcsin\left(\frac{x - 3}{3}\right) + C$$

$$53. \int \frac{4}{4x^2 + 4x + 65} dx = \int \frac{1}{[x + (1/2)]^2 + 16} dx = \frac{1}{4} \arctan\left[\frac{x + (1/2)}{4}\right] + C = \frac{1}{4} \arctan\left(\frac{2x + 1}{8}\right) + C$$

$$55. \frac{ds}{dt} = \frac{t}{\sqrt{1 - t^4}}, \left(0, -\frac{1}{2}\right)$$

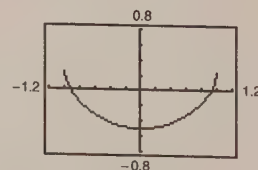
(a)

(b) $u = t^2, du = 2t dt$

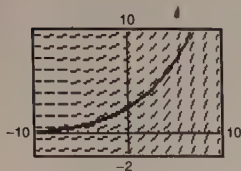
$$\int \frac{t}{\sqrt{1 - t^4}} dt = \frac{1}{2} \int \frac{2t}{\sqrt{1 - (t^2)^2}} dt = \frac{1}{2} \arcsin t^2 + C$$

$$\left(0, -\frac{1}{2}\right): -\frac{1}{2} = \frac{1}{2} \arcsin 0 + C \Rightarrow C = -\frac{1}{2}$$

$$s = \frac{1}{2} \arcsin t^2 - \frac{1}{2}$$



57.



$$y = 3e^{0.2x}$$

$$61. \frac{dy}{dx} = \frac{\sec^2 x}{4 + \tan^2 x}$$

$$\text{Let } u = \tan x, du = \sec^2 x dx.$$

$$y = \int \frac{\sec^2 x}{4 + \tan^2 x} dx = \frac{1}{2} \arctan\left(\frac{\tan x}{2}\right) + C$$

$$65. \text{ Let } u = -x^2, du = -2x dx.$$

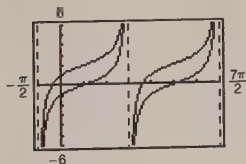
$$\begin{aligned} \int_0^1 x e^{-x^2} dx &= -\frac{1}{2} \int_0^1 e^{-x^2} (-2x) dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^1 \\ &= \frac{1}{2} (1 - e^{-1}) \approx 0.316 \end{aligned}$$

$$69. \text{ Let } u = 3x, du = 3 dx.$$

$$\begin{aligned} \int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2} dx &= \frac{1}{3} \int_0^{2/\sqrt{3}} \frac{3}{4 + (3x)^2} dx \\ &= \left[\frac{1}{6} \arctan\left(\frac{3x}{2}\right) \right]_0^{2/\sqrt{3}} \\ &= \frac{\pi}{18} \approx 0.175 \end{aligned}$$

$$73. \int \frac{1}{1 + \sin \theta} d\theta = \tan \theta - \sec \theta + C \left(\text{or } \frac{-2}{1 + \tan(\theta/2)} \right)$$

The antiderivatives are vertical translations of each other.



$$77. \text{ Log Rule: } \int \frac{du}{u} = \ln|u| + C, u = x^2 + 1.$$

$$\begin{aligned} 59. y &= \int (1 + e^x)^2 dx = \int (e^{2x} + 2e^x + 1) dx \\ &= \frac{1}{2} e^{2x} + 2e^x + x + C \end{aligned}$$

$$63. \text{ Let } u = 2x, du = 2 dx.$$

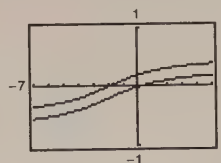
$$\begin{aligned} \int_0^{\pi/4} \cos 2x dx &= \frac{1}{2} \int_0^{\pi/4} \cos 2x (2) dx \\ &= \left[\frac{1}{2} \sin 2x \right]_0^{\pi/4} = \frac{1}{2} \end{aligned}$$

$$67. \text{ Let } u = x^2 + 9, du = 2x dx.$$

$$\begin{aligned} \int_0^4 \frac{2x}{\sqrt{x^2 + 9}} dx &= \int_0^4 (x^2 + 9)^{-1/2} (2x) dx \\ &= \left[2\sqrt{x^2 + 9} \right]_0^4 = 4 \end{aligned}$$

$$71. \int \frac{1}{x^2 + 4x + 13} dx = \frac{1}{3} \arctan\left(\frac{x+2}{3}\right) + C$$

The antiderivatives are vertical translations of each other.



$$75. \text{ Power Rule: } \int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1.$$

$$u = x^2 + 1, n = 3$$

$$79. \text{ The are equivalent because}$$

$$e^{x+C_1} = e^x \cdot e^{C_1} = C e^x, C = e^{C_1}$$

81. $\sin x + \cos x = a \sin(x + b)$

$$\sin x + \cos x = a \sin x \cos b + a \cos x \sin b$$

$$\sin x + \cos x = (a \cos b) \sin x + (a \sin b) \cos x$$

Equate coefficients of like terms to obtain the following.

$$1 = a \cos b \quad \text{and} \quad 1 = a \sin b$$

Thus, $a = 1/\cos b$. Now, substitute for a in $1 = a \sin b$.

$$1 = \left(\frac{1}{\cos b}\right) \sin b$$

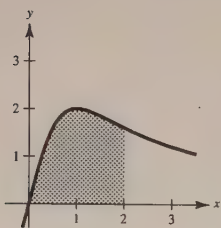
$$1 = \tan b \Rightarrow b = \frac{\pi}{4}$$

$$\text{Since } b = \frac{\pi}{4}, a = \frac{1}{\cos(\pi/4)} = \sqrt{2}. \text{ Thus, } \sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right).$$

$$\int \frac{dx}{\sin x + \cos x} = \int \frac{dx}{\sqrt{2} \sin(x + (\pi/4))} = \frac{1}{\sqrt{2}} \int \csc\left(x + \frac{\pi}{4}\right) dx = -\frac{1}{\sqrt{2}} \ln \left| \csc\left(x + \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right) \right| + C$$

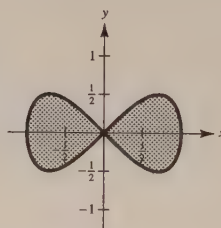
83. $\int_0^2 \frac{4x}{x^2 + 1} dx \approx 3$

Matches (a).



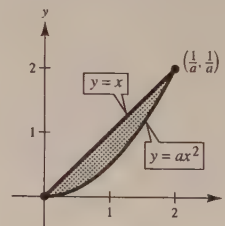
85. Let $u = 1 - x^2$, $du = -2x dx$.

$$\begin{aligned}
 A &= 4 \int_0^1 x \sqrt{1 - x^2} dx \\
 &= -2 \int_0^1 (1 - x^2)^{1/2} (-2x) dx \\
 &= \left[-\frac{4}{3} (1 - x^2)^{3/2} \right]_0^1 = \frac{4}{3}
 \end{aligned}$$



$$\begin{aligned}
 87. \int_0^{1/a} (x - ax^2) dx &= \left[\frac{1}{2}x^2 - \frac{a}{3}x^3 \right]_0^{1/a} \\
 &= \frac{1}{6a^2}
 \end{aligned}$$

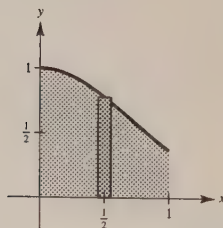
$$\text{Let } \frac{1}{6a^2} = \frac{2}{3}, 12a^2 = 3, a = \frac{1}{2}.$$



89. (a) Shell Method:

$$\text{Let } u = -x^2, du = -2x dx.$$

$$\begin{aligned}
 V &= 2\pi \int_0^1 x e^{-x^2} dx \\
 &= -\pi \int_0^1 e^{-x^2} (-2x) dx \\
 &= \left[-\pi e^{-x^2} \right]_0^1 \\
 &= \pi(1 - e^{-1}) \approx 1.986
 \end{aligned}$$



(b) Shell Method:

$$\begin{aligned}
 V &= 2\pi \int_0^b x e^{-x^2} dx \\
 &= \left[-\pi e^{-x^2} \right]_0^b \\
 &= \pi(1 - e^{-b^2}) = \frac{4}{3} \\
 e^{-b^2} &= \frac{3\pi - 4}{3\pi} \\
 b &= \sqrt{\ln\left(\frac{3\pi}{3\pi - 4}\right)} \\
 &\approx 0.743
 \end{aligned}$$

$$91. A = \int_0^4 \frac{5}{\sqrt{25-x^2}} dx = \left[5 \arcsin \frac{x}{5} \right]_0^4 = 5 \arcsin \frac{4}{5}$$

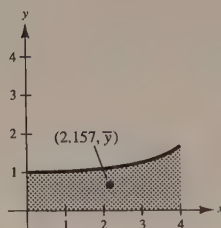
$$\bar{x} = \frac{1}{A} \int_0^4 x \left(\frac{5}{\sqrt{25-x^2}} \right) dx$$

$$= \frac{1}{5 \arcsin(4/5)} \left(-\frac{5}{2} \right) \int_0^4 (25-x^2)^{-1/2} (-2x) dx$$

$$= \frac{1}{5 \arcsin(4/5)} (-5) \left[(25-x^2)^{1/2} \right]_0^4$$

$$= -\frac{1}{\arcsin(4/5)} [3-5]$$

$$= \frac{2}{\arcsin(4/5)} \approx 2.157$$



$$93. y = \tan(\pi x)$$

$$y' = \pi \sec^2(\pi x)$$

$$1 + (y')^2 = 1 + \pi^2 \sec^4(\pi x)$$

$$s = \int_0^{1/4} \sqrt{1 + \pi^2 \sec^4(\pi x)} dx$$

$$\approx 1.0320$$

Section 7.2 Integration by Parts

$$1. \frac{d}{dx} [\sin x - x \cos x] = \cos x - (-x \sin x + \cos x) = x \sin x. \text{ Matches (b)}$$

$$3. \frac{d}{dx} [x^2 e^x - 2x e^x + 2e^x] = x^2 e^x + 2x e^x - 2x e^x - 2e^x + 2e^x = x^2 e^x. \text{ Matches (c)}$$

$$5. \int x e^{2x} dx$$

$$u = x, dv = e^{2x} dx$$

$$7. \int (\ln x)^2 dx$$

$$u = (\ln x)^2, dv = dx$$

$$9. \int x \sec^2 x dx$$

$$u = x, dv = \sec^2 x dx$$

$$11. dv = e^{-2x} dx \Rightarrow v = \int e^{-2x} dx = -\frac{1}{2} e^{-2x}$$

$$u = x \Rightarrow du = dx$$

$$\int x e^{-2x} dx = -\frac{1}{2} x e^{-2x} - \int -\frac{1}{2} e^{-2x} dx$$

$$= -\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + C = \frac{-1}{4e^{2x}} (2x + 1) + C$$

13. Use integration by parts three times.

$$(1) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x \quad (2) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x \quad (3) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = x^3 \Rightarrow du = 3x^2 dx \quad u = x^2 \Rightarrow du = 2x dx \quad u = x \Rightarrow du = dx$$

$$\int x^3 e^x dx = x^3 e^x - 3 \int x^2 e^x dx = x^3 e^x - 3x^2 e^x + 6 \int x e^x dx$$

$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C = e^x(x^3 - 3x^2 + 6x - 6) + C$$

$$15. \int x^2 e^{x^3} dx = \frac{1}{3} \int e^{x^3} (3x^2) dx = \frac{1}{3} e^{x^3} + C$$

$$17. \, dv = t \, dt \Rightarrow v = \int t \, dt = \frac{t^2}{2}$$

$$u = \ln(t+1) \Rightarrow du = \frac{1}{t+1} dt$$

$$\int t \ln(t+1) dt = \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \int \frac{t^2}{t+1} dt$$

$$= \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \int \left(t - 1 + \frac{1}{t+1} \right) dt$$

$$= \frac{t^2}{2} \ln(t+1) - \frac{1}{2} \left[\frac{t^2}{2} - t + \ln(t+1) \right] + C$$

$$= \frac{1}{4} [2(t^2 - 1) \ln|t+1| - t^2 + 2t] + C$$

$$19. \text{ Let } u = \ln x, du = \frac{1}{x} dx.$$

$$\int \frac{(\ln x)^2}{x} dx = \int (\ln x)^2 \left(\frac{1}{x} \right) dx = \frac{(\ln x)^3}{3} + C$$

$$21. \, dv = \frac{1}{(2x+1)^2} dx \Rightarrow v = \int (2x+1)^{-2} dx$$

$$= -\frac{1}{2(2x+1)}$$

$$u = xe^{2x} \Rightarrow du = (2xe^{2x} + e^{2x}) dx$$

$$= e^{2x}(2x+1) dx$$

$$\int \frac{xe^{2x}}{(2x+1)^2} dx = -\frac{xe^{2x}}{2(2x+1)} + \int \frac{e^{2x}}{2} dx$$

$$= \frac{-xe^{2x}}{2(2x+1)} + \frac{e^{2x}}{4} + C$$

$$= \frac{e^{2x}}{4(2x+1)} + C$$

23. Use integration by parts twice.

$$(1) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$\int (x^2 - 1)e^x dx = \int x^2 e^x dx - \int e^x dx = x^2 e^x - 2 \int x e^x dx - e^x$$

$$= x^2 e^x - 2 \left[x e^x - \int e^x dx \right] - e^x = x^2 e^x - 2x e^x + e^x + C = (x-1)^2 e^x + C$$

$$(2) \, dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = x \Rightarrow du = dx$$

$$25. dv = \sqrt{x-1} dx \Rightarrow v = \int (x-1)^{1/2} dx = \frac{2}{3}(x-1)^{3/2}$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x\sqrt{x-1} dx &= \frac{2}{3}x(x-1)^{3/2} - \frac{2}{3} \int (x-1)^{3/2} dx \\ &= \frac{2}{3}x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C \\ &= \frac{2(x-1)^{3/2}}{15}(3x+2) + C \end{aligned}$$

$$27. dv = \cos x dx \Rightarrow v = \int \cos x dx = \sin x$$

$$u = x \Rightarrow du = dx$$

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C$$

29. Use integration by parts three times.

$$(1) u = x^3, du = 3x^2, dv = \sin x dx, v = -\cos x$$

$$\int x^3 \sin x dx = -x^3 \cos x + 3 \int x^2 \cos x dx$$

$$(2) u = x^2, du = 2x dx, dv = \cos x dx, v = \sin x$$

$$\begin{aligned} \int x^3 \sin x dx &= -x^3 \cos x + 3 \left[x^2 \sin x - 2 \int x \sin x dx \right] \\ &= -x^3 \cos x + 3x^2 \sin x - 6 \int x \sin x dx \end{aligned}$$

$$(3) u = x, du = dx, dv = \sin x dx, v = -\cos x$$

$$\begin{aligned} \int x^3 \sin x dx &= -x^3 \cos x + 3x^2 \sin x - 6 \left[-x \cos x + \int \cos x dx \right] \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C \end{aligned}$$

$$31. u = t, du = dt, dv = \csc t \cot t dt, v = -\csc t$$

$$\begin{aligned} \int t \csc t \cot t dt &= -t \csc t + \int \csc t dt \\ &= -t \csc t - \ln |\csc t + \cot t| + C \end{aligned}$$

$$33. dv = dx \Rightarrow v = \int dx = x$$

$$u = \arctan x \Rightarrow du = \frac{1}{1+x^2} dx$$

$$\begin{aligned} \int \arctan x dx &= x \arctan x - \int \frac{x}{1+x^2} dx \\ &= x \arctan x - \frac{1}{2} \ln(1+x^2) + C \end{aligned}$$

35. Use integration by parts twice.

$$(1) dv = e^{2x} dx \Rightarrow v = \frac{1}{2} e^{2x}$$

$$u = \sin x \Rightarrow du = \cos x dx$$

$$(2) dv = e^{2x} dx \Rightarrow v = \frac{1}{2} e^{2x}$$

$$u = \cos x \Rightarrow du = -\sin x dx$$

$$\int e^{2x} \sin x dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \left(\frac{1}{2} e^{2x} \cos x + \frac{1}{2} \int e^{2x} \sin x dx \right)$$

$$\frac{5}{4} \int e^{2x} \sin x dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x$$

$$\int e^{2x} \sin x dx = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + C$$

37. $y' = xe^{x^2}$

$$y = \int xe^{x^2} dx = \frac{1}{2}e^{x^2} + C$$

39. Use integration by parts twice.

$$(1) dv = \frac{1}{\sqrt{2+3t}} dt \Rightarrow v = \int (2+3t)^{-1/2} dt = \frac{2}{3}\sqrt{2+3t}$$

$$u = t^2 \Rightarrow du = 2t dt$$

$$(2) dv = \sqrt{2+3t} dt \Rightarrow v = \int (2+3t)^{1/2} dt = \frac{2}{9}(2+3t)^{3/2}$$

$$u = t \Rightarrow du = dt$$

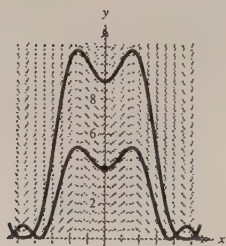
$$\begin{aligned} y &= \int \frac{t^2}{\sqrt{2+3t}} dt = \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3} \int t\sqrt{2+3t} dt \\ &= \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3} \left[\frac{2t}{9}(2+3t)^{3/2} - \frac{2}{9} \int (2+3t)^{3/2} dt \right] \\ &= \frac{2t^2\sqrt{2+3t}}{3} - \frac{8t}{27}(2+3t)^{3/2} + \frac{16}{405}(2+3t)^{5/2} + C \\ &= \frac{2\sqrt{2+3t}}{405}(27t^2 - 24t + 32) + C \end{aligned}$$

41. $(\cos y)y' = 2x$

$$\int \cos y dy = \int 2x dx$$

$$\sin y = x^2 + C$$

43. (a)



(b) $\frac{dy}{dx} = x\sqrt{y} \cos x, (0, 4)$

$$\int \frac{dy}{\sqrt{y}} = \int x \cos x dx$$

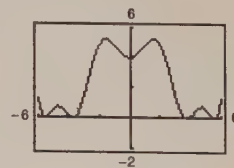
$$\int y^{-1/2} dy = \int x \cos x dx \quad (u = x, du = dx, dv = \cos x dx, v = \sin x)$$

$$2y^{1/2} = x \sin x - \int \sin x dx$$

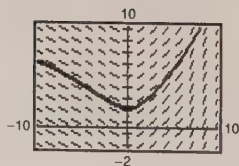
$$= x \sin x + \cos x + C$$

$$(0, 4): 2(4)^{1/2} = 0 + 1 + C \Rightarrow C = 3$$

$$2\sqrt{y} = x \sin x + \cos x + 3$$



45. $\frac{dy}{dx} = \frac{x}{y}e^{x/8}, y(0) = 2$



47. $u = x, du = dx, dv = e^{-x/2} dx, v = -2e^{-x/2}$

$$\int x e^{-x/2} dx = -2x e^{-x/2} + \int 2e^{-x/2} dx = -2x e^{-x/2} - 4e^{-x/2} + C$$

$$\begin{aligned}\text{Thus, } \int_0^4 x e^{-x/2} dx &= \left[-2x e^{-x/2} - 4e^{-x/2} \right]_0^4 \\ &= -8e^{-2} - 4e^{-2} + 4 \\ &= -12e^{-2} + 4 \approx 2.376.\end{aligned}$$

49. See Exercise 27.

$$\int_0^{\pi/2} x \cos x dx = \left[x \sin x + \cos x \right]_0^{\pi/2} = \frac{\pi}{2} - 1$$

51. $u = \arccos x, du = -\frac{1}{\sqrt{1-x^2}} dx, dv = dx, v = x$

$$\int \arccos x dx = x \arccos x + \int \frac{x}{\sqrt{1-x^2}} dx = x \arccos x - \sqrt{1-x^2} + C$$

$$\begin{aligned}\text{Thus, } \int_0^{1/2} \arccos x dx &= \left[x \arccos x - \sqrt{1-x^2} \right]_0^{1/2} \\ &= \frac{1}{2} \arccos\left(\frac{1}{2}\right) - \sqrt{\frac{3}{4}} + 1 \\ &= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1 \approx 0.658.\end{aligned}$$

53. Use integration by parts twice.

$$(1) dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = \sin x \Rightarrow du = \cos x dx$$

$$(2) dv = e^x dx \Rightarrow v = \int e^x dx = e^x$$

$$u = \cos x \Rightarrow du = -\sin x dx$$

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

$$2 \int e^x \sin x dx = e^x (\sin x - \cos x)$$

$$\int e^x \sin x dx = \frac{e^x}{2} (\sin x - \cos x) + C$$

$$\text{Thus, } \int_0^1 e^x \sin x dx = \left[\frac{e^x}{2} (\sin x - \cos x) \right]_0^1 = \frac{e}{2} (\sin 1 - \cos 1) + \frac{1}{2} = \frac{e(\sin 1 - \cos 1) + 1}{2} \approx 0.909.$$

55. $dv = x^2 dx, v = \frac{x^3}{3}, u = \ln x, du = \frac{1}{x} dx$

$$\int x^2 \ln x dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \left(\frac{1}{x} \right) dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx$$

$$\text{Hence, } \int_1^2 x^2 \ln x dx = \left[\frac{x^3}{3} \ln x - \frac{1}{9} x^3 \right]_1^2$$

$$= \frac{8}{3} \ln 2 - \frac{8}{9} + \frac{1}{9} = \frac{8}{3} \ln 2 - \frac{7}{9} \approx 1.071.$$

$$57. dv = x dx, v = \frac{x^2}{2}, u = \operatorname{arcsec} x, du = \frac{1}{x\sqrt{x^2-1}} dx$$

$$\begin{aligned}\int x \operatorname{arcsec} x dx &= \frac{x^2}{2} \operatorname{arcsec} x - \int \frac{x^2/2}{x\sqrt{x^2-1}} dx \\ &= \frac{x^2}{2} \operatorname{arcsec} x - \frac{1}{4} \int \frac{2x}{\sqrt{x^2-1}} dx \\ &= \frac{x^2}{2} \operatorname{arcsec} x - \frac{1}{2} \sqrt{x^2-1} + C\end{aligned}$$

Hence,

$$\begin{aligned}\int_2^4 x \operatorname{arcsec} x dx &= \left[\frac{x^2}{2} \operatorname{arcsec} x - \frac{1}{2} \sqrt{x^2-1} \right]_2^4 \\ &= \left(8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2} \right) - \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \\ &= 8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2} + \frac{\sqrt{3}}{2} - \frac{2\pi}{3} \\ &\approx 7.380.\end{aligned}$$

$$\begin{aligned}59. \int x^2 e^{2x} dx &= x^2 \left(\frac{1}{2} e^{2x} \right) - (2x) \left(\frac{1}{4} e^{2x} \right) + 2 \left(\frac{1}{8} e^{2x} \right) + C \\ &= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + C \\ &= \frac{1}{4} e^{2x} (2x^2 - 2x + 1) + C\end{aligned}$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^2	e^{2x}
-	$2x$	$\frac{1}{2} e^{2x}$
+	2	$\frac{1}{4} e^{2x}$
-	0	$\frac{1}{8} e^{2x}$

$$\begin{aligned}61. \int x^3 \sin x dx &= x^3(-\cos x) - 3x^2(-\sin x) + 6x \cos x - 6 \sin x + C \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C \\ &= (3x^2 - 6) \sin x - (x^3 - 6x) \cos x + C\end{aligned}$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x^3	$\sin x$
-	$3x^2$	$-\cos x$
+	$6x$	$-\sin x$
-	6	$\cos x$
+	0	$\sin x$

$$63. \int x \sec^2 x dx = x \tan x + \ln|\cos x| + C$$

Alternate signs	u and its derivatives	v' and its antiderivatives
+	x	$\sec^2 x$
-	1	$\tan x$
+	0	$-\ln \cos x $

65. Integration by parts is based on the product rule.

67. No. Substitution.

69. Yes. $u = x^2, dv = e^{2x} dx$

$$71. \text{ Yes. Let } u = x \text{ and } du = \frac{1}{\sqrt{x+1}} dx.$$

(Substitution also works. Let $u = \sqrt{x+1}$)

$$73. \int t^3 e^{-4t} dt = -\frac{e^{-4t}}{128} (32t^3 + 24t^2 + 12t + 3) + C$$

$$75. \int_0^{\pi/2} e^{-2x} \sin 3x dx = \left[\frac{e^{-2x}(-2 \sin 3x - 3 \cos 3x)}{13} \right]_0^{\pi/2} = \frac{1}{13} (2e^{-\pi} + 3) \approx 0.2374$$

$$77. (a) \, dv = \sqrt{2x-3} \, dx \Rightarrow v = \int (2x-3)^{1/2} dx = \frac{1}{3}(2x-3)^{3/2}$$

$$u = 2x \Rightarrow du = 2 \, dx$$

$$\begin{aligned} \int 2x\sqrt{2x-3} \, dx &= \frac{2}{3}x(2x-3)^{3/2} - \frac{2}{3} \int (2x-3)^{3/2} dx \\ &= \frac{2}{3}x(2x-3)^{3/2} - \frac{2}{15}(2x-3)^{5/2} + C \\ &= \frac{2}{15}(2x-3)^{3/2}(3x+3) + C = \frac{2}{5}(2x-3)^{3/2}(x+1) + C \end{aligned}$$

$$(b) \, u = 2x - 3 \Rightarrow x = \frac{u+3}{2} \text{ and } dx = \frac{1}{2} du$$

$$\begin{aligned} \int 2x\sqrt{2x-3} \, dx &= \int 2\left(\frac{u+3}{2}\right)u^{1/2}\left(\frac{1}{2}\right) du = \frac{1}{2} \int (u^{3/2} + 3u^{1/2}) du = \frac{1}{2} \left[\frac{2}{5}u^{5/2} + 2u^{3/2} \right] + C \\ &= \frac{1}{5}u^{3/2}(u+5) + C = \frac{1}{5}(2x-3)^{3/2}[(2x-3)+5] + C = \frac{2}{5}(2x-3)^{3/2}(x+1) + C \end{aligned}$$

$$79. (a) \, dv = \frac{x}{\sqrt{4+x^2}} dx \Rightarrow v = \int (4+x^2)^{-1/2} x \, dx = \sqrt{4+x^2}$$

$$u = x^2 \Rightarrow du = 2x \, dx$$

$$\begin{aligned} \int \frac{x^3}{\sqrt{4+x^2}} dx &= x^2\sqrt{4+x^2} - 2 \int x\sqrt{4+x^2} dx \\ &= x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)^{3/2} + C = \frac{1}{3}\sqrt{4+x^2}(x^2-8) + C \end{aligned}$$

$$(b) \, u = 4 + x^2 \Rightarrow x^2 = u - 4 \text{ and } 2x \, dx = du \Rightarrow x \, dx = \frac{1}{2} du$$

$$\begin{aligned} \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int \frac{x^2}{\sqrt{4+x^2}} x \, dx = \int \frac{u-4}{\sqrt{u}} \frac{1}{2} du \\ &= \frac{1}{2} \int (u^{1/2} - 4u^{-1/2}) du = \frac{1}{2} \left(\frac{2}{3}u^{3/2} - 8u^{1/2} \right) + C \\ &= \frac{1}{3}u^{1/2}(u-12) + C = \frac{1}{3}\sqrt{4+x^2}[(4+x^2)-12] + C = \frac{1}{3}\sqrt{4+x^2}(x^2-8) + C \end{aligned}$$

$$81. \, n = 0: \int \ln x \, dx = x(\ln x - 1) + C$$

$$n = 1: \int x \ln x \, dx = \frac{x^2}{4}(2 \ln x - 1) + C$$

$$n = 2: \int x^2 \ln x \, dx = \frac{x^3}{9}(3 \ln x - 1) + C$$

$$n = 3: \int x^3 \ln x \, dx = \frac{x^4}{16}(4 \ln x - 1) + C$$

$$n = 4: \int x^4 \ln x \, dx = \frac{x^5}{25}(5 \ln x - 1) + C$$

$$\text{In general, } \int x^n \ln x \, dx = \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1] + C. \text{ (See Exercise 85.)}$$

83. $dv = \sin x \, dx \Rightarrow v = -\cos x$

$u = x^n \Rightarrow du = nx^{n-1} \, dx$

$$\int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

85. $dv = x^n \, dx \Rightarrow v = \frac{x^{n+1}}{n+1}$

$u = \ln x \Rightarrow du = \frac{1}{x} \, dx$

$$\begin{aligned} \int x^n \ln x \, dx &= \frac{x^{n+1}}{n+1} \ln x - \int \frac{x^n}{n+1} \, dx \\ &= \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + C \\ &= \frac{x^{n+1}}{(n+1)^2} [(n+1) \ln x - 1] + C \end{aligned}$$

87. Use integration by parts twice.

(1) $dv = e^{ax} \, dx \Rightarrow v = \frac{1}{a} e^{ax}$

$u = \sin bx \Rightarrow du = b \cos bx \, dx$

$$\begin{aligned} \int e^{ax} \sin bx \, dx &= \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \int e^{ax} \cos bx \, dx \\ &= \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \left[\frac{e^{ax} \cos bx}{a} + \frac{b}{a} \int e^{ax} \sin bx \, dx \right] = \frac{e^{ax} \sin bx}{a} - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx \, dx \end{aligned}$$

Therefore, $\left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2}$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C.$$

(2) $dv = e^{ax} \, dx \Rightarrow v = \frac{1}{a} e^{ax}$

$u = \cos bx \Rightarrow du = -b \sin bx \, dx$

89. $n = 3$ (Use formula in Exercise 85.)

$$\int x^3 \ln x \, dx = \frac{x^4}{16} [4 \ln x - 1] + C$$

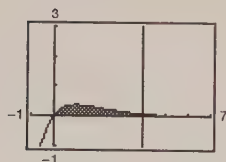
91. $a = 2, b = 3$ (Use formula in Exercise 88.)

$$\int e^{2x} \cos 3x \, dx = \frac{e^{2x}(2 \cos 3x + 3 \sin 3x)}{13} + C$$

93. $dv = e^{-x} \, dx \Rightarrow v = -e^{-x}$

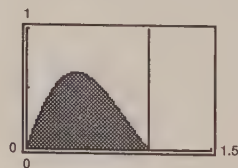
$u = x \Rightarrow du = dx$

$$\begin{aligned} A &= \int_0^4 x e^{-x} \, dx = \left[-x e^{-x} \right]_0^4 + \int_0^4 e^{-x} \, dx = \frac{-4}{e^4} - \left[e^{-x} \right]_0^4 \\ &= 1 - \frac{5}{e^4} \approx 0.908 \end{aligned}$$



95. $A = \int_0^1 e^{-x} \sin(\pi x) \, dx$

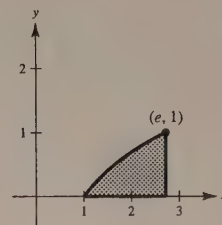
$$\begin{aligned} &= \left[\frac{e^{-x}(-\sin \pi x - \pi \cos \pi x)}{1 + \pi^2} \right]_0^1 \\ &= \frac{1}{1 + \pi^2} \left(\frac{\pi}{e} + \pi \right) = \frac{\pi}{1 + \pi^2} \left(\frac{1}{e} + 1 \right) \\ &\approx 0.395 \text{ (See Exercise 87.)} \end{aligned}$$



97. (a) $A = \int_1^e \ln x \, dx = \left[-x + x \ln x \right]_1^e = 1$ (See Exercise 4.)

(b) $R(x) = \ln x, r(x) = 0$

$$\begin{aligned} V &= \pi \int_1^e (\ln x)^2 \, dx \\ &= \pi \left[x(\ln x)^2 - 2x \ln x + 2x \right]_1^e \quad (\text{Use integration by parts twice, see Exercise 7.}) \\ &= \pi(e - 2) \approx 2.257 \end{aligned}$$



(c) $p(x) = x, h(x) = \ln x$

$$\begin{aligned} V &= 2\pi \int_1^e x \ln x \, dx = 2\pi \left[\frac{x^2}{4}(-1 + 2 \ln x) \right]_1^e \\ &= \frac{(e^2 + 1)\pi}{2} \approx 13.177 \quad (\text{See Exercise 85.}) \end{aligned}$$

(d) $\bar{x} = \frac{\int_1^e x \ln x \, dx}{1} = \frac{e^2 + 1}{4} \approx 2.097$

$$\bar{y} = \frac{\frac{1}{2} \int_1^e (\ln x)^2 \, dx}{1} = \frac{e - 2}{2} \approx 0.359$$

$$(\bar{x}, \bar{y}) = \left(\frac{e^2 + 1}{4}, \frac{e - 2}{2} \right) \approx (2.097, 0.359)$$

99. Average value $= \frac{1}{\pi} \int_0^\pi e^{-4t}(\cos 2t + 5 \sin 2t) \, dt$

$$\begin{aligned} &= \frac{1}{\pi} \left[e^{-4t} \left(\frac{-4 \cos 2t + 2 \sin 2t}{20} \right) + 5e^{-4t} \left(\frac{-4 \sin 2t - 2 \cos 2t}{20} \right) \right]_0^\pi \quad (\text{From Exercises 87 and 88}) \\ &= \frac{7}{10\pi} (1 - e^{-4\pi}) \approx 0.223 \end{aligned}$$

101. $c(t) = 100,000 + 4000t, r = 5\%, t_1 = 10$

$$P = \int_0^{10} (100,000 + 4000t)e^{-0.05t} \, dt = 4000 \int_0^{10} (25 + t)e^{-0.05t} \, dt$$

Let $u = 25 + t, dv = e^{-0.05t} \, dt, du = dt, v = -\frac{100}{5}e^{-0.05t}$

$$\begin{aligned} P &= 4000 \left\{ \left[(25 + t) \left(-\frac{100}{5}e^{-0.05t} \right) \right]_0^{10} + \frac{100}{5} \int_0^{10} e^{-0.05t} \, dt \right\} \\ &= 4000 \left\{ \left[(25 + t) \left(-\frac{100}{5}e^{-0.05t} \right) \right]_0^{10} - \left[\frac{10,000}{25}e^{-0.05t} \right]_0^{10} \right\} \approx \$931,265 \end{aligned}$$

103. $\int_{-\pi}^{\pi} x \sin nx \, dx = \left[-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right]_{-\pi}^{\pi}$

$$\begin{aligned} &= -\frac{\pi}{n} \cos \pi n - \frac{\pi}{n} \cos(-\pi n) \\ &= -\frac{2\pi}{n} \cos \pi n \\ &= \begin{cases} -(2\pi/n), & \text{if } n \text{ is even} \\ (2\pi/n), & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

105. Let $u = x$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

$$\begin{aligned} I_1 &= \int_0^1 x \sin\left(\frac{n\pi}{2}x\right) dx = \left[\frac{-2x}{n\pi} \cos\left(\frac{n\pi}{2}x\right) \right]_0^1 + \frac{2}{n\pi} \int_0^1 \cos\left(\frac{n\pi}{2}x\right) dx \\ &= -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right) \right]_0^1 \\ &= -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

Let $u = (-x + 2)$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = -dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

$$\begin{aligned} I_2 &= \int_1^2 (-x + 2) \sin\left(\frac{n\pi}{2}x\right) dx = \left[\frac{-2(-x + 2)}{n\pi} \cos\left(\frac{n\pi}{2}x\right) \right]_1^2 - \frac{2}{n\pi} \int_1^2 \cos\left(\frac{n\pi}{2}x\right) dx \\ &= \frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) - \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right) \right]_1^2 \\ &= \frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

$$h(I_1 + I_2) = b_n = h \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) \right] = \frac{8h}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

107. Shell Method:

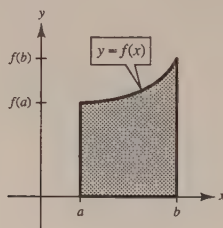
$$V = 2\pi \int_a^b x f(x) dx$$

$$dv = x dx \Rightarrow v = \frac{x^2}{2}$$

$$u = f(x) \Rightarrow du = f'(x) dx$$

$$V = 2\pi \left[\frac{x^2}{2} f(x) - \int \frac{x^2}{2} f'(x) dx \right]_a^b$$

$$= \pi \left[(b^2 f(b) - a^2 f(a)) - \int_a^b x^2 f'(x) dx \right]$$



Disk Method:

$$\begin{aligned} V &= \pi \int_0^{f(a)} (b^2 - a^2) dy + \pi \int_{f(a)}^{f(b)} [b^2 - [f^{-1}(y)]^2] dy \\ &= \pi(b^2 - a^2)f(a) + \pi b^2(f(b) - f(a)) - \pi \int_{f(a)}^{f(b)} [f^{-1}(y)]^2 dy \\ &= \pi \left[(b^2 f(b) - a^2 f(a)) - \int_{f(a)}^{f(b)} [f^{-1}(y)]^2 dy \right] \end{aligned}$$

Since $x = f^{-1}(y)$, we have $f(x) = y$ and $f'(x)dx = dy$. When $y = f(a)$, $x = a$. When $y = f(b)$, $x = b$. Thus,

$$\int_{f(a)}^{f(b)} [f^{-1}(y)]^2 dy = \int_a^b x^2 f'(x) dx$$

and the volumes are the same.

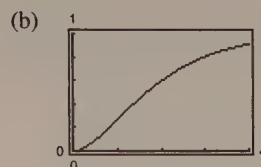
109. $f'(x) = xe^{-x}$

(a) $f(x) = \int xe^{-x} dx = -xe^{-x} - e^{-x} + C$

(Parts: $u = x$, $dv = e^{-x} dx$)

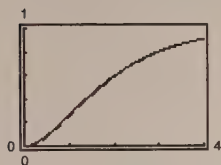
$f(0) = 0 = -1 + C \Rightarrow C = 1$

$f(x) = -xe^{-x} - e^{-x} + 1$



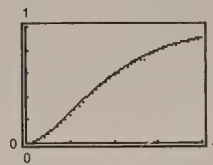
(c) You obtain the points

n	x_n	y_n
0	0	0
1	0.05	0
2	0.10	2.378×10^{-3}
3	0.15	0.0069
4	0.20	0.0134
\vdots	\vdots	\vdots
80	4.0	0.9064



(d) You obtain the points

n	x_n	y_n
0	0	0
1	0.1	0
2	0.2	0.0090484
3	0.3	0.025423
4	0.4	0.047648
\vdots	\vdots	\vdots
40	4.0	0.9039



(e) $f(4) = 0.9084$

The approximations are tangent line approximations. The results in (c) are better because Δx is smaller.

Section 7.3 Trigonometric Integrals

1. $f(x) = \sin^4 x + \cos^4 x$

$$\begin{aligned}
 \text{(a) } \sin^4 x + \cos^4 x &= \left(\frac{1 - \cos 2x}{2} \right)^2 + \left(\frac{1 + \cos 2x}{2} \right)^2 \\
 &= \frac{1}{4} [1 - 2 \cos 2x + \cos^2 2x + 1 + 2 \cos 2x + \cos^2 2x] \\
 &= \frac{1}{4} \left[2 + 2 \frac{1 + \cos 4x}{2} \right] \\
 &= \frac{1}{4} [3 + \cos 4x]
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } \sin^4 x + \cos^4 x &= (\sin^2 x)^2 + \cos^4 x \\
 &= (1 - \cos^2 x)^2 + \cos^4 x \\
 &= 1 - 2 \cos^2 x + 2 \cos^4 x
 \end{aligned}$$

$$\begin{aligned}
 \text{(c) } \sin^4 x + \cos^4 x &= \sin^4 x + 2 \sin^2 x \cos^2 x + \cos^4 x - 2 \sin^2 x \cos^2 x \\
 &= (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x \\
 &= 1 - 2 \sin^2 x \cos^2 x
 \end{aligned}$$

—CONTINUED—

1. —CONTINUED—

$$(d) 1 - 2 \sin^2 x \cos^2 x = 1 - (2 \sin x \cos x)(\sin x \cos x)$$

$$= 1 - (\sin 2x)\left(\frac{1}{2} \sin 2x\right)$$

$$= 1 - \frac{1}{2} \sin^2(2x)$$

(e) Four ways. There is often more than one way to rewrite a trigonometric expression.

$$3. \text{ Let } u = \cos x, du = -\sin x dx.$$

$$\begin{aligned} \int \cos^3 x \sin x dx &= -\int \cos^3 x (-\sin x) dx \\ &= -\frac{1}{4} \cos^4 x + C \end{aligned}$$

$$5. \text{ Let } u = \sin 2x, du = 2 \cos 2x dx.$$

$$\begin{aligned} \int \sin^5 2x \cos 2x dx &= \frac{1}{2} \int \sin^5 2x (2 \cos 2x) dx \\ &= \frac{1}{12} \sin^6 2x + C \end{aligned}$$

$$7. \text{ Let } u = \cos x, du = -\sin x dx.$$

$$\begin{aligned} \int \sin^5 x \cos^2 x dx &= \int \sin x (1 - \cos^2 x)^2 \cos^2 x dx \\ &= -\int (\cos^2 x - 2 \cos^4 x + \cos^6 x)(-\sin x) dx = -\frac{1}{3} \cos^3 x + \frac{2}{5} \cos^5 x - \frac{1}{7} \cos^7 x + C \end{aligned}$$

$$\begin{aligned} 9. \int \cos^3 \theta \sqrt{\sin \theta} d\theta &= \int \cos \theta (1 - \sin^2 \theta)(\sin \theta)^{1/2} d\theta \\ &= \int [(\sin \theta)^{1/2} - (\sin \theta)^{5/2}] \cos \theta d\theta \\ &= \frac{2}{3} (\sin \theta)^{3/2} - \frac{2}{7} (\sin \theta)^{7/2} + C \end{aligned}$$

$$\begin{aligned} 11. \int \cos^2 3x dx &= \int \frac{1 + \cos 6x}{2} dx \\ &= \frac{1}{2} \left(x + \frac{1}{6} \sin 6x \right) + C \\ &= \frac{1}{12} (6x + \sin 6x) + C \end{aligned}$$

$$\begin{aligned} 13. \int \sin^2 \alpha \cdot \cos^2 \alpha d\alpha &= \int \frac{1 - \cos 2\alpha}{2} \cdot \frac{1 + \cos 2\alpha}{2} d\alpha \\ &= \frac{1}{4} \int (1 - \cos^2 2\alpha) d\alpha \\ &= \frac{1}{4} \int \left(1 - \frac{1 + \cos 4\alpha}{2} \right) d\alpha \\ &= \frac{1}{8} \int (1 - \cos 4\alpha) d\alpha \\ &= \frac{1}{8} \left[\alpha - \frac{1}{4} \sin 4\alpha \right] + C \\ &= \frac{1}{32} [4\alpha - \sin 4\alpha] + C \end{aligned}$$

15. Integration by parts.

$$dv = \sin^2 x \, dx = \frac{1 - \cos 2x}{2} \Rightarrow v = \frac{x}{2} - \frac{\sin 2x}{4} = \frac{1}{4}(2x - \sin 2x)$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned} \int x \sin^2 x \, dx &= \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4} \int (2x - \sin 2x) \, dx \\ &= \frac{1}{4}x(2x - \sin 2x) - \frac{1}{4} \left(x^2 + \frac{1}{2} \cos 2x \right) + C = \frac{1}{8}(2x^2 - 2x \sin 2x - \cos 2x) + C \end{aligned}$$

17. Let $u = \sin x$, $du = \cos x \, dx$.

$$\begin{aligned} \int_0^{\pi/2} \cos^3 x \, dx &= \int_0^{\pi/2} (1 - \sin^2 x) \cos x \, dx \\ &= \left[\sin x - \frac{1}{3} \sin^3 x \right]_0^{\pi/2} = \frac{2}{3} \end{aligned}$$

19. Let $u = \sin x$, $du = \cos x \, dx$.

$$\begin{aligned} \int_0^{\pi/2} \cos^7 x \, dx &= \int_0^{\pi/2} (1 - \sin^2 x)^3 \cos x \, dx = \int_0^{\pi/2} (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x) \cos x \, dx \\ &= \left[\sin x - \sin^3 x + \frac{3}{5} \sin^5 x - \frac{1}{7} \sin^7 x \right]_0^{\pi/2} = \frac{16}{35} \end{aligned}$$

$$21. \int \sec(3x) \, dx = \frac{1}{3} \ln |\sec 3x + \tan 3x| + C$$

$$\begin{aligned} 23. \int \sec^4 5x \, dx &= \int (1 + \tan^2 5x) \sec^2 5x \, dx \\ &= \frac{1}{5} \left(\tan 5x + \frac{\tan^3 5x}{3} \right) + C \\ &= \frac{\tan 5x}{15} (3 + \tan^2 5x) + C \end{aligned}$$

$$25. dv = \sec^2 \pi x \, dx \Rightarrow v = \frac{1}{\pi} \tan \pi x$$

$$u = \sec \pi x \Rightarrow du = \pi \sec \pi x \tan \pi x \, dx$$

$$\int \sec^3 \pi x \, dx = \frac{1}{\pi} \sec \pi x \tan \pi x - \int \sec \pi x \tan^2 \pi x \, dx = \frac{1}{\pi} \sec \pi x \tan \pi x - \int \sec \pi x (\sec^2 \pi x - 1) \, dx$$

$$2 \int \sec^3 \pi x \, dx = \frac{1}{\pi} (\sec \pi x \tan \pi x + \ln |\sec \pi x + \tan \pi x|) + C_1$$

$$\int \sec^3 \pi x \, dx = \frac{1}{2\pi} (\sec \pi x \tan \pi x + \ln |\sec \pi x + \tan \pi x|) + C$$

$$\begin{aligned} 27. \int \tan^5 \frac{x}{4} \, dx &= \int \left(\sec^2 \frac{x}{4} - 1 \right) \tan^3 \frac{x}{4} \, dx \\ &= \int \tan^3 \frac{x}{4} \sec^2 \frac{x}{4} \, dx - \int \tan^3 \frac{x}{4} \, dx \\ &= \tan^4 \frac{x}{4} - \int \left(\sec^2 \frac{x}{4} - 1 \right) \tan \frac{x}{4} \, dx \\ &= \tan^4 \frac{x}{4} - 2 \tan^2 \frac{x}{4} - 4 \ln \left| \cos \frac{x}{4} \right| + C \end{aligned}$$

$$29. u = \tan x, du = \sec^2 x \, dx$$

$$\int \sec^2 x \tan x \, dx = \frac{1}{2} \tan^2 x + C$$

$$31. \int \tan^2 x \sec^2 x \, dx = \frac{\tan^3 x}{3} + C$$

$$35. \text{ Let } u = \sec x, du = \sec x \tan x \, dx.$$

$$\begin{aligned} \int \sec^3 x \tan x \, dx &= \int \sec^2 x (\sec x \tan x) \, dx \\ &= \frac{1}{3} \sec^3 x + C \end{aligned}$$

$$\begin{aligned} 39. r &= \int \sin^4(\pi\theta) \, d\theta = \frac{1}{4} \int [1 - \cos(2\pi\theta)]^2 \, d\theta \\ &= \frac{1}{4} \int [1 - 2\cos(2\pi\theta) + \cos^2(2\pi\theta)] \, d\theta \\ &= \frac{1}{4} \int \left[1 - 2\cos(2\pi\theta) + \frac{1 + \cos(4\pi\theta)}{2} \right] \, d\theta \\ &= \frac{1}{4} \left[\theta - \frac{1}{\pi} \sin(2\pi\theta) + \frac{\theta}{2} + \frac{1}{8\pi} \sin(4\pi\theta) \right] + C \\ &= \frac{1}{32\pi} [12\pi\theta - 8\sin(2\pi\theta) + \sin(4\pi\theta)] + C \end{aligned}$$

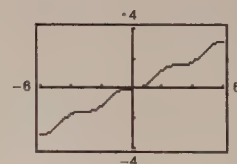
43. (a)



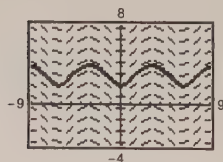
$$(b) \frac{dy}{dx} = \sin^2 x, (0, 0)$$

$$\begin{aligned} y &= \int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx \\ &= \frac{1}{2}x - \frac{\sin 2x}{4} + C \end{aligned}$$

$$(0, 0): 0 = C, y = \frac{1}{2}x - \frac{\sin 2x}{4}$$



$$45. \frac{dy}{dx} = \frac{3 \sin x}{y}, y(0) = 2$$



$$\begin{aligned} 47. \int \sin 3x \cos 2x \, dx &= \frac{1}{2} \int (\sin 5x + \sin x) \, dx \\ &= -\frac{1}{2} \left(\frac{1}{5} \cos 5x + \cos x \right) + C \\ &= -\frac{1}{10} (\cos 5x + 5 \cos x) + C \end{aligned}$$

$$\begin{aligned} 49. \int \sin \theta \sin 3\theta \, d\theta &= \frac{1}{2} \int (\cos 2\theta - \cos 4\theta) \, d\theta \\ &= \frac{1}{2} \left(\frac{1}{2} \sin 2\theta - \frac{1}{4} \sin 4\theta \right) + C \\ &= \frac{1}{8} (2 \sin 2\theta - \sin 4\theta) + C \end{aligned}$$

$$\begin{aligned} 51. \int \cot^3 2x \, dx &= \int (\csc^2 2x - 1) \cot 2x \, dx \\ &= -\frac{1}{2} \int \cot 2x (-2 \csc^2 2x) \, dx - \frac{1}{2} \int \frac{2 \cos 2x}{\sin 2x} \, dx \\ &= -\frac{1}{4} \cot^2 2x - \frac{1}{2} \ln |\sin 2x| + C \\ &= \frac{1}{4} (\ln |\csc^2 2x| - \cot^2 2x) + C \end{aligned}$$

53. Let $u = \cot \theta$, $du = -\csc^2 \theta d\theta$.

$$\begin{aligned}\int \csc^4 \theta d\theta &= \int \csc^2 \theta (1 + \cot^2 \theta) d\theta \\ &= \int \csc^2 \theta d\theta + \int \csc^2 \theta \cot^2 \theta d\theta \\ &= -\cot \theta - \frac{1}{3} \cot^3 \theta + C\end{aligned}$$

57. $\int \frac{1}{\sec x \tan x} dx = \int \frac{\cos^2 x}{\sin x} dx = \int \frac{1 - \sin^2 x}{\sin x} dx$

$$= \int (\csc x - \sin x) dx$$

$$= \ln|\csc x - \cot x| + \cos x + C$$

59. $\int (\tan^4 t - \sec^4 t) dt = \int (\tan^2 t + \sec^2 t)(\tan^2 t - \sec^2 t) dt \quad (\tan^2 t - \sec^2 t = -1)$

$$= -\int (\tan^2 t + \sec^2 t) dt = -\int (2 \sec^2 t - 1) dt = -2 \tan t + t + C$$

61. $\int_{-\pi}^{\pi} \sin^2 x dx = 2 \int_0^{\pi} \frac{1 - \cos 2x}{2} dx$

$$= \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \pi$$

63. $\int_0^{\pi/4} \tan^3 x dx = \int_0^{\pi/4} (\sec^2 x - 1) \tan x dx$

$$= \int_0^{\pi/4} \sec^2 x \tan x dx - \int_0^{\pi/4} \frac{\sin x}{\cos x} dx$$

$$= \left[\frac{1}{2} \tan^2 x + \ln|\cos x| \right]_0^{\pi/4}$$

$$= \frac{1}{2} (1 - \ln 2)$$

65. Let $u = 1 + \sin t$, $du = \cos t dt$.

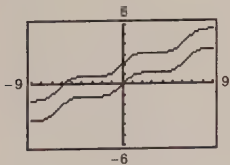
$$\int_0^{\pi/2} \frac{\cos t}{1 + \sin t} dt = \left[\ln|1 + \sin t| \right]_0^{\pi/2} = \ln 2$$

67. Let $u = \sin x$, $du = \cos x dx$.

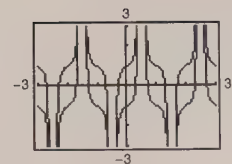
$$\int_{-\pi/2}^{\pi/2} \cos^3 x dx = 2 \int_0^{\pi/2} (1 - \sin^2 x) \cos x dx$$

$$= 2 \left[\sin x - \frac{1}{3} \sin^3 x \right]_0^{\pi/2} = \frac{4}{3}$$

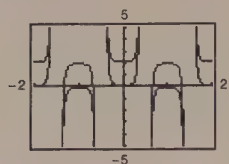
69. $\int \cos^4 \frac{x}{2} dx = \frac{1}{16} [6x + 8 \sin x + \sin 2x] + C$



71. $\int \sec^5 \pi x dx = \frac{1}{4\pi} \left\{ \sec^3 \pi x \tan \pi x + \frac{3}{2} [\sec \pi x \tan \pi x + \ln|\sec \pi x + \tan \pi x|] \right\} + C$



$$73. \int \sec^5 \pi x \tan \pi x \, dx = \frac{1}{5\pi} \sec^5 \pi x + C$$



$$75. \int_0^{\pi/4} \sin 2\theta \sin 3\theta \, d\theta = \frac{1}{2} \left[\sin \theta - \frac{1}{5} \sin 5\theta \right]_0^{\pi/4} = \frac{3\sqrt{2}}{10}$$

$$77. \int_0^{\pi/2} \sin^4 x \, dx = \frac{1}{4} \left[\frac{3x}{2} - \sin 2x + \frac{1}{8} \sin 4x \right]_0^{\pi/2} = \frac{3\pi}{16}$$

79. (a) Save one sine factor and convert the remaining sine factors to cosine. Then expand and integrate.
- (b) Save one cosine factor and convert the remaining cosine factors to sine. Then expand and integrate.
- (c) Make repeated use of the power reducing formula to convert the integrand to odd powers of the cosine.

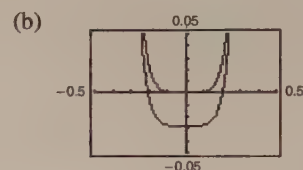
$$81. (a) \text{ Let } u = \tan 3x, du = 3 \sec^2 3x \, dx.$$

$$\begin{aligned} \int \sec^4 3x \tan^3 3x \, dx &= \int \sec^2 3x \tan^3 3x \sec^2 3x \, dx \\ &= \frac{1}{3} \int (\tan^2 3x + 1) \tan^3 3x (3 \sec^2 3x) \, dx \\ &= \frac{1}{3} \int (\tan^5 3x + \tan^3 3x) (3 \sec^2 3x) \, dx \\ &= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_1 \end{aligned}$$

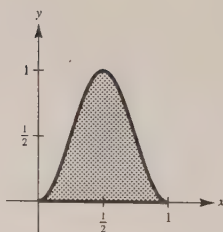
$$\text{Or let } u = \sec 3x, du = 3 \sec 3x \tan 3x \, dx.$$

$$\begin{aligned} \int \sec^4 3x \tan^3 3x \, dx &= \int \sec^3 3x \tan^2 3x \sec 3x \tan 3x \, dx \\ &= \frac{1}{3} \int \sec^3 3x (\sec^2 3x - 1) (3 \sec 3x \tan 3x) \, dx \\ &= \frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C \end{aligned}$$

$$\begin{aligned} (c) \frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C &= \frac{(1 + \tan^2 3x)^3}{18} - \frac{(1 + \tan^2 3x)^2}{12} + C \\ &= \frac{1}{18} \tan^6 3x + \frac{1}{6} \tan^4 3x + \frac{1}{6} \tan^2 3x + \frac{1}{18} - \frac{1}{12} \tan^4 3x - \frac{1}{6} \tan^2 3x - \frac{1}{12} + C \\ &= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + \left(\frac{1}{18} - \frac{1}{12} \right) + C \\ &= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_2 \end{aligned}$$



$$\begin{aligned} 83. A &= \int_0^1 \sin^2(\pi x) \, dx \\ &= \int_0^1 \frac{1 - \cos(2\pi x)}{2} \, dx \\ &= \left[\frac{x}{2} - \frac{1}{4\pi} \sin(2\pi x) \right]_0^1 \\ &= \frac{1}{2} \end{aligned}$$



$$85. (a) V = \pi \int_0^{\pi} \sin^2 x \, dx = \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \, dx = \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \frac{\pi^2}{2}$$

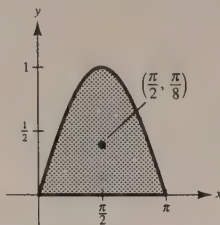
$$(b) A = \int_0^{\pi} \sin x \, dx = \left[-\cos x \right]_0^{\pi} = 1 + 1 = 2$$

Let $u = x$, $dv = \sin x \, dx$, $du = dx$, $v = -\cos x$.

$$\bar{x} = \frac{1}{A} \int_0^{\pi} x \sin x \, dx = \frac{1}{2} \left[\left[-x \cos x \right]_0^{\pi} + \int_0^{\pi} \cos x \, dx \right] = \frac{1}{2} \left[-x \cos x + \sin x \right]_0^{\pi} = \frac{\pi}{2}$$

$$\begin{aligned} \bar{y} &= \frac{1}{2A} \int_0^{\pi} \sin^2 x \, dx \\ &= \frac{1}{8} \int_0^{\pi} (1 - \cos 2x) \, dx \\ &= \frac{1}{8} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \frac{\pi}{8} \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi}{2}, \frac{\pi}{8} \right)$$



$$87. dv = \sin x \, dx \Rightarrow v = -\cos x$$

$$u = \sin^{n-1} x \Rightarrow du = (n-1) \sin^{n-2} x \cos x \, dx$$

$$\begin{aligned} \int \sin^n x \, dx &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx \end{aligned}$$

$$\text{Therefore, } n \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx$$

$$\int \sin^n x \, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

$$89. \text{ Let } u = \sin^{n-1} x, du = (n-1) \sin^{n-2} x \cos x \, dx, dv = \cos^m x \sin x \, dx, v = \frac{-\cos^{m+1} x}{m+1}.$$

$$\begin{aligned} \int \cos^m x \sin^n x \, dx &= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^{m+2} x \, dx \\ &= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x (1 - \sin^2 x) \, dx \\ &= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x \, dx - \frac{n-1}{m+1} \int \sin^n x \cos^m x \, dx \end{aligned}$$

$$\frac{m+n}{m+1} \int \cos^m x \sin^n x \, dx = \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{n-2} x \cos^m x \, dx$$

$$\int \cos^m x \sin^n x \, dx = \frac{-\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \, dx$$

$$\begin{aligned}
 91. \int \sin^5 x \, dx &= -\frac{\sin^4 x \cos x}{5} + \frac{4}{5} \int \sin^3 x \, dx \\
 &= -\frac{\sin^4 x \cos x}{5} + \frac{4}{5} \left[-\frac{\sin^2 x \cos x}{3} + \frac{2}{3} \int \sin x \, dx \right] \\
 &= -\frac{1}{5} \sin^4 x \cos x - \frac{4}{15} \sin^2 x \cos x - \frac{8}{15} \cos x + C \\
 &= -\frac{\cos x}{15} [3 \sin^4 x + 4 \sin^2 x + 8] + C
 \end{aligned}$$

$$\begin{aligned}
 93. \int \sec^4\left(\frac{2\pi x}{5}\right) dx &= \frac{5}{2\pi} \int \sec^4\left(\frac{2\pi x}{5}\right) \frac{2\pi}{5} dx \\
 &= \frac{5}{2\pi} \left[\frac{1}{3} \sec^2\left(\frac{2\pi x}{5}\right) \tan\left(\frac{2\pi x}{5}\right) + \frac{2}{3} \int \sec^2\left(\frac{2\pi x}{5}\right) \frac{2\pi}{5} dx \right] \\
 &= \frac{5}{6\pi} \left[\sec^2\left(\frac{2\pi x}{5}\right) \tan\left(\frac{2\pi x}{5}\right) + 2 \tan\left(\frac{2\pi x}{5}\right) \right] + C \\
 &= \frac{5}{6\pi} \tan\left(\frac{2\pi x}{5}\right) \left[\sec^2\left(\frac{2\pi x}{5}\right) + 2 \right] + C
 \end{aligned}$$

95. (a) $f(t) = a_0 + a_1 \cos \frac{\pi t}{6} + b_1 \sin \frac{\pi t}{6}$ where:

$$a_0 = \frac{1}{12} \int_0^{12} f(t) \, dt$$

$$a_1 = \frac{1}{6} \int_0^{12} f(t) \cos \frac{\pi t}{6} \, dt$$

$$b_1 = \frac{1}{6} \int_0^{12} f(t) \sin \frac{\pi t}{6} \, dt$$

$$\begin{aligned}
 a_0 &\approx \frac{12-0}{3(12)^2} [30.9 + 4(32.2) + 2(41.1) + 4(53.7) + 2(64.6) + 4(74.0) + 2(78.2) + 4(77.0) + 2(71.0) + \\
 &\quad 4(60.1) + 2(47.1) + 4(35.7) + 30.9] \approx 55.46
 \end{aligned}$$

$$\begin{aligned}
 a_1 &\approx \frac{12-0}{6(3)(12)} \left[30.9 \cos 0 + 4 \left(32.2 \cos \frac{\pi}{6} \right) + 2 \left(41.1 \cos \frac{\pi}{3} \right) + 4 \left(53.7 \cos \frac{\pi}{2} \right) + 2 \left(64.6 \cos \frac{2\pi}{3} \right) + \right. \\
 &\quad 4 \left(74.0 \cos \frac{5\pi}{6} \right) + 2(78.2 \cos \pi) + 4 \left(77.0 \cos \frac{7\pi}{6} \right) + 2 \left(71.0 \cos \frac{4\pi}{3} \right) + \\
 &\quad \left. 4 \left(60.1 \cos \frac{3\pi}{2} \right) + 2 \left(47.1 \cos \frac{5\pi}{3} \right) + 4 \left(35.7 \cos \frac{11\pi}{6} \right) + 30.9 \cos 2\pi \right] \approx -23.88
 \end{aligned}$$

$$\begin{aligned}
 b_1 &\approx \frac{12-0}{6(3)(12)} \left[30.9 \sin 0 + 4 \left(32.2 \sin \frac{\pi}{6} \right) + 2 \left(41.1 \sin \frac{\pi}{3} \right) + 4 \left(53.7 \sin \frac{\pi}{2} \right) + 2 \left(64.6 \sin \frac{2\pi}{3} \right) + \right. \\
 &\quad 4 \left(74.0 \sin \frac{5\pi}{6} \right) + 2(78.2 \sin \pi) + 4 \left(77.0 \sin \frac{7\pi}{6} \right) + 2 \left(71.0 \sin \frac{4\pi}{3} \right) + \\
 &\quad \left. 4 \left(60.1 \sin \frac{3\pi}{2} \right) + 2 \left(47.1 \sin \frac{5\pi}{3} \right) + 4 \left(35.7 \sin \frac{11\pi}{6} \right) + 30.9 \sin 2\pi \right] \approx -3.34
 \end{aligned}$$

$$H(t) \approx 55.46 - 23.88 \cos \frac{\pi t}{6} - 3.34 \sin \frac{\pi t}{6}$$

—CONTINUED—

95. —CONTINUED—

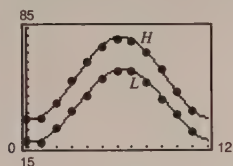
$$(b) a_0 \approx \frac{12-0}{3(12)^2} [18.0 + 4(17.7) + 2(25.8) + 4(36.1) + 2(45.4) + 4(55.2) + 2(59.9) + 4(59.4) + 2(53.1) + 4(43.2) + 2(34.3) + 4(24.2) + 18.0] \approx 39.34$$

$$a_1 \approx \frac{12-0}{6(3)(12)} \left[18.0 \cos 0 + 4 \left(17.7 \cos \frac{\pi}{6} \right) + 2 \left(25.8 \cos \frac{\pi}{3} \right) + 4 \left(36.1 \cos \frac{\pi}{2} \right) + 2 \left(45.4 \cos \frac{2\pi}{3} \right) + 4 \left(55.2 \cos \frac{5\pi}{6} \right) + 2(59.9 \cos \pi) + 4 \left(59.4 \cos \frac{7\pi}{6} \right) + 2 \left(53.1 \cos \frac{4\pi}{3} \right) + 4 \left(43.2 \cos \frac{3\pi}{2} \right) + 2 \left(34.3 \cos \frac{5\pi}{3} \right) + 4 \left(24.2 \cos \frac{11\pi}{6} \right) + 18 \cos 2\pi \right] \approx -20.78$$

$$b_1 \approx \frac{12-0}{6(3)(12)} \left[18.0 \sin 0 + 4 \left(17.7 \sin \frac{\pi}{6} \right) + 2 \left(25.8 \sin \frac{\pi}{3} \right) + 4 \left(36.1 \sin \frac{\pi}{2} \right) + 2 \left(45.4 \sin \frac{2\pi}{3} \right) + 4 \left(55.2 \sin \frac{5\pi}{6} \right) + 2(59.9 \sin \pi) + 4 \left(59.4 \sin \frac{7\pi}{6} \right) + 2 \left(53.1 \sin \frac{4\pi}{3} \right) + 4 \left(43.2 \sin \frac{3\pi}{2} \right) + 2 \left(34.3 \sin \frac{5\pi}{3} \right) + 4 \left(24.2 \sin \frac{11\pi}{6} \right) + 18 \sin 2\pi \right] \approx -4.33$$

$$L(t) \approx 39.34 - 20.78 \cos \frac{\pi t}{6} - 4.33 \sin \frac{\pi t}{6}$$

(c) The difference between the maximum and minimum temperatures is greatest in the summer.



$$97. \int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx = \frac{1}{2} \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_{-\pi}^{\pi} = 0, \quad (m \neq n)$$

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x] dx \\ &= \frac{1}{2} \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi} = 0, \quad (m \neq n) \end{aligned}$$

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(mx) \cos(nx) dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\sin(m+n)x + \sin(m-n)x] dx \\ &= -\frac{1}{2} \left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n} \right]_{-\pi}^{\pi}, \quad (m \neq n) \\ &= -\frac{1}{2} \left[\left(\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n} \right) - \left(\frac{\cos(m+n)(-\pi)}{m+n} + \frac{\cos(m-n)(-\pi)}{m-n} \right) \right] \\ &= 0, \text{ since } \cos(-\theta) = \cos\theta. \end{aligned}$$

$$\int_{-\pi}^{\pi} \sin(mx) \cos(mx) dx = \frac{1}{m} \left[\frac{\sin^2(mx)}{2} \right]_{-\pi}^{\pi} = 0$$

Section 7.4 Trigonometric Substitution

$$\begin{aligned}
 1. \frac{d}{dx} \left[4 \ln \left| \frac{\sqrt{x^2 + 16} - 4}{x} \right| + \sqrt{x^2 + 16} + C \right] &= \frac{d}{dx} \left[4 \ln \left| \sqrt{x^2 + 16} - 4 \right| - 4 \ln |x| + \sqrt{x^2 + 16} + C \right] \\
 &= 4 \left[\frac{x/\sqrt{x^2 + 16}}{\sqrt{x^2 + 16} - 4} \right] - \frac{4}{x} + \frac{x}{\sqrt{x^2 + 16}} \\
 &= \frac{4x}{\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} - \frac{4}{x} + \frac{x}{\sqrt{x^2 + 16}} \\
 &= \frac{4x^2 - 4\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4) + x^2(\sqrt{x^2 + 16} - 4)}{x\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} \\
 &= \frac{4x^2 - 4(x^2 + 16) + 16\sqrt{x^2 + 16} + x^2\sqrt{x^2 + 16} - 4x^2}{x\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} \\
 &= \frac{\sqrt{x^2 + 16}(x^2 + 16) - 4(x^2 + 16)}{x\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} \\
 &= \frac{(x^2 + 16)(\sqrt{x^2 + 16} - 4)}{x\sqrt{x^2 + 16}(\sqrt{x^2 + 16} - 4)} = \frac{\sqrt{x^2 + 16}}{x}
 \end{aligned}$$

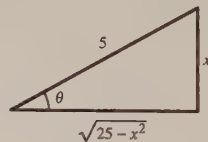
Indefinite integral: $\int \frac{\sqrt{x^2 + 16}}{x} dx$ Matches (b)

$$\begin{aligned}
 3. \frac{d}{dx} \left[8 \arcsin \frac{x}{4} - \frac{x\sqrt{16 - x^2}}{2} + C \right] &= 8 \frac{1/4}{\sqrt{1 - (x/4)^2}} - \frac{x(1/2)(16 - x^2)^{-1/2}(-2x) + \sqrt{16 - x^2}}{2} \\
 &= \frac{8}{\sqrt{16 - x^2}} + \frac{x^2}{2\sqrt{16 - x^2}} - \frac{\sqrt{16 - x^2}}{2} \\
 &= \frac{16}{2\sqrt{16 - x^2}} + \frac{x^2}{2\sqrt{16 - x^2}} - \frac{(16 - x^2)}{2\sqrt{16 - x^2}} = \frac{x^2}{\sqrt{16 - x^2}}
 \end{aligned}$$

Matches (a)

5. Let $x = 5 \sin \theta$, $dx = 5 \cos \theta d\theta$, $\sqrt{25 - x^2} = 5 \cos \theta$.

$$\begin{aligned}
 \int \frac{1}{(25 - x^2)^{3/2}} dx &= \int \frac{5 \cos \theta}{(5 \cos \theta)^3} d\theta \\
 &= \frac{1}{25} \int \sec^2 \theta d\theta \\
 &= \frac{1}{25} \tan \theta + C \\
 &= \frac{x}{25\sqrt{25 - x^2}} + C
 \end{aligned}$$

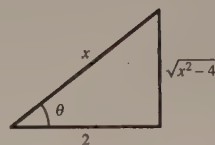


7. Same substitution as in Exercise 5

$$\begin{aligned}
 \int \frac{\sqrt{25 - x^2}}{x} dx &= \int \frac{25 \cos^2 \theta d\theta}{5 \sin \theta} = 5 \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta = 5 \int (\csc \theta - \sin \theta) d\theta \\
 &= 5 [\ln |\csc \theta - \cot \theta| + \cos \theta] + C = 5 \ln \left| \frac{5 - \sqrt{25 - x^2}}{x} \right| + \sqrt{25 - x^2} + C
 \end{aligned}$$

9. Let $x = 2 \sec \theta$, $dx = 2 \sec \theta \tan \theta d\theta$, $\sqrt{x^2 - 4} = 2 \tan \theta$.

$$\begin{aligned}\int \frac{1}{\sqrt{x^2 - 4}} dx &= \int \frac{2 \sec \theta \tan \theta d\theta}{2 \tan \theta} = \int \sec \theta d\theta = \ln|\sec \theta + \tan \theta| + C_1 \\ &= \ln \left| \frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2} \right| + C_1 \\ &= \ln|x + \sqrt{x^2 - 4}| - \ln 2 + C_1 = \ln|x + \sqrt{x^2 - 4}| + C\end{aligned}$$



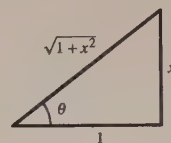
11. Same substitution as in Exercise 9

$$\begin{aligned}\int x^3 \sqrt{x^2 - 4} dx &= \int (8 \sec^3 \theta)(2 \tan \theta)(2 \sec \theta \tan \theta) d\theta = 32 \int \tan^2 \theta \sec^4 \theta d\theta \\ &= 32 \int \tan^2 \theta (1 + \tan^2 \theta) \sec^2 \theta d\theta = 32 \left(\frac{\tan^3 \theta}{3} + \frac{\tan^5 \theta}{5} \right) + C \\ &= \frac{32}{15} \tan^3 \theta [5 + 3 \tan^2 \theta] + C = \frac{32}{15} \frac{(x^2 - 4)^{3/2}}{8} \left[5 + 3 \frac{(x^2 - 4)}{4} \right] + C \\ &= \frac{1}{15} (x^2 - 4)^{3/2} [20 + 3(x^2 - 4)] + C = \frac{1}{15} (x^2 - 4)^{3/2} (3x^2 + 8) + C\end{aligned}$$

13. Let $x = \tan \theta$, $dx = \sec^2 \theta d\theta$, $\sqrt{1 + x^2} = \sec \theta$.

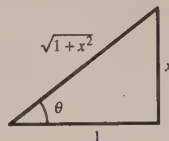
$$\int x \sqrt{1 + x^2} dx = \int \tan \theta (\sec \theta) \sec^2 \theta d\theta = \frac{\sec^3 \theta}{3} + C = \frac{1}{3} (1 + x^2)^{3/2} + C$$

Note: This integral could have been evaluated with the Power Rule.



15. Same substitution as in Exercise 13

$$\begin{aligned}\int \frac{1}{(1 + x^2)^2} dx &= \int \frac{1}{(\sqrt{1 + x^2})^4} dx \\ &= \int \frac{\sec^2 \theta d\theta}{\sec^4 \theta} \\ &= \int \cos^2 \theta d\theta = \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2} \right] \\ &= \frac{1}{2} [\theta + \sin \theta \cos \theta] + C \\ &= \frac{1}{2} \left[\arctan x + \left(\frac{x}{\sqrt{1 + x^2}} \right) \left(\frac{1}{\sqrt{1 + x^2}} \right) \right] + C \\ &= \frac{1}{2} \left[\arctan x + \frac{x}{1 + x^2} \right] + C\end{aligned}$$



17. Let $u = 3x$, $a = 2$, and $du = 3 dx$.

$$\begin{aligned}\int \sqrt{4 + 9x^2} dx &= \frac{1}{3} \int \sqrt{(2)^2 + (3x)^2} 3 dx \\ &= \frac{1}{3} \left(\frac{1}{2} \right) (3x \sqrt{4 + 9x^2} + 4 \ln|3x + \sqrt{4 + 9x^2}|) + C \\ &= \frac{1}{2} x \sqrt{4 + 9x^2} + \frac{2}{3} \ln|3x + \sqrt{4 + 9x^2}| + C\end{aligned}$$

$$19. \int \frac{x}{\sqrt{x^2 + 9}} dx = \frac{1}{2} \int (x^2 + 9)^{-1/2} (2x) dx$$

$$= \sqrt{x^2 + 9} + C$$

(Power Rule)

$$21. \int \frac{1}{\sqrt{16 - x^2}} dx = \arcsin\left(\frac{x}{4}\right) + C$$

$$23. \text{ Let } x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4 - x^2} = 2 \cos \theta.$$

$$\int \sqrt{16 - 4x^2} dx = 2 \int \sqrt{4 - x^2} dx$$

$$= 2 \int 2 \cos \theta (2 \cos \theta d\theta)$$

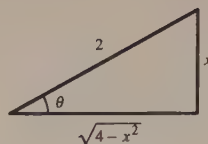
$$= 8 \int \cos^2 \theta d\theta$$

$$= 4 \int (1 + \cos 2\theta) d\theta$$

$$= 4 \left[\theta + \frac{1}{2} \sin 2\theta \right] + C$$

$$= 4\theta + 4 \sin \theta \cos \theta + C$$

$$= 4 \arcsin\left(\frac{x}{2}\right) + x\sqrt{4 - x^2} + C$$



$$25. \text{ Let } x = 3 \sec \theta, dx = 3 \sec \theta \tan \theta d\theta,$$

$$\sqrt{x^2 - 9} = 3 \tan \theta.$$

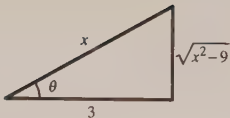
$$\int \frac{1}{\sqrt{x^2 - 9}} dx = \int \frac{3 \sec \theta \tan \theta d\theta}{3 \tan \theta}$$

$$= \int \sec \theta d\theta$$

$$= \ln |\sec \theta + \tan \theta| + C_1$$

$$= \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| + C_1$$

$$= \ln |x + \sqrt{x^2 - 9}| + C$$



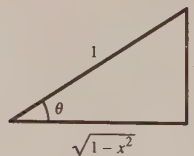
$$27. \text{ Let } x = \sin \theta, dx = \cos \theta d\theta, \sqrt{1 - x^2} = \cos \theta.$$

$$\int \frac{\sqrt{1 - x^2}}{x^4} dx = \int \frac{\cos \theta (\cos \theta d\theta)}{\sin^4 \theta}$$

$$= \int \cot^2 \theta \csc^2 \theta d\theta$$

$$= -\frac{1}{3} \cot^3 \theta + C$$

$$= \frac{-(1 - x^2)^{3/2}}{3x^3} + C$$



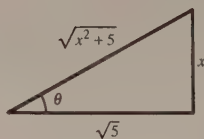
$$29. \text{ Same substitutions as in Exercise 28}$$

$$\int \frac{1}{x\sqrt{4x^2 + 9}} dx = \int \frac{(3/2) \sec^2 \theta d\theta}{(3/2) \tan \theta 3 \sec \theta}$$

$$= \frac{1}{3} \int \csc \theta d\theta = -\frac{1}{3} \ln |\csc \theta + \cot \theta| + C = -\frac{1}{3} \ln \left| \frac{\sqrt{4x^2 + 9} + 3}{2x} \right| + C$$

31. Let $x = \sqrt{5} \tan \theta$, $dx = \sqrt{5} \sec^2 \theta d\theta$, $x^2 + 5 = 5 \sec^2 \theta$.

$$\begin{aligned} \int \frac{-5x}{(x^2 + 5)^{3/2}} dx &= \int \frac{-5\sqrt{5} \tan \theta}{(5 \sec^2 \theta)^{3/2}} \sqrt{5} \sec^2 \theta d\theta \\ &= -\sqrt{5} \int \frac{\tan \theta}{\sec \theta} d\theta \\ &= -\sqrt{5} \int \sin \theta d\theta \\ &= \sqrt{5} \cos \theta + C \\ &= \sqrt{5} \frac{\sqrt{5}}{\sqrt{x^2 + 5}} + C \\ &= \frac{5}{\sqrt{x^2 + 5}} + C \end{aligned}$$

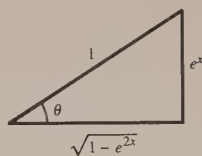


33. Let $u = 1 + e^{2x}$, $du = 2e^{2x} dx$.

$$\int e^{2x} \sqrt{1 + e^{2x}} dx = \frac{1}{2} \int (1 + e^{2x})^{1/2} (2e^{2x}) dx = \frac{1}{3} (1 + e^{2x})^{3/2} + C$$

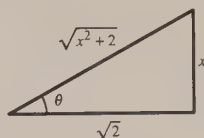
35. Let $e^x = \sin \theta$, $e^x dx = \cos \theta d\theta$, $\sqrt{1 - e^{2x}} = \cos \theta$.

$$\begin{aligned} \int e^x \sqrt{1 - e^{2x}} dx &= \int \cos^2 \theta d\theta \\ &= \frac{1}{2} \int (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2} \right] \\ &= \frac{1}{2} (\theta + \sin \theta \cos \theta) + C = \frac{1}{2} (\arcsin e^x + e^x \sqrt{1 - e^{2x}}) + C \end{aligned}$$



37. Let $x = \sqrt{2} \tan \theta$, $dx = \sqrt{2} \sec^2 \theta d\theta$, $x^2 + 2 = 2 \sec^2 \theta$.

$$\begin{aligned} \int \frac{1}{4 + 4x^2 + x^4} dx &= \int \frac{1}{(x^2 + 2)^2} dx \\ &= \int \frac{\sqrt{2} \sec^2 \theta d\theta}{4 \sec^4 \theta} \\ &= \frac{\sqrt{2}}{4} \int \cos^2 \theta d\theta \\ &= \frac{\sqrt{2}}{4} \left(\frac{1}{2} \right) \int (1 + \cos 2\theta) d\theta \\ &= \frac{\sqrt{2}}{8} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\ &= \frac{\sqrt{2}}{8} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{1}{4} \left[\frac{x}{x^2 + 2} + \frac{1}{\sqrt{2}} \arctan \frac{x}{\sqrt{2}} \right] + C \end{aligned}$$



39. Since $x > \frac{1}{2}$,

$$u = \operatorname{arcsec} 2x, \Rightarrow du = \frac{1}{x\sqrt{4x^2-1}} dx, dv = dx \Rightarrow v = x$$

$$\int \operatorname{arcsec} 2x dx = x \operatorname{arcsec} 2x - \int \frac{1}{\sqrt{4x^2-1}} dx$$

$$2x = \sec \theta, dx = \frac{1}{2} \sec \theta \tan \theta d\theta, \sqrt{4x^2-1} = \tan \theta$$

$$\int \operatorname{arcsec} 2x dx = x \operatorname{arcsec} 2x - \int \frac{(1/2) \sec \theta \tan \theta d\theta}{\tan \theta} = x \operatorname{arcsec} 2x - \frac{1}{2} \int \sec \theta d\theta$$

$$= x \operatorname{arcsec} 2x - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C = x \operatorname{arcsec} 2x - \frac{1}{2} \ln |2x + \sqrt{4x^2-1}| + C.$$

$$41. \int \frac{1}{\sqrt{4x-x^2}} dx = \int \frac{1}{\sqrt{4-(x-2)^2}} dx = \arcsin\left(\frac{x-2}{2}\right) + C$$

43. Let $x+2 = 2 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$, $\sqrt{(x+2)^2+4} = 2 \sec \theta$.

$$\int \frac{x}{\sqrt{x^2+4x+8}} dx = \int \frac{x}{\sqrt{(x+2)^2+4}} dx = \int \frac{(2 \tan \theta - 2)(2 \sec^2 \theta) d\theta}{2 \sec \theta}$$

$$= 2 \int (\tan \theta - 1)(\sec \theta) d\theta$$

$$= 2[\sec \theta - \ln |\sec \theta + \tan \theta|] + C_1$$

$$= 2 \left[\frac{\sqrt{(x+2)^2+4}}{2} - \ln \left| \frac{\sqrt{(x+2)^2+4}}{2} + \frac{x+2}{2} \right| \right] + C_1$$

$$= \sqrt{x^2+4x+8} - 2[\ln |\sqrt{x^2+4x+8} + (x+2)| - \ln 2] + C_1$$

$$= \sqrt{x^2+4x+8} - 2 \ln |\sqrt{x^2+4x+8} + (x+2)| + C$$

45. Let $t = \sin \theta$, $dt = \cos \theta d\theta$, $1-t^2 = \cos^2 \theta$.

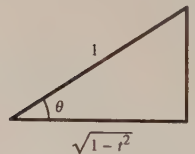
$$(a) \int \frac{t^2}{(1-t^2)^{3/2}} dt = \int \frac{\sin^2 \theta \cos \theta d\theta}{\cos^3 \theta}$$

$$= \int \tan^2 \theta d\theta$$

$$= \int (\sec^2 \theta - 1) d\theta$$

$$= \tan \theta - \theta + C$$

$$= \frac{t}{\sqrt{1-t^2}} - \arcsin t + C$$



$$\text{Thus, } \int_0^{\sqrt{3}/2} \frac{t^2}{(1-t^2)^{3/2}} dt = \left[\frac{t}{\sqrt{1-t^2}} - \arcsin t \right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}/2}{\sqrt{1/4}} - \arcsin \frac{\sqrt{3}}{2} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$

(b) When $t = 0$, $\theta = 0$. When $t = \sqrt{3}/2$, $\theta = \pi/3$. Thus,

$$\int_0^{\sqrt{3}/2} \frac{t^2}{(1-t^2)^{3/2}} dt = \left[\tan \theta - \theta \right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$

47. (a) Let $x = 3 \tan \theta$, $dx = 3 \sec^2 \theta d\theta$, $\sqrt{x^2 + 9} = 3 \sec \theta$.

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2 + 9}} dx &= \int \frac{(27 \tan^3 \theta)(3 \sec^2 \theta d\theta)}{3 \sec \theta} \\ &= 27 \int (\sec^2 \theta - 1) \sec \theta \tan \theta d\theta \\ &= 27 \left[\frac{1}{3} \sec^3 \theta - \sec \theta \right] + C = 9[\sec^3 \theta - 3 \sec \theta] + C \\ &= 9 \left[\left(\frac{\sqrt{x^2 + 9}}{3} \right)^3 - 3 \left(\frac{\sqrt{x^2 + 9}}{3} \right) \right] + C = \frac{1}{3} (x^2 + 9)^{3/2} - 9\sqrt{x^2 + 9} + C \end{aligned}$$

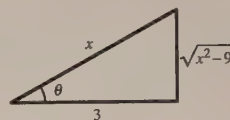
$$\begin{aligned} \text{Thus, } \int_0^3 \frac{x^3}{\sqrt{x^2 + 9}} dx &= \left[\frac{1}{3} (x^2 + 9)^{3/2} - 9\sqrt{x^2 + 9} \right]_0^3 \\ &= \left(\frac{1}{3} (54\sqrt{2}) - 27\sqrt{2} \right) - (9 - 27) \\ &= 18 - 9\sqrt{2} = 9(2 - \sqrt{2}) \approx 5.272. \end{aligned}$$

- (b) When $x = 0$, $\theta = 0$. When $x = 3$, $\theta = \pi/4$. Thus,

$$\int_0^3 \frac{x^3}{\sqrt{x^2 + 9}} dx = 9 \left[\sec^3 \theta - 3 \sec \theta \right]_0^{\pi/4} = 9(2\sqrt{2} - 3\sqrt{2}) - 9(1 - 3) = 9(2 - \sqrt{2}) \approx 5.272.$$

49. (a) Let $x = 3 \sec \theta$, $dx = 3 \sec \theta \tan \theta d\theta$, $\sqrt{x^2 - 9} = 3 \tan \theta$.

$$\begin{aligned} \int \frac{x^2}{\sqrt{x^2 - 9}} dx &= \int \frac{9 \sec^2 \theta}{3 \tan \theta} 3 \sec \theta \tan \theta d\theta \\ &= 9 \int \sec^3 \theta d\theta \\ &= 9 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta d\theta \right] \quad (7.3 \text{ Exercise } 90) \\ &= \frac{9}{2} [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|] \\ &= \frac{9}{2} \left[\frac{x}{3} \cdot \frac{\sqrt{x^2 - 9}}{3} + \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| \right] \end{aligned}$$



Hence,

$$\begin{aligned} \int_4^6 \frac{x^2}{\sqrt{x^2 - 9}} dx &= \frac{9}{2} \left[\frac{x\sqrt{x^2 - 9}}{9} + \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| \right]_4^6 \\ &= \frac{9}{2} \left[\left(\frac{6\sqrt{27}}{9} + \ln \left| 2 + \frac{\sqrt{27}}{3} \right| \right) - \left(\frac{4\sqrt{7}}{9} + \ln \left| \frac{4}{3} + \frac{\sqrt{7}}{3} \right| \right) \right] \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \left(\ln \left(\frac{6 + \sqrt{27}}{3} \right) - \ln \left(\frac{4 + \sqrt{7}}{3} \right) \right) \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \ln \left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}} \right) \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \ln \left(\frac{(4 - \sqrt{7})(2 + \sqrt{3})}{3} \right) \approx 12.644. \end{aligned}$$

49. —CONTINUED—

(b) When $x = 4$, $\theta = \operatorname{arcsec}\left(\frac{4}{3}\right)$.

When $x = 6$, $\theta = \operatorname{arcsec}(2) = \frac{\pi}{3}$.

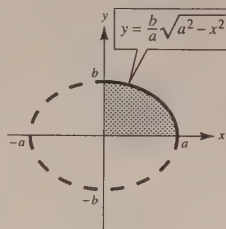
$$\begin{aligned}\int_4^6 \frac{x^2}{\sqrt{x^2-9}} dx &= \frac{9}{2} \left[\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]_{\operatorname{arcsec}(4/3)}^{\pi/3} \\ &= \frac{9}{2} \left[2 \cdot \sqrt{3} + \ln |2 + \sqrt{3}| \right] - \frac{9}{2} \left[\frac{4}{3} \frac{\sqrt{7}}{3} + \ln \left| \frac{4}{3} + \frac{\sqrt{7}}{3} \right| \right] \\ &= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2} \ln \left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}} \right) \approx 12.644\end{aligned}$$

51. $\int \frac{x^2}{\sqrt{x^2 + 10x + 9}} dx = \frac{1}{2} \sqrt{x^2 + 10x + 9} (x - 15) + 33 \ln |(x + 5) + \sqrt{x^2 + 10x + 9}| + C$

53. $\int \frac{x^2}{\sqrt{x^2 - 1}} dx = \frac{1}{2} (x\sqrt{x^2 - 1} + \ln |x + \sqrt{x^2 - 1}|) + C$ 55. (a) $u = a \sin \theta$ (b) $u = a \tan \theta$ (c) $u = a \sec \theta$

57. $A = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx$

$$\begin{aligned}&= \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} dx \\ &= \left[\frac{4b}{a} \left(\frac{1}{2} \left(a^2 \arcsin \frac{x}{a} + x\sqrt{a^2 - x^2} \right) \right) \right]_0^a \\ &= \frac{2b}{a} \left(a^2 \left(\frac{\pi}{2} \right) \right) \\ &= \pi ab\end{aligned}$$



Note: See Theorem 7.2 for $\int \sqrt{a^2 - x^2} dx$.

59. $x^2 + y^2 = a^2$

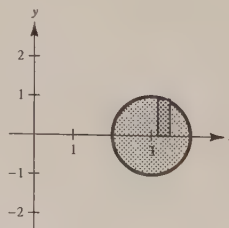
$$x = \pm \sqrt{a^2 - y^2}$$

$$\begin{aligned}A &= 2 \int_h^a \sqrt{a^2 - y^2} dy = \left[a^2 \arcsin \left(\frac{y}{a} \right) + y\sqrt{a^2 - y^2} \right]_h^a \quad (\text{Theorem 7.2}) \\ &= \left(a^2 \frac{\pi}{2} \right) - \left(a^2 \arcsin \left(\frac{h}{a} \right) + h\sqrt{a^2 - h^2} \right) \\ &= \frac{a^2 \pi}{2} - a^2 \arcsin \left(\frac{h}{a} \right) - h\sqrt{a^2 - h^2}\end{aligned}$$

61. Let $x - 3 = \sin \theta$, $dx = \cos \theta d\theta$, $\sqrt{1 - (x - 3)^2} = \cos \theta$.

Shell Method:

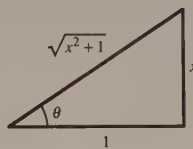
$$\begin{aligned}V &= 4\pi \int_2^4 x \sqrt{1 - (x - 3)^2} dx \\ &= 4\pi \int_{-\pi/2}^{\pi/2} (3 + \sin \theta) \cos^2 \theta d\theta \\ &= 4\pi \left[\frac{3}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta + \int_{-\pi/2}^{\pi/2} \cos^2 \theta \sin \theta d\theta \right] \\ &= 4\pi \left[\frac{3}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) - \frac{1}{3} \cos^3 \theta \right]_{-\pi/2}^{\pi/2} = 6\pi^2\end{aligned}$$



63. $y = \ln x, y' = \frac{1}{x}, 1 + (y')^2 = 1 + \frac{1}{x^2} = \frac{x^2 + 1}{x^2}$

Let $x = \tan \theta, dx = \sec^2 \theta d\theta, \sqrt{x^2 + 1} = \sec \theta$.

$$\begin{aligned} s &= \int_1^5 \sqrt{\frac{x^2 + 1}{x^2}} dx = \int_1^5 \frac{\sqrt{x^2 + 1}}{x} dx \\ &= \int_a^b \frac{\sec \theta}{\tan \theta} \sec^2 \theta d\theta = \int_a^b \frac{\sec \theta}{\tan \theta} (1 + \tan^2 \theta) d\theta \\ &= \int_a^b (\csc \theta + \sec \theta \tan \theta) d\theta \\ &= \left[-\ln|\csc \theta + \cot \theta| + \sec \theta \right]_a^b \\ &= \left[-\ln \left| \frac{\sqrt{x^2 + 1}}{x} + \frac{1}{x} \right| + \sqrt{x^2 + 1} \right]_1^5 \\ &= \left[-\ln \left(\frac{\sqrt{26} + 1}{5} \right) + \sqrt{26} \right] - \left[-\ln(\sqrt{2} + 1) + \sqrt{2} \right] \\ &= \ln \left[\frac{5(\sqrt{2} + 1)}{\sqrt{26} + 1} \right] + \sqrt{26} - \sqrt{2} \approx 4.367 \text{ or } \ln \left[\frac{\sqrt{26} - 1}{5(\sqrt{2} - 1)} \right] + \sqrt{26} - \sqrt{2} \end{aligned}$$



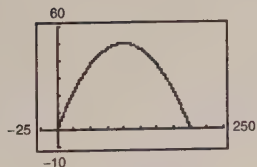
65. Length of one arch of sine curve: $y = \sin x, y' = \cos x$

$$L_1 = \int_0^\pi \sqrt{1 + \cos^2 x} dx$$

Length of one arch of cosine curve: $y = \cos x, y' = -\sin x$

$$\begin{aligned} L_2 &= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x} dx \\ &= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos^2 \left(x - \frac{\pi}{2} \right)} dx \quad u = x - \frac{\pi}{2}, du = dx \\ &= \int_{-\pi}^0 \sqrt{1 + \cos^2 u} du \\ &= \int_0^\pi \sqrt{1 + \cos^2 u} du = L_1 \end{aligned}$$

67. (a)



(b) $y = 0$ for $x = 200$ (range)

(c) $y = x - 0.005x^2, y' = 1 - 0.01x, 1 + (y')^2 = 1 + (1 - 0.01x)^2$

Let $u = 1 - 0.01x, du = -0.01 dx, a = 1$. (See Theorem 7.2.)

$$\begin{aligned} s &= \int_0^{200} \sqrt{1 + (1 - 0.01x)^2} dx = -100 \int_0^{200} \sqrt{(1 - 0.01x)^2 + 1} (-0.01) dx \\ &= -50 \left[(1 - 0.01x) \sqrt{(1 - 0.01x)^2 + 1} + \ln |(1 - 0.01x) + \sqrt{(1 - 0.01x)^2 + 1}| \right]_0^{200} \\ &= -50 \left[(-\sqrt{2} + \ln |-1 + \sqrt{2}|) - (\sqrt{2} + \ln |1 + \sqrt{2}|) \right] \\ &= 100\sqrt{2} + 50 \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) \approx 229.559 \end{aligned}$$

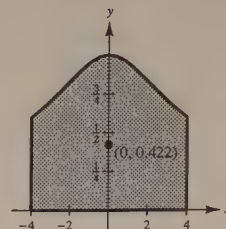
69. Let $x = 3 \tan \theta$, $dx = 3 \sec^2 \theta d\theta$, $\sqrt{x^2 + 9} = 3 \sec \theta$.

$$\begin{aligned} A &= 2 \int_0^4 \frac{3}{\sqrt{x^2 + 9}} dx = 6 \int_0^4 \frac{dx}{\sqrt{x^2 + 9}} = 6 \int_a^b \frac{3 \sec^2 \theta d\theta}{3 \sec \theta} \\ &= 6 \int_a^b \sec \theta d\theta = \left[6 \ln |\sec \theta + \tan \theta| \right]_a^b = \left[6 \ln \left| \frac{\sqrt{x^2 + 9} + x}{3} \right| \right]_0^4 = 6 \ln 3 \end{aligned}$$

$\bar{x} = 0$ (by symmetry)

$$\begin{aligned} \bar{y} &= \frac{1}{2} \left(\frac{1}{A} \right) \int_{-4}^4 \left(\frac{3}{\sqrt{x^2 + 9}} \right)^2 dx \\ &= \frac{9}{12 \ln 3} \int_{-4}^4 \frac{1}{x^2 + 9} dx \\ &= \frac{3}{4 \ln 3} \left[\frac{1}{3} \arctan \frac{x}{3} \right]_{-4}^4 \\ &= \frac{2}{4 \ln 3} \arctan \frac{4}{3} \approx 0.422 \end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{1}{2 \ln 3} \arctan \frac{4}{3} \right) \approx (0, 0.422)$$



71. $y = x^2$, $y' = 2x$, $1 + (y')^2 = 1 + 4x^2$

$$2x = \tan \theta, dx = \frac{1}{2} \sec^2 \theta d\theta, \sqrt{1 + 4x^2} = \sec \theta$$

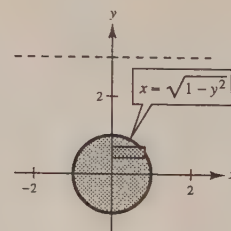
(For $\int \sec^5 \theta d\theta$ and $\int \sec^3 \theta d\theta$, see Exercise 90 in Section 7.3)

$$\begin{aligned} S &= 2\pi \int_0^{\sqrt{2}} x^2 \sqrt{1 + 4x^2} dx = 2\pi \int_a^b \left(\frac{\tan \theta}{2} \right)^2 (\sec \theta) \left(\frac{1}{2} \sec^2 \theta \right) d\theta \\ &= \frac{\pi}{4} \int_a^b \sec^3 \theta \tan^2 \theta d\theta = \frac{\pi}{4} \left[\int_a^b \sec^5 \theta d\theta - \int_a^b \sec^3 \theta d\theta \right] \\ &= \frac{\pi}{4} \left[\frac{1}{4} \left[\sec^3 \theta \tan \theta + \frac{3}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right] - \frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right] \Big|_a^b \\ &= \frac{\pi}{4} \left[\frac{1}{4} [(1 + 4x^2)^{3/2} (2x)] - \frac{1}{8} [(1 + 4x^2)^{1/2} (2x) + \ln |\sqrt{1 + 4x^2} + 2x|] \right] \Big|_0^{\sqrt{2}} \\ &= \frac{\pi}{4} \left[\frac{54\sqrt{2}}{4} - \frac{6\sqrt{2}}{8} - \frac{1}{8} \ln(3 + 2\sqrt{2}) \right] \\ &= \frac{\pi}{4} \left(\frac{51\sqrt{2}}{4} - \frac{\ln(3 + 2\sqrt{2})}{8} \right) = \frac{\pi}{32} [102\sqrt{2} - \ln(3 + 2\sqrt{2})] \approx 13.989 \end{aligned}$$

73. (a) Area of representative rectangle: $2\sqrt{1 - y^2} \Delta y$

Pressure: $2(62.4)(3 - y)\sqrt{1 - y^2} \Delta y$

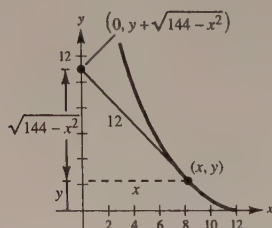
$$\begin{aligned} F &= 124.8 \int_{-1}^1 (3 - y)\sqrt{1 - y^2} dy \\ &= 124.8 \left[3 \int_{-1}^1 \sqrt{1 - y^2} dy - \int_{-1}^1 y\sqrt{1 - y^2} dy \right] \\ &= 124.8 \left[\frac{3}{2} (\arcsin y + y\sqrt{1 - y^2}) + \frac{1}{2} \left(\frac{2}{3} \right) (1 - y^2)^{3/2} \right]_{-1}^1 \\ &= (62.4)3[\arcsin 1 - \arcsin(-1)] = 187.2\pi \text{ lb} \end{aligned}$$



$$\begin{aligned} \text{(b) } F &= 124.8 \int_{-1}^1 (d - y)\sqrt{1 - y^2} dy = 124.8d \int_{-1}^1 \sqrt{1 - y^2} dy - 124.8 \int_{-1}^1 y\sqrt{1 - y^2} dy \\ &= 124.8 \left(\frac{d}{2} \right) \left[\arcsin y + y\sqrt{1 - y^2} \right]_{-1}^1 - 124.8(0) = 62.4\pi d \text{ lb} \end{aligned}$$

$$75. (a) m = \frac{dy}{dx} = \frac{y - (y + \sqrt{144 - x^2})}{x - 0}$$

$$= -\frac{\sqrt{144 - x^2}}{x}$$



$$(b) y = -\int \frac{\sqrt{144 - x^2}}{x} dx$$

$$\text{Let } x = 12 \sin \theta, dx = 12 \cos \theta d\theta, \sqrt{144 - x^2} = 12 \cos \theta.$$

$$y = -\int \frac{12 \cos \theta}{12 \sin \theta} 12 \cos \theta d\theta = -12 \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta$$

$$= -12 \int (\csc \theta - \sin \theta) d\theta = -12 \ln |\csc \theta - \cot \theta| - 12 \cos \theta + C$$

$$= -12 \ln \left| \frac{12}{x} - \frac{\sqrt{144 - x^2}}{x} \right| - 12 \left(\frac{\sqrt{144 - x^2}}{12} \right) + C$$

$$= -12 \ln \left| \frac{12 - \sqrt{144 - x^2}}{x} \right| - \sqrt{144 - x^2} + C$$

$$\text{When } x = 12, y = 0 \Rightarrow C = 0. \text{ Thus, } y = -12 \ln \left(\frac{12 - \sqrt{144 - x^2}}{x} \right) - \sqrt{144 - x^2}.$$

$$\text{Note: } \frac{12 - \sqrt{144 - x^2}}{x} > 0 \text{ for } 0 < x \leq 12$$

(c) Vertical asymptote: $x = 0$

$$(d) y + \sqrt{144 - x^2} = 12 \Rightarrow y = 12 - \sqrt{144 - x^2}$$

Thus,

$$12 - \sqrt{144 - x^2} = -12 \ln \left(\frac{12 - \sqrt{144 - x^2}}{x} \right) - \sqrt{144 - x^2}$$

$$-1 = \ln \left(\frac{12 - \sqrt{144 - x^2}}{x} \right)$$

$$xe^{-1} = 12 - \sqrt{144 - x^2}$$

$$(xe^{-1} - 12)^2 = (-\sqrt{144 - x^2})^2$$

$$x^2 e^{-2} - 24xe^{-1} + 144 = 144 - x^2$$

$$x^2(e^{-2} + 1) - 24xe^{-1} = 0$$

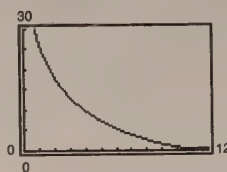
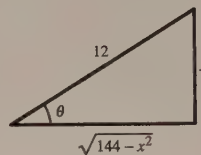
$$x[x(e^{-2} + 1) - 24e^{-1}] = 0$$

$$x = 0 \text{ or } x = \frac{24e^{-1}}{e^{-2} + 1} \approx 7.77665.$$

Therefore,

$$s = \int_{7.77665}^{12} \sqrt{1 + \left(-\frac{\sqrt{144 - x^2}}{x} \right)^2} dx = \int_{7.77665}^{12} \sqrt{\frac{x^2 + (144 - x^2)}{x^2}} dx$$

$$= \int_{7.77665}^{12} \frac{12}{x} dx = \left[12 \ln |x| \right]_{7.77665}^{12} = 12(\ln 12 - \ln 7.77665) \approx 5.2 \text{ meters.}$$



77. True

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos \theta d\theta}{\cos \theta} = \int d\theta$$

79. False

$$\int_0^{\sqrt{3}} \frac{dx}{(\sqrt{1+x^2})^3} = \int_0^{\pi/3} \frac{\sec^2 \theta d\theta}{\sec^3 \theta} = \int_0^{\pi/3} \cos \theta d\theta$$

81. Let $u = a \sin \theta$, $du = a \cos \theta d\theta$, $\sqrt{a^2 - u^2} = a \cos \theta$.

$$\begin{aligned} \int \sqrt{a^2 - u^2} du &= \int a^2 \cos^2 \theta d\theta = a^2 \int \frac{1 + \cos 2\theta}{2} d\theta \\ &= \frac{a^2}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C = \frac{a^2}{2} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{a^2}{2} \left[\arcsin \frac{u}{a} + \left(\frac{u}{a} \right) \left(\frac{\sqrt{a^2 - u^2}}{a} \right) \right] + C = \frac{1}{2} \left[a^2 \arcsin \frac{u}{a} + u \sqrt{a^2 - u^2} \right] + C \end{aligned}$$

Let $u = a \sec \theta$, $du = a \sec \theta \tan \theta d\theta$, $\sqrt{u^2 - a^2} = a \tan \theta$.

$$\begin{aligned} \int \sqrt{u^2 - a^2} du &= \int a \tan \theta (a \sec \theta \tan \theta) d\theta = a^2 \int \tan^2 \theta \sec \theta d\theta \\ &= a^2 \int (\sec^2 \theta - 1) \sec \theta d\theta = a^2 \int (\sec^3 \theta - \sec \theta) d\theta \\ &= a^2 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \int \sec \theta d\theta \right] - a^2 \int \sec \theta d\theta = a^2 \left[\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] \\ &= \frac{a^2}{2} \left[\frac{u}{a} \cdot \frac{\sqrt{u^2 - a^2}}{a} - \ln \left| \frac{u}{a} + \frac{\sqrt{u^2 - a^2}}{a} \right| \right] + C_1 \\ &= \frac{1}{2} [u \sqrt{u^2 - a^2} - a^2 \ln |u + \sqrt{u^2 - a^2}|] + C \end{aligned}$$

Let $u = a \tan \theta$, $du = a \sec^2 \theta d\theta$, $\sqrt{u^2 + a^2} = a \sec \theta d\theta$.

$$\begin{aligned} \int \sqrt{u^2 + a^2} du &= \int (a \sec \theta)(a \sec^2 \theta) d\theta \\ &= a^2 \int \sec^3 \theta d\theta = a^2 \left[\frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C_1 \\ &= \frac{a^2}{2} \left[\frac{\sqrt{u^2 + a^2}}{a} \cdot \frac{u}{a} + \ln \left| \frac{\sqrt{u^2 + a^2}}{a} + \frac{u}{a} \right| \right] + C_1 = \frac{1}{2} [u \sqrt{u^2 + a^2} + a^2 \ln |u + \sqrt{u^2 + a^2}|] + C \end{aligned}$$

Section 7.5 Partial Fractions

$$1. \frac{5}{x^2 - 10x} = \frac{5}{x(x-10)} = \frac{A}{x} + \frac{B}{x-10}$$

$$3. \frac{2x-3}{x^3+10x} = \frac{2x-3}{x(x^2+10)} = \frac{A}{x} + \frac{Bx+C}{x^2+10}$$

$$5. \frac{16x}{x^3-10x^2} = \frac{16x}{x^2(x-10)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-10}$$

$$7. \frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$1 = A(x-1) + B(x+1)$$

$$\text{When } x = -1, 1 = -2A, A = -\frac{1}{2}.$$

$$\text{When } x = 1, 1 = 2B, B = \frac{1}{2}.$$

$$\begin{aligned} \int \frac{1}{x^2-1} dx &= -\frac{1}{2} \int \frac{1}{x+1} dx + \frac{1}{2} \int \frac{1}{x-1} dx \\ &= -\frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + C \\ &= \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C \end{aligned}$$

$$9. \frac{3}{x^2 + x - 2} = \frac{3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$3 = A(x+2) + B(x-1)$$

When $x = 1$, $3 = 3A$, $A = 1$.

When $x = -2$, $3 = -3B$, $B = -1$.

$$\begin{aligned} \int \frac{3}{x^2 + x - 2} dx &= \int \frac{1}{x-1} dx - \int \frac{1}{x+2} dx \\ &= \ln|x-1| - \ln|x+2| + C \\ &= \ln\left|\frac{x-1}{x+2}\right| + C \end{aligned}$$

$$11. \frac{5-x}{2x^2+x-1} = \frac{5-x}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1}$$

$$5-x = A(x+1) + B(2x-1)$$

When $x = \frac{1}{2}$, $\frac{9}{2} = \frac{3}{2}A$, $A = 3$.

When $x = -1$, $6 = -3B$, $B = -2$.

$$\begin{aligned} \int \frac{5-x}{2x^2+x-1} dx &= 3 \int \frac{1}{2x-1} dx - 2 \int \frac{1}{x+1} dx \\ &= \frac{3}{2} \ln|2x-1| - 2 \ln|x+1| + C \end{aligned}$$

$$13. \frac{x^2 + 12x + 12}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

$$x^2 + 12x + 12 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$$

When $x = 0$, $12 = -4A$, $A = -3$. When $x = -2$, $-8 = 8B$, $B = -1$. When $x = 2$, $40 = 8C$, $C = 5$.

$$\begin{aligned} \int \frac{x^2 + 12x + 12}{x^3 - 4x} dx &= 5 \int \frac{1}{x-2} dx - \int \frac{1}{x+2} dx - 3 \int \frac{1}{x} dx \\ &= 5 \ln|x-2| - \ln|x+2| - 3 \ln|x| + C \end{aligned}$$

$$15. \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} = 2x + \frac{x+5}{(x-4)(x+2)} = 2x + \frac{A}{x-4} + \frac{B}{x+2}$$

$$x+5 = A(x+2) + B(x-4)$$

When $x = 4$, $9 = 6A$, $A = \frac{3}{2}$. When $x = -2$, $3 = -6B$, $B = -\frac{1}{2}$.

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} dx &= \int \left[2x + \frac{3/2}{x-4} - \frac{1/2}{x+2} \right] dx \\ &= x^2 + \frac{3}{2} \ln|x-4| - \frac{1}{2} \ln|x+2| + C \end{aligned}$$

$$17. \frac{4x^2 + 2x - 1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$4x^2 + 2x - 1 = Ax(x+1) + B(x+1) + Cx^2$$

When $x = 0$, $B = -1$. When $x = -1$, $C = 1$. When $x = 1$, $A = 3$.

$$\begin{aligned} \int \frac{4x^2 + 2x - 1}{x^3 + x^2} dx &= \int \left[\frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1} \right] dx = 3 \ln|x| + \frac{1}{x} + \ln|x+1| + C \\ &= \frac{1}{x} + \ln|x^4 + x^3| + C \end{aligned}$$

$$19. \frac{x^2 + 3x - 4}{x^3 - 4x^2 + 4x} = \frac{x^2 + 3x - 4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$x^2 + 3x - 4 = A(x-2)^2 + Bx(x-2) + Cx$$

When $x = 0$, $-4 = 4A \Rightarrow A = -1$. When $x = 2$, $6 = 2C \Rightarrow C = 3$. When $x = 1$, $0 = -1 - B + 3 \Rightarrow B = 2$.

$$\begin{aligned} \int \frac{x^2 + 3x - 4}{x^3 - 4x^2 + 4x} dx &= \int \frac{-1}{x} dx + \int \frac{2}{(x-2)} dx + \int \frac{3}{(x-2)^2} dx \\ &= -\ln|x| + 2 \ln|x-2| - \frac{3}{(x-2)} + C \end{aligned}$$

$$21. \frac{x^2 - 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$$x^2 - 1 = A(x^2 + 1) + (Bx + C)x$$

When $x = 0$, $A = -1$. When $x = 1$, $0 = -2 + B + C$. When $x = -1$, $0 = -2 + B - C$.

Solving these equations we have $A = -1$, $B = 2$, $C = 0$.

$$\begin{aligned} \int \frac{x^2 - 1}{x^3 + x} dx &= -\int \frac{1}{x} dx + \int \frac{2x}{x^2 + 1} dx \\ &= \ln|x^2 + 1| - \ln|x| + C \\ &= \ln\left|\frac{x^2 + 1}{x}\right| + C \end{aligned}$$

$$23. \frac{x^2}{x^4 - 2x^2 - 8} = \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{Cx + D}{x^2 + 2}$$

$$x^2 = A(x + 2)(x^2 + 2) + B(x - 2)(x^2 + 2) + (Cx + D)(x + 2)(x - 2)$$

When $x = 2$, $4 = 24A$. When $x = -2$, $4 = -24B$. When $x = 0$, $0 = 4A - 4B - 4D$, and when $x = 1$, $1 = 9A - 3B - 3C - 3D$. Solving these equations we have $A = \frac{1}{6}$, $B = -\frac{1}{6}$, $C = 0$, $D = \frac{1}{3}$.

$$\begin{aligned} \int \frac{x^2}{x^4 - 2x^2 - 8} dx &= \frac{1}{6} \left[\int \frac{1}{x - 2} dx - \int \frac{1}{x + 2} dx + 2 \int \frac{1}{x^2 + 2} dx \right] \\ &= \frac{1}{6} \left[\ln\left|\frac{x - 2}{x + 2}\right| + \sqrt{2} \arctan \frac{x}{\sqrt{2}} \right] + C \end{aligned}$$

$$25. \frac{x}{(2x - 1)(2x + 1)(4x^2 + 1)} = \frac{A}{2x - 1} + \frac{B}{2x + 1} + \frac{Cx + D}{4x^2 + 1}$$

$$x = A(2x + 1)(4x^2 + 1) + B(2x - 1)(4x^2 + 1) + (Cx + D)(2x - 1)(2x + 1)$$

When $x = \frac{1}{2}$, $\frac{1}{2} = 4A$. When $x = -\frac{1}{2}$, $-\frac{1}{2} = -4B$. When $x = 0$, $0 = A - B - D$, and when $x = 1$, $1 = 15A + 5B + 3C + 3D$. Solving these equations we have $A = \frac{1}{8}$, $B = \frac{1}{8}$, $C = -\frac{1}{2}$, $D = 0$.

$$\begin{aligned} \int \frac{x}{16x^4 - 1} dx &= \frac{1}{8} \left[\int \frac{1}{2x - 1} dx + \int \frac{1}{2x + 1} dx - 4 \int \frac{x}{4x^2 + 1} dx \right] \\ &= \frac{1}{16} \ln\left|\frac{4x^2 - 1}{4x^2 + 1}\right| + C \end{aligned}$$

$$27. \frac{x^2 + 5}{(x + 1)(x^2 - 2x + 3)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - 2x + 3}$$

$$x^2 + 5 = A(x^2 - 2x + 3) + (Bx + C)(x + 1)$$

$$= (A + B)x^2 + (-2A + B + C)x + (3A + C)$$

When $x = -1$, $A = 1$. By equating coefficients of like terms, we have $A + B = 1$, $-2A + B + C = 0$, $3A + C = 5$. Solving these equations we have $A = 1$, $B = 0$, $C = 2$.

$$\begin{aligned} \int \frac{x^2 + 5}{x^3 - x^2 + x + 3} dx &= \int \frac{1}{x + 1} dx + 2 \int \frac{1}{(x - 1)^2 + 2} dx \\ &= \ln|x + 1| + \sqrt{2} \arctan\left(\frac{x - 1}{\sqrt{2}}\right) + C \end{aligned}$$

$$29. \frac{3}{(2x+1)(x+2)} = \frac{A}{2x+1} + \frac{B}{x+2}$$

$$3 = A(x+2) + B(2x+1)$$

When $x = -\frac{1}{2}$, $A = 2$. When $x = -2$, $B = -1$.

$$\begin{aligned} \int_0^1 \frac{3}{2x^2 + 5x + 2} dx &= \int_0^1 \frac{2}{2x+1} dx - \int_0^1 \frac{1}{x+2} dx \\ &= \left[\ln|2x+1| - \ln|x+2| \right]_0^1 \\ &= \ln 2 \end{aligned}$$

$$31. \frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

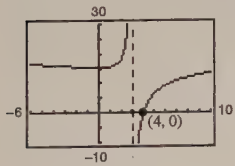
$$x+1 = A(x^2+1) + (Bx+C)x$$

When $x = 0$, $A = 1$. When $x = 1$, $2 = 2A + B + C$. When $x = -1$, $0 = 2A + B - C$. Solving these equations we have $A = 1$, $B = -1$, $C = 1$.

$$\begin{aligned} \int_1^2 \frac{x+1}{x(x^2+1)} dx &= \int_1^2 \frac{1}{x} dx - \int_1^2 \frac{x}{x^2+1} dx + \int_1^2 \frac{1}{x^2+1} dx \\ &= \left[\ln|x| - \frac{1}{2} \ln(x^2+1) + \arctan x \right]_1^2 \\ &= \frac{1}{2} \ln \frac{8}{5} - \frac{\pi}{4} + \arctan 2 \\ &\approx 0.557 \end{aligned}$$

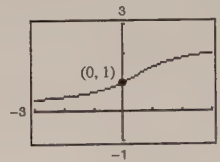
$$33. \int \frac{3x dx}{x^2 - 6x + 9} = 3 \ln|x-3| - \frac{9}{x-3} + C$$

$$(4, 0): 3 \ln|4-3| - \frac{9}{4-3} + C = 0 \Rightarrow C = 9$$



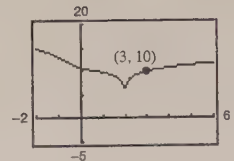
$$35. \int \frac{x^2 + x + 2}{(x^2 + 2)^2} dx = \frac{\sqrt{2}}{2} \arctan \frac{x}{\sqrt{2}} - \frac{1}{2(x^2 + 2)} + C$$

$$(0, 1): 0 - \frac{1}{4} + C = 1 \Rightarrow C = \frac{5}{4}$$



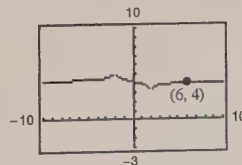
$$37. \int \frac{2x^2 - 2x + 3}{x^3 - x^2 - x - 2} dx = \ln|x-2| + \frac{1}{2} \ln|x^2 + x + 1| - \sqrt{3} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$(3, 10): 0 + \frac{1}{2} \ln 13 - \sqrt{3} \arctan \frac{7}{\sqrt{3}} + C = 10 \Rightarrow C = 10 - \frac{1}{2} \ln 13 + \sqrt{3} \arctan \frac{7}{\sqrt{3}}$$



$$39. \int \frac{1}{x^2 - 4} dx = \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C$$

$$(6, 4): \frac{1}{4} \ln \left| \frac{4}{8} \right| + C = 4 \Rightarrow C = 4 - \frac{1}{4} \ln \frac{1}{2} = 4 + \frac{1}{4} \ln 2$$



41. Let
- $u = \cos x$
- $du = -\sin x dx$
- .

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

When $u = 0$, $A = -1$. When $u = 1$, $B = 1$, $u = \cos x$,
 $du = -\sin x dx$.

$$\int \frac{\sin x}{\cos x(\cos x - 1)} dx = - \int \frac{1}{u(u-1)} du$$

$$= \int \frac{1}{u} du - \int \frac{1}{u-1} du$$

$$= \ln|u| - \ln|u-1| + C$$

$$= \ln \left| \frac{u}{u-1} \right| + C$$

$$= \ln \left| \frac{\cos x}{\cos x - 1} \right| + C$$

45. Let
- $u = e^x$
- ,
- $du = e^x dx$
- .

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$1 = A(u+4) + B(u-1)$$

When $u = 1$, $A = \frac{1}{5}$. When $u = -4$, $B = -\frac{1}{5}$, $u = e^x$,
 $du = e^x dx$.

$$\int \frac{e^x}{(e^x-1)(e^x+4)} dx = \int \frac{1}{(u-1)(u+4)} du$$

$$= \frac{1}{5} \left(\int \frac{1}{u-1} du - \int \frac{1}{u+4} du \right)$$

$$= \frac{1}{5} \ln \left| \frac{u-1}{u+4} \right| + C$$

$$= \frac{1}{5} \ln \left| \frac{e^x-1}{e^x+4} \right| + C$$

- 49.
- $\frac{x}{(a+bx)^2} = \frac{A}{a+bx} + \frac{B}{(a+bx)^2}$

$$x = A(a+bx) + B$$

When $x = -a/b$, $B = -a/b$.

When $x = 0$, $0 = aA + B \Rightarrow A = 1/b$.

$$\int \frac{x}{(a+bx)^2} dx = \int \left(\frac{1/b}{a+bx} + \frac{-a/b}{(a+bx)^2} \right) dx$$

$$= \frac{1}{b} \int \frac{1}{a+bx} dx - \frac{a}{b} \int \frac{1}{(a+bx)^2} dx$$

$$= \frac{1}{b^2} \ln|a+bx| + \frac{a}{b^2} \left(\frac{1}{a+bx} \right) + C$$

$$= \frac{1}{b^2} \left(\frac{a}{a+bx} + \ln|a+bx| \right) + C$$

$$43. \int \frac{3 \cos x}{\sin^2 x + \sin x - 2} dx = 3 \int \frac{1}{u^2 + u - 2} du$$

$$= \ln \left| \frac{u-1}{u+2} \right| + C$$

$$= \ln \left| \frac{-1 + \sin x}{2 + \sin x} \right| + C$$

(From Exercise 9 with $u = \sin x$, $du = \cos x dx$)

$$47. \frac{1}{x(a+bx)} = \frac{A}{x} + \frac{B}{a+bx}$$

$$1 = A(a+bx) + Bx$$

When $x = 0$, $1 = aA \Rightarrow A = 1/a$.

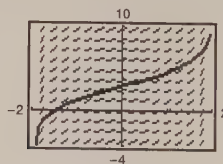
When $x = -a/b$, $1 = -(a/b)B \Rightarrow B = -b/a$.

$$\int \frac{1}{x(a+bx)} dx = \frac{1}{a} \int \left(\frac{1}{x} - \frac{b}{a+bx} \right) dx$$

$$= \frac{1}{a} (\ln|x| - \ln|a+bx|) + C$$

$$= \frac{1}{a} \ln \left| \frac{x}{a+bx} \right| + C$$

- 51.
- $\frac{dy}{dx} = \frac{6}{4-x^2}$
- ,
- $y(0) = 3$



53. Dividing x^3 by $x - 5$.

55. (a) Substitution: $u = x^2 + 2x - 8$

(b) Partial fractions

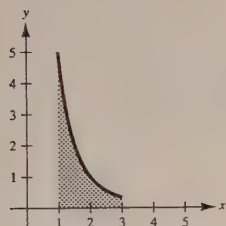
(c) Trigonometric substitution (tan) or inverse tangent rule

$$\begin{aligned}
 57. \text{ Average Cost} &= \frac{1}{80 - 75} \int_{75}^{80} \frac{124p}{(10 + p)(100 - p)} dp \\
 &= \frac{1}{5} \int_{75}^{80} \left(\frac{-124}{(10 + p)11} + \frac{1240}{(100 - p)11} \right) dp \\
 &= \frac{1}{5} \left[\frac{-124}{11} \ln(10 + p) - \frac{1240}{11} \ln(100 - p) \right]_{75}^{80} \\
 &\approx \frac{1}{5} (24.51) = 4.9
 \end{aligned}$$

Approximately \$490,000.

$$59. A = \int_1^3 \frac{10}{x(x^2 + 1)} dx \approx 3$$

Matches (c)



$$61. \frac{1}{(x+1)(n-x)} = \frac{A}{x+1} + \frac{B}{n-x}, A = B = \frac{1}{n+1}$$

$$\frac{1}{n+1} \int \left(\frac{1}{x+1} + \frac{1}{n-x} \right) dx = kt + C$$

$$\frac{1}{n+1} \ln \left| \frac{x+1}{n-x} \right| = kt + C$$

$$\text{When } t = 0, x = 0, C = \frac{1}{n+1} \ln \frac{1}{n}$$

$$\frac{1}{n+1} \ln \left| \frac{x+1}{n-x} \right| = kt + \frac{1}{n+1} \ln \frac{1}{n}$$

$$\frac{1}{n+1} \left[\ln \left| \frac{x+1}{n-x} \right| - \ln \frac{1}{n} \right] = kt$$

$$\ln \frac{nx + n}{n - x} = (n+1)kt$$

$$\frac{nx + n}{n - x} = e^{(n+1)kt}$$

$$x = \frac{n[e^{(n+1)kt} - 1]}{n + e^{(n+1)kt}} \quad \text{Note: } \lim_{t \rightarrow \infty} x = n$$

$$\begin{aligned}
 63. \quad \frac{x}{1+x^4} &= \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} \\
 x &= (Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1) \\
 &= (A+C)x^3 + (B+D-\sqrt{2}A+\sqrt{2}C)x^2 + (A+C-\sqrt{2}B+\sqrt{2}D)x + (B+D)
 \end{aligned}$$

$$0 = A + C \Rightarrow C = -A$$

$$0 = B + D - \sqrt{2}A + \sqrt{2}C \quad \left\{ \begin{array}{l} -2\sqrt{2}A = 0 \Rightarrow A = 0 \text{ and } C = 0 \end{array} \right.$$

$$1 = A + C - \sqrt{2}B + \sqrt{2}D \quad \left\{ \begin{array}{l} -2\sqrt{2}B = 1 \Rightarrow B = -\frac{\sqrt{2}}{4} \text{ and } D = \frac{\sqrt{2}}{4} \end{array} \right.$$

$$0 = B + D \Rightarrow D = -B$$

Thus,

$$\begin{aligned}
 \int_0^1 \frac{x}{1+x^4} dx &= \int_0^1 \left[\frac{-\sqrt{2}/4}{x^2+\sqrt{2}x+1} + \frac{\sqrt{2}/4}{x^2-\sqrt{2}x+1} \right] dx \\
 &= \frac{\sqrt{2}}{4} \int_0^1 \left[\frac{-1}{[x+(\sqrt{2}/2)]^2+(1/2)} + \frac{1}{[x-(\sqrt{2}/2)]^2+(1/2)} \right] dx \\
 &= \frac{\sqrt{2}}{4} \cdot \frac{1}{1/\sqrt{2}} \left[-\arctan\left(\frac{x+(\sqrt{2}/2)}{1/\sqrt{2}}\right) + \arctan\left(\frac{x-(\sqrt{2}/2)}{1/\sqrt{2}}\right) \right]_0^1 \\
 &= \frac{1}{2} \left[-\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x-1) \right]_0^1 \\
 &= \frac{1}{2} [(-\arctan(\sqrt{2}+1) + \arctan(\sqrt{2}-1)) - (-\arctan 1 + \arctan(-1))] \\
 &= \frac{1}{2} \left[\arctan(\sqrt{2}-1) - \arctan(\sqrt{2}+1) + \frac{\pi}{4} + \frac{\pi}{4} \right].
 \end{aligned}$$

Since $\arctan x - \arctan y = \arctan[(x-y)/(1+xy)]$, we have:

$$\int_0^1 \frac{x}{1+x^4} dx = \frac{1}{2} \left[\arctan\left(\frac{(\sqrt{2}-1) - (\sqrt{2}+1)}{1 + (\sqrt{2}-1)(\sqrt{2}+1)}\right) + \frac{\pi}{2} \right] = \frac{1}{2} \left[\arctan\left(\frac{-2}{2}\right) + \frac{\pi}{2} \right] = \frac{1}{2} \left[-\frac{\pi}{4} + \frac{\pi}{2} \right] = \frac{\pi}{8}$$

Section 7.6 Integration by Tables and Other Integration Techniques

1. By Formula 6: $\int \frac{x^2}{1+x} dx = -\frac{x}{2}(2-x) + \ln|1+x| + C$

3. By Formula 26: $\int e^x \sqrt{1+e^{2x}} dx = \frac{1}{2} [e^x \sqrt{e^{2x}+1} + \ln|e^x + \sqrt{e^{2x}+1}|] + C$
 $u = e^x, du = e^x dx$

5. By Formula 44: $\int \frac{1}{x^2 \sqrt{1-x^2}} dx = -\frac{\sqrt{1-x^2}}{x} + C$

7. By Formulas 50 and 48: $\int \sin^4(2x) dx = \frac{1}{2} \int \sin^4(2x)(2) dx$

$$= \frac{1}{2} \left[\frac{-\sin^3(2x) \cos(2x)}{4} + \frac{3}{4} \int \sin^2(2x)(2) dx \right]$$

$$= \frac{1}{2} \left[\frac{-\sin^3(2x) \cos(2x)}{4} + \frac{3}{8} (2x - \sin 2x \cos 2x) \right] + C$$

$$= \frac{1}{16} (6x - 3 \sin 2x \cos 2x - 2 \sin^3 2x \cos 2x) + C$$

9. By Formula 57: $\int \frac{1}{\sqrt{x}(1 - \cos \sqrt{x})} dx = 2 \int \frac{1}{1 - \cos \sqrt{x}} \left(\frac{1}{2\sqrt{x}} \right) dx$

$$= -2(\cot \sqrt{x} + \csc \sqrt{x}) + C$$

$$u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx$$

11. By Formula 84:

$$\int \frac{1}{1 + e^{2x}} dx = x - \frac{1}{2} \ln(1 + e^{2x}) + C$$

13. By Formula 89:

$$\int x^3 \ln x dx = \frac{x^4}{16} (4 \ln|x| - 1) + C$$

15. (a) By Formulas 83 and 82: $\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$

$$= x^2 e^x - 2[(x - 1)e^x + C_1]$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

(b) Integration by parts: $u = x^2, du = 2x dx, dv = e^x dx, v = e^x$

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$$

Parts again: $u = 2x, du = 2 dx, dv = e^x dx, v = e^x$

$$\int x^2 e^x dx = x^2 e^x - \left[2x e^x - \int 2e^x dx \right] = x^2 e^x - 2x e^x + 2e^x + C$$

17. (a) By Formula 12, $a = b = 1, u = x$, and

$$\int \frac{1}{x^2(x+1)} dx = \frac{-1}{1} \left(\frac{1}{x} + \frac{1}{1} \ln \left| \frac{x}{1+x} \right| \right) + C$$

$$= \frac{-1}{x} - \ln \left| \frac{x}{1+x} \right| + C$$

$$= \frac{-1}{x} + \ln \left| \frac{x+1}{x} \right| + C$$

(b) Partial fractions:

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$1 = Ax(x+1) + B(x+1) + Cx^2$$

$$x = 0: 1 = B$$

$$x = -1: 1 = C$$

$$x = 1: 1 = 2A + 2 + 1 \Rightarrow A = -1$$

$$\int \frac{1}{x^2(x+1)} dx = \int \left[\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right] dx$$

$$= -\ln|x| - \frac{1}{x} + \ln|x+1| + C$$

$$= -\frac{1}{x} - \ln \left| \frac{x}{x+1} \right| + C$$

19. By Formula 81: $\int x e^{x^2} = \frac{1}{2} e^{x^2} + C$

21. By Formula 79: $\int x \operatorname{arcsec}(x^2 + 1) dx = \frac{1}{2} \int \operatorname{arcsec}(x^2 + 1)(2x) dx$
 $= \frac{1}{2} [(x^2 + 1) \operatorname{arcsec}(x^2 + 1) - \ln((x^2 + 1) + \sqrt{x^4 + 2x^2})] + C$
 $u = x^2 + 1, du = 2x dx$

23. By Formula 89: $\int x^2 \ln x dx = \frac{x^3}{9}(-1 + 3 \ln|x|) + C$

25. By Formula 35: $\int \frac{1}{x^2 \sqrt{x^2 - 4}} dx = \frac{\sqrt{x^2 - 4}}{4x} + C$

27. By Formula 4: $\int \frac{2x}{(1 - 3x)^2} dx = 2 \int \frac{x}{(1 - 3x)^2} dx = \frac{2}{9} \left(\ln|1 - 3x| + \frac{1}{1 - 3x} \right) + C$

29. By Formula 76:

$$\int e^x \arccos e^x dx = e^x \arccos e^x - \sqrt{1 - e^{2x}} + C$$

$$u = e^x, du = e^x dx$$

31. By Formula 73:

$$\int \frac{x}{1 - \sec x^2} dx = \frac{1}{2} \int \frac{2x}{1 - \sec x^2} dx$$

$$= \frac{1}{2} (x^2 + \cot x^2 + \csc x^2) + C$$

33. By Formula 23: $\int \frac{\cos x}{1 + \sin^2 x} dx = \arctan(\sin x) + C$
 $u = \sin x, du = \cos x dx$

35. By Formula 14: $\int \frac{\cos \theta}{3 + 2 \sin \theta + \sin^2 \theta} d\theta = \frac{\sqrt{2}}{2} \arctan\left(\frac{1 + \sin \theta}{\sqrt{2}}\right) + C$
 $u = \sin \theta, du = \cos \theta d\theta$

37. By Formula 35: $\int \frac{1}{x^2 \sqrt{2 + 9x^2}} dx = 3 \int \frac{3}{(3x)^2 \sqrt{(\sqrt{2})^2 + (3x)^2}} dx$
 $= -\frac{3\sqrt{2 + 9x^2}}{6x} + C$
 $= -\frac{\sqrt{2 + 9x^2}}{2x} + C$

39. By Formulas 54 and 55:

$$\int t^3 \cos t dt = t^3 \sin t - 3 \int t^2 \sin t dt$$

$$= t^3 \sin t - 3 \left[-t^2 \cos t + 2 \int t \cos t dt \right]$$

$$= t^3 \sin t + 3t^2 \cos t - 6 \left[t \sin t - \int \sin t dt \right]$$

$$= t^3 \sin t + 3t^2 \cos t - 6t \sin t - 6 \cos t + C$$

41. By Formula 3: $\int \frac{\ln x}{x(3 + 2 \ln x)} dx = \frac{1}{4}(2 \ln|x| - 3 \ln|3 + 2 \ln|x||) + C$

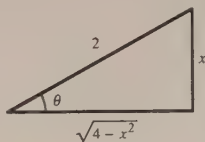
$$u = \ln x, du = \frac{1}{x} dx$$

43. By Formulas 1, 25, and 33:
$$\begin{aligned} \int \frac{x}{(x^2 - 6x + 10)^2} dx &= \frac{1}{2} \int \frac{2x - 6 + 6}{(x^2 - 6x + 10)^2} dx \\ &= \frac{1}{2} \int (x^2 - 6x + 10)^{-2} (2x - 6) dx + 3 \int \frac{1}{[(x - 3)^2 + 1]^2} dx \\ &= -\frac{1}{2(x^2 - 6x + 10)} + \frac{3}{2} \left[\frac{x - 3}{x^2 - 6x + 10} + \arctan(x - 3) \right] + C \\ &= \frac{3x - 10}{2(x^2 - 6x + 10)} + \frac{3}{2} \arctan(x - 3) + C \end{aligned}$$

45. By Formula 31:
$$\begin{aligned} \int \frac{x}{\sqrt{x^4 - 6x^2 + 5}} dx &= \frac{1}{2} \int \frac{2x}{\sqrt{(x^2 - 3)^2 - 4}} dx \\ &= \frac{1}{2} \ln|x^2 - 3 + \sqrt{x^4 - 6x^2 + 5}| + C \end{aligned}$$

$$u = x^2 - 3, du = 2x dx$$

47.
$$\begin{aligned} \int \frac{x^3}{\sqrt{4 - x^2}} dx &= \int \frac{8 \sin^3 \theta (2 \cos \theta d\theta)}{2 \cos \theta} \\ &= 8 \int (1 - \cos^2 \theta) \sin \theta d\theta \\ &= 8 \int [\sin \theta - \cos^2 \theta (\sin \theta)] d\theta \\ &= -8 \cos \theta + \frac{8 \cos^3 \theta}{3} + C \\ &= \frac{-\sqrt{4 - x^2}}{3} (x^2 + 8) + C \end{aligned}$$



$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4 - x^2} = 2 \cos \theta$$

49. By Formula 8:
$$\begin{aligned} \int \frac{e^{3x}}{(1 + e^x)^3} dx &= \int \frac{(e^x)^2}{(1 + e^x)^3} (e^x) dx \\ &= \frac{2}{1 + e^x} - \frac{1}{2(1 + e^x)^2} + \ln|1 + e^x| + C \end{aligned}$$

$$u = e^x, du = e^x dx$$

51.
$$\begin{aligned} \frac{u^2}{(a + bu)^2} &= \frac{1}{b^2} - \frac{(2a/b)u + (a^2/b^2)}{(a + bu)^2} = \frac{1}{b^2} + \frac{A}{a + bu} + \frac{B}{(a + bu)^2} \\ -\frac{2a}{b}u - \frac{a^2}{b^2} &= A(a + bu) + B = (aA + B) + bAu \end{aligned}$$

Equating the coefficients of like terms we have $aA + B = -a^2/b^2$ and $bA = -2a/b$. Solving these equations we have $A = -2a/b^2$ and $B = a^2/b^2$.

$$\begin{aligned} \int \frac{u^2}{(a + bu)^2} du &= \frac{1}{b^2} \int du - \frac{2a(1/b)}{b^2} \int \frac{1}{a + bu} b du + \frac{a^2(1/b)}{b^2} \int \frac{1}{(a + bu)^2} b du = \frac{1}{b^2} u - \frac{2a}{b^3} \ln|a + bu| - \frac{a^2}{b^3} \left(\frac{1}{a + bu} \right) + C \\ &= \frac{1}{b^3} \left(bu - \frac{a^2}{a + bu} - 2a \ln|a + bu| \right) + C \end{aligned}$$

53. When we have $u^2 + a^2$:

$$u = a \tan \theta$$

$$du = a \sec^2 \theta d\theta$$

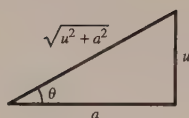
$$u^2 + a^2 = a^2 \sec^2 \theta$$

$$\int \frac{1}{(u^2 + a^2)^{3/2}} du = \int \frac{a \sec^2 \theta d\theta}{a^3 \sec^3 \theta}$$

$$= \frac{1}{a^2} \int \cos \theta d\theta$$

$$= \frac{1}{a^2} \sin \theta + C$$

$$= \frac{u}{a^2 \sqrt{u^2 + a^2}} + C$$



When we have $u^2 - a^2$:

$$u = a \sec \theta$$

$$du = a \sec \theta \tan \theta d\theta$$

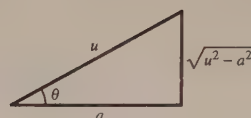
$$u^2 - a^2 = a^2 \tan^2 \theta$$

$$\int \frac{1}{(u^2 - a^2)^{3/2}} du = \int \frac{a \sec \theta \tan \theta d\theta}{a^3 \tan^3 \theta}$$

$$= \frac{1}{a^2} \int \frac{\cos \theta}{\sin^2 \theta} d\theta$$

$$= -\frac{1}{a^2} \csc \theta + C$$

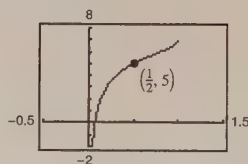
$$= \frac{-u}{a^2 \sqrt{u^2 - a^2}} + C$$



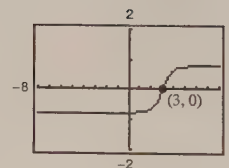
$$\begin{aligned} 55. \int (\arctan u) du &= u \arctan u - \frac{1}{2} \int \frac{2u}{1 + u^2} du \\ &= u \arctan u - \frac{1}{2} \ln(1 + u^2) + C \\ &= u \arctan u - \ln \sqrt{1 + u^2} + C \end{aligned}$$

$$w = \arctan u, dv = du, dw = \frac{du}{1 + u^2}, v = u$$

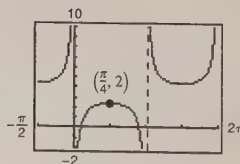
$$\begin{aligned} 57. \int \frac{1}{x^{3/2} \sqrt{1-x}} dx &= \frac{-2\sqrt{1-x}}{\sqrt{x}} + C \\ \left(\frac{1}{2}, 5\right): \frac{-2\sqrt{1/2}}{\sqrt{1/2}} + C &= 5 \Rightarrow C = 7 \\ y &= \frac{-2\sqrt{1-x}}{\sqrt{x}} + 7 \end{aligned}$$



$$\begin{aligned} 59. \int \frac{1}{(x^2 - 6x + 10)^2} dx &= \frac{1}{2} \left[\tan^{-1}(x-3) + \frac{x-3}{x^2 - 6x + 10} \right] + C \\ (3, 0): \frac{1}{2} \left[0 + \frac{0}{10} \right] + C &= 0 \Rightarrow C = 0 \\ y &= \frac{1}{2} \left[\tan^{-1}(x-3) + \frac{x-3}{x^2 - 6x + 10} \right] \end{aligned}$$



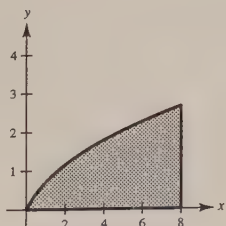
$$\begin{aligned} 61. \int \frac{1}{\sin \theta \tan \theta} d\theta &= -\csc \theta + C \\ \left(\frac{\pi}{4}, 2\right): -\frac{2}{\sqrt{2}} + C &= 2 \Rightarrow C = 2 + \sqrt{2} \\ y &= -\csc \theta + 2 + \sqrt{2} \end{aligned}$$



$$\begin{aligned}
 63. \int \frac{1}{2 - 3 \sin \theta} d\theta &= \int \left[\frac{\frac{2 du}{1 + u^2}}{2 - 3 \left(\frac{2u}{1 + u^2} \right)} \right] \\
 &= \int \frac{2}{2(1 + u^2) - 6u} du \\
 &= \int \frac{1}{u^2 - 3u + 1} du \\
 &= \int \frac{1}{\left(u - \frac{3}{2}\right)^2 - \frac{5}{4}} du \\
 &= \frac{1}{\sqrt{5}} \ln \left| \frac{\left(u - \frac{3}{2}\right) - \frac{\sqrt{5}}{2}}{\left(u - \frac{3}{2}\right) + \frac{\sqrt{5}}{2}} \right| + C \\
 &= \frac{1}{\sqrt{5}} \ln \left| \frac{2u - 3 - \sqrt{5}}{2u - 3 + \sqrt{5}} \right| + C \\
 &= \frac{1}{\sqrt{5}} \ln \left| \frac{2 \tan\left(\frac{\theta}{2}\right) - 3 - \sqrt{5}}{2 \tan\left(\frac{\theta}{2}\right) - 3 + \sqrt{5}} \right| + C \\
 u &= \tan \frac{\theta}{2}
 \end{aligned}$$

$$\begin{aligned}
 67. \int \frac{\sin \theta}{3 - 2 \cos \theta} d\theta &= \frac{1}{2} \int \frac{2 \sin \theta}{3 - 2 \cos \theta} d\theta \\
 &= \frac{1}{2} \ln |u| + C \\
 &= \frac{1}{2} \ln(3 - 2 \cos \theta) + C \\
 u &= 3 - 2 \cos \theta, du = 2 \sin \theta d\theta
 \end{aligned}$$

$$\begin{aligned}
 71. A &= \int_0^8 \frac{x}{\sqrt{x+1}} dx \\
 &= \left[\frac{-2(2-x)}{3} \sqrt{x+1} \right]_0^8 \\
 &= 12 - \left(-\frac{4}{3} \right) \\
 &= \frac{40}{3} \approx 13.333 \text{ square units}
 \end{aligned}$$



$$\begin{aligned}
 65. \int_0^{\pi/2} \frac{1}{1 + \sin \theta + \cos \theta} d\theta &= \int_0^1 \left[\frac{\frac{2 du}{1 + u^2}}{1 + \frac{2u}{1 + u^2} + \frac{1 - u^2}{1 + u^2}} \right] \\
 &= \int_0^1 \frac{1}{1 + u} du \\
 &= \left[\ln |1 + u| \right]_0^1 \\
 &= \ln 2 \\
 u &= \tan \frac{\theta}{2}
 \end{aligned}$$

$$\begin{aligned}
 69. \int \frac{\cos \sqrt{\theta}}{\sqrt{\theta}} d\theta &= 2 \int \cos \sqrt{\theta} \left(\frac{1}{2\sqrt{\theta}} \right) d\theta \\
 &= 2 \sin \sqrt{\theta} + C \\
 u &= \sqrt{\theta}, du = \frac{1}{2\sqrt{\theta}} d\theta
 \end{aligned}$$

73. Arctangent Formula, Formula 23,

$$\int \frac{1}{u^2 + 1} du, u = e^x$$

 75. Substitution: $u = x^2, du = 2x dx$
 Then Formula 81.

77. Cannot be integrated.

79. Answers will vary. For example,

$$\int (2x)e^{2x} dx$$

 can be integrated by first letting
 $u = 2x$ and then using Formula 82.

$$81. W = \int_0^5 2000xe^{-x} dx$$

$$= -2000 \int_0^5 -xe^{-x} dx$$

$$= 2000 \int_0^5 (-x)e^{-x}(-1) dx$$

$$= 2000 \left[(-x)e^{-x} - e^{-x} \right]_0^5$$

$$= 2000 \left(-\frac{6}{e^5} + 1 \right)$$

$$\approx 1919.145 \text{ ft} \cdot \text{lbs}$$

$$83. (a) V = 20(2) \int_0^3 \frac{2}{\sqrt{1+y^2}} dy$$

$$= \left[80 \ln|y + \sqrt{1+y^2}| \right]_0^3$$

$$= 80 \ln(3 + \sqrt{10})$$

$$\approx 145.5 \text{ cubic feet}$$

$$W = 148(80 \ln(3 + \sqrt{10}))$$

$$= 11,840 \ln(3 + \sqrt{10})$$

$$\approx 21,530.4 \text{ lb}$$

(b) By symmetry, $\bar{x} = 0$.

$$M = \rho(2) \int_0^3 \frac{2}{\sqrt{1+y^2}} dy = \left[4\rho \ln|y + \sqrt{1+y^2}| \right]_0^3 = 4\rho \ln(3 + \sqrt{10})$$

$$M_x = 2\rho \int_0^3 \frac{2y}{\sqrt{1+y^2}} dy = \left[4\rho \sqrt{1+y^2} \right]_0^3 = 4\rho(\sqrt{10} - 1)$$

$$\bar{y} = \frac{M_x}{M} = \frac{4\rho(\sqrt{10} - 1)}{4\rho \ln(3 + \sqrt{10})} \approx 1.19$$

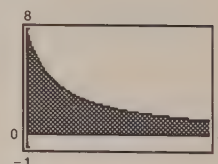
Centroid: $(\bar{x}, \bar{y}) \approx (0, 1.19)$.

$$85. (a) \int_0^4 \frac{k}{2+3x} dx = 10$$

$$k = \frac{10}{\int_0^4 \frac{1}{2+3x} dx} \approx \frac{10}{0.6486}$$

$$= 15.417 \left(= \frac{30}{\ln 7} \right)$$

$$(b) \int_0^4 \frac{15.417}{2+3x} dx$$

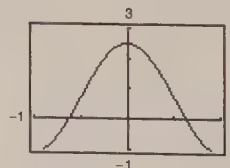


87. False. You might need to convert your integral using substitution or algebra.

Section 7.7 Indeterminate Forms and L'Hôpital's Rule

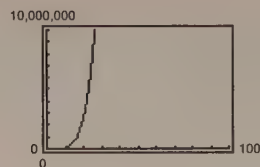
$$1. \lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 2x} \approx 2.5 \left(\text{exact: } \frac{5}{2} \right)$$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.4132	2.4991	2.500	2.500	2.4991	2.4132



$$3. \lim_{x \rightarrow \infty} x^5 e^{-x/100} \approx 0$$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	0.9900	90,484	3.7×10^9	4.5×10^{10}	0	0



$$5. (a) \lim_{x \rightarrow 3} \frac{2(x-3)}{x^2-9} = \lim_{x \rightarrow 3} \frac{2(x-3)}{(x+3)(x-3)} = \lim_{x \rightarrow 3} \frac{2}{x+3} = \frac{1}{3}$$

$$(b) \lim_{x \rightarrow 3} \frac{2(x-3)}{x^2-9} = \lim_{x \rightarrow 3} \frac{(d/dx)[2(x-3)]}{(d/dx)[x^2-9]} = \lim_{x \rightarrow 3} \frac{2}{2x} = \frac{2}{6} = \frac{1}{3}$$

$$7. (a) \lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x \rightarrow 3} \frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)} = \lim_{x \rightarrow 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{4}$$

$$(b) \lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x \rightarrow 3} \frac{(d/dx)[\sqrt{x+1}-2]}{(d/dx)[x-3]} = \lim_{x \rightarrow 3} \frac{1/(2\sqrt{x+1})}{1} = \frac{1}{4}$$

$$9. (a) \lim_{x \rightarrow \infty} \frac{5x^2-3x+1}{3x^2-5} = \lim_{x \rightarrow \infty} \frac{5-(3/x)+(1/x^2)}{3-(5/x^2)} = \frac{5}{3}$$

$$(b) \lim_{x \rightarrow \infty} \frac{5x^2-3x+1}{3x^2-5} = \lim_{x \rightarrow \infty} \frac{(d/dx)[5x^2-3x+1]}{(d/dx)[3x^2-5]} = \lim_{x \rightarrow \infty} \frac{10x-3}{6x} = \lim_{x \rightarrow \infty} \frac{(d/dx)[10x-3]}{(d/dx)[6x]} = \lim_{x \rightarrow \infty} \frac{10}{6} = \frac{5}{3}$$

$$11. \lim_{x \rightarrow 2} \frac{x^2-x-2}{x-2} = \lim_{x \rightarrow 2} \frac{2x-1}{1} = 3$$

$$13. \lim_{x \rightarrow 0} \frac{\sqrt{4-x^2}-2}{x} = \lim_{x \rightarrow 0} \frac{-x/\sqrt{4-x^2}}{1} = 0$$

$$15. \lim_{x \rightarrow 0} \frac{e^x - (1-x)}{x} = \lim_{x \rightarrow 0} \frac{e^x + 1}{1} = 2$$

17. Case 1: $n = 1$

$$\lim_{x \rightarrow 0^+} \frac{e^x - (1+x)}{x} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{1} = 0$$

Case 2: $n = 2$

$$\lim_{x \rightarrow 0^+} \frac{e^x - (1+x)}{x^2} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{2x} = \lim_{x \rightarrow 0^+} \frac{e^x}{2} = \frac{1}{2}$$

Case 3: $n \geq 3$

$$\lim_{x \rightarrow 0^+} \frac{e^x - (1+x)}{x^n} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{nx^{n-1}} = \lim_{x \rightarrow 0^+} \frac{e^x}{n(n-1)x^{n-2}} = \infty$$

$$19. \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \rightarrow 0} \frac{2 \cos 2x}{3 \cos 3x} = \frac{2}{3}$$

$$21. \lim_{x \rightarrow 0} \frac{\arcsin x}{x} = \lim_{x \rightarrow 0} \frac{1/\sqrt{1-x^2}}{1} = 1$$

$$23. \lim_{x \rightarrow \infty} \frac{3x^2-2x+1}{2x^2+3} = \lim_{x \rightarrow \infty} \frac{6x-2}{4x} = \lim_{x \rightarrow \infty} \frac{6}{4} = \frac{3}{2}$$

$$25. \lim_{x \rightarrow \infty} \frac{x^2+2x+3}{x-1} = \lim_{x \rightarrow \infty} \frac{2x+2}{1} = \infty$$

$$27. \lim_{x \rightarrow \infty} \frac{x^3}{e^{x/2}} = \lim_{x \rightarrow \infty} \frac{3x^2}{(1/2)e^{x/2}} = \lim_{x \rightarrow \infty} \frac{6x}{(1/4)e^{x/2}} = \lim_{x \rightarrow \infty} \frac{6}{(1/8)e^{x/2}} = 0$$

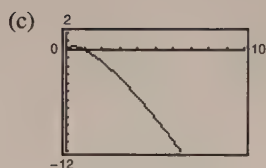
$$29. \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + (1/x^2)}} = 1$$

Note: L'Hôpital's Rule does not work on this limit.
See Exercise 79.

$$33. \lim_{x \rightarrow \infty} \frac{\ln x}{x^2} = \lim_{x \rightarrow \infty} \frac{1/x}{2x} = \lim_{x \rightarrow \infty} \frac{1}{2x^2} = 0$$

$$37. (a) \lim_{x \rightarrow 0^+} (-x \ln x) = (-0)(-\infty) = (0)(\infty)$$

$$\begin{aligned} (b) \lim_{x \rightarrow 0^+} (-x \ln x) &= \lim_{x \rightarrow 0^+} \frac{\ln x}{-1/x} \\ &= \lim_{x \rightarrow 0^+} \frac{1/x}{1/x^2} \\ &= \lim_{x \rightarrow 0^+} x = 0 \end{aligned}$$



$$41. (a) \lim_{x \rightarrow 0^+} x^{1/x} = 0^\infty = 0, \text{ not indeterminate} \\ (\text{See Exercise 95})$$

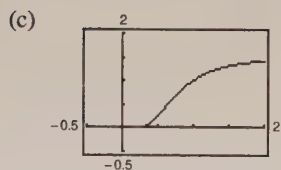
$$(b) \text{ Let } y = x^{1/x}$$

$$\ln y = \ln x^{1/x} = \frac{1}{x} \ln x.$$

Since $x \rightarrow 0^+$, $\frac{1}{x} \ln x \rightarrow (\infty)(-\infty) = -\infty$. Hence,

$$\ln y \rightarrow -\infty \Rightarrow y \rightarrow 0^+.$$

Therefore, $\lim_{x \rightarrow 0^+} x^{1/x} = 0$.



$$45. (a) \lim_{x \rightarrow 0^+} (1+x)^{1/x} = 1^\infty$$

$$(b) \text{ Let } y = \lim_{x \rightarrow 0^+} (1+x)^{1/x}.$$

$$\begin{aligned} \ln y &= \lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x} \\ &= \lim_{x \rightarrow 0^+} \left(\frac{1/(1+x)}{1} \right) = 1 \end{aligned}$$

Thus, $\ln y = 1 \Rightarrow y = e^1 = e$.

Therefore, $\lim_{x \rightarrow 0^+} (1+x)^{1/x} = e$.

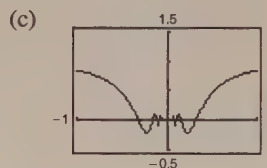
$$31. \lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0 \text{ by Squeeze Theorem}$$

$$\left(\frac{\cos x}{x} \leq \frac{1}{x} \right)$$

$$35. \lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$$

$$39. (a) \lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right) = (\infty)(0)$$

$$\begin{aligned} (b) \lim_{x \rightarrow \infty} x \sin \frac{1}{x} &= \lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} \\ &= \lim_{x \rightarrow \infty} \frac{(-1/x^2) \cos(1/x)}{-1/x^2} \\ &= \lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right) = 1 \end{aligned}$$



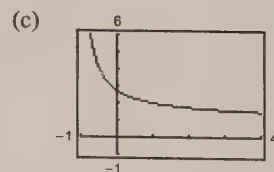
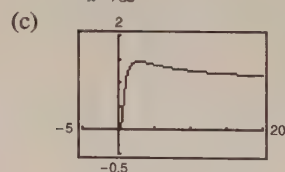
$$43. (a) \lim_{x \rightarrow \infty} x^{1/x} = \infty^0$$

$$(b) \text{ Let } y = \lim_{x \rightarrow \infty} x^{1/x}.$$

$$\ln y = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \left(\frac{1/x}{1} \right) = 0$$

Thus, $\ln y = 0 \Rightarrow y = e^0 = 1$. Therefore,

$$\lim_{x \rightarrow \infty} x^{1/x} = 1.$$



47. (a) $\lim_{x \rightarrow 0^+} [3(x)^{x/2}]^4 = 0^0$

(b) Let $y = \lim_{x \rightarrow 0^+} 3(x)^{x/2}$.

$$\ln y = \lim_{x \rightarrow 0^+} \left[\ln 3 + \frac{x}{2} \ln x \right]$$

$$= \lim_{x \rightarrow 0^+} \left[\ln 3 + \frac{\ln x}{2/x} \right]$$

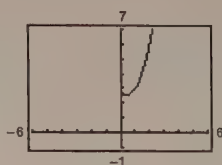
$$= \lim_{x \rightarrow 0^+} \ln 3 + \lim_{x \rightarrow 0^+} \frac{1/x}{-2/x^2}$$

$$= \lim_{x \rightarrow 0^+} \ln 3 - \lim_{x \rightarrow 0^+} \frac{x}{2}$$

$$= \ln 3$$

Hence, $\lim_{x \rightarrow 0^+} 3(x)^{x/2} = 3$.

(c)



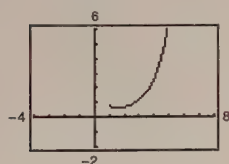
49. (a) $\lim_{x \rightarrow 1^+} (\ln x)^{x-1} = 0^0$

(b) Let $y = \lim_{x \rightarrow 1^+} (\ln x)^{x-1}$

$$= \lim_{x \rightarrow 1^+} (x-1) \ln x = 0$$

Hence, $\lim_{x \rightarrow 1^+} (\ln x)^{x-1} = 1$

(c)



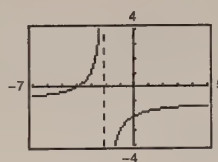
51. (a) $\lim_{x \rightarrow 2^+} \left(\frac{8}{x^2 - 4} - \frac{x}{x-2} \right) = \infty - \infty$

(b) $\lim_{x \rightarrow 2^+} \left(\frac{8}{x^2 - 4} - \frac{x}{x-2} \right) = \lim_{x \rightarrow 2^+} \frac{8 - x(x+2)}{x^2 - 4}$

$$= \lim_{x \rightarrow 2^+} \frac{(2-x)(4+x)}{(x+2)(x-2)}$$

$$= \lim_{x \rightarrow 2^+} \frac{-(x+4)}{x+2} = \frac{-3}{2}$$

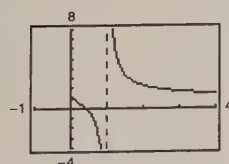
(c)



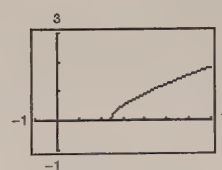
53. (a) $\lim_{x \rightarrow 1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1} \right) = \infty - \infty$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow 1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1} \right) &= \lim_{x \rightarrow 1^+} \frac{3x - 3 - 2 \ln x}{(x-1) \ln x} \\ &= \lim_{x \rightarrow 1^+} \frac{3 - (2/x)}{[(x-1)/x] + \ln x} = \infty \end{aligned}$$

(c)

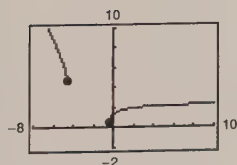


55. (a)



$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow 3} \frac{x-3}{\ln(2x-5)} &= \lim_{x \rightarrow 3} \frac{1}{2/(2x-5)} \\ &= \lim_{x \rightarrow 3} \frac{2x-5}{2} = \frac{1}{2} \end{aligned}$$

57. (a)



$$\text{(b)} \quad \lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x + 2} - x) = \lim_{x \rightarrow \infty} (\sqrt{x^2 + 5x + 2} - x) \left(\frac{\sqrt{x^2 + 5x + 2} + x}{\sqrt{x^2 + 5x + 2} + x} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{(x^2 + 5x + 2) - x^2}{\sqrt{x^2 + 5x + 2} + x}$$

$$= \lim_{x \rightarrow \infty} \frac{5x + 2}{\sqrt{x^2 + 5x + 2} + x}$$

$$= \lim_{x \rightarrow \infty} \frac{5 + (2/x)}{\sqrt{1 + (5/x) + (2/x^2)} + 1} = \frac{5}{2}$$

59. $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^\infty, 0^0, \infty - \infty, \infty^0$

63. $\lim_{x \rightarrow \infty} \frac{x^2}{e^{5x}} = \lim_{x \rightarrow \infty} \frac{2x}{5e^{5x}} = \lim_{x \rightarrow \infty} \frac{2}{25e^{5x}} = 0$

65. $\lim_{x \rightarrow \infty} \frac{(\ln x)^3}{x} = \lim_{x \rightarrow \infty} \frac{3(\ln x)^2(1/x)}{1}$
 $= \lim_{x \rightarrow \infty} \frac{3(\ln x)^2}{x}$
 $= \lim_{x \rightarrow \infty} \frac{6(\ln x)(1/x)}{1}$
 $= \lim_{x \rightarrow \infty} \frac{6(\ln x)}{x} = \lim_{x \rightarrow \infty} \frac{6}{x} = 0$

69.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$	2.811	4.498	0.720	0.036	0.001	0.000

71. $y = x^{1/x}, x > 0$

Horizontal asymptote: $y = 1$ (See Exercise 43)

$$\ln y = \frac{1}{x} \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} \left(\frac{1}{x} \right) + (\ln x) \left(-\frac{1}{x^2} \right)$$

$$\frac{dy}{dx} = x^{1/x} \left(\frac{1}{x^2} \right) (1 - \ln x) = x^{(1/x)-2} (1 - \ln x) = 0$$

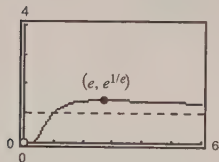
Critical number: $x = e$

Intervals: $(0, e)$ (e, ∞)

Sign of dy/dx : $+$ $-$

$y = f(x)$: Increasing Decreasing

Relative maximum: $(e, e^{1/e})$



75. $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x} = \frac{0}{1} = 0$

Limit is not of the form $0/0$ or ∞/∞ .
 L'Hôpital's Rule does not apply.

61. (a) Let $f(x) = x^2 - 25$ and $g(x) = x - 5$.

(b) Let $f(x) = (x - 5)^2$ and $g(x) = x^2 - 25$.

(c) Let $f(x) = x^2 - 25$ and $g(x) = (x - 5)^3$.

67. $\lim_{x \rightarrow \infty} \frac{(\ln x)^n}{x^m} = \lim_{x \rightarrow \infty} \frac{n(\ln x)^{n-1}/x}{mx^{m-1}}$
 $= \lim_{x \rightarrow \infty} \frac{n(\ln x)^{n-1}}{mx^m}$
 $= \lim_{x \rightarrow \infty} \frac{n(n-1)(\ln x)^{n-2}}{m^2 x^m}$
 $= \dots = \lim_{x \rightarrow \infty} \frac{n!}{m^n x^m} = 0$

73. $y = 2xe^{-x}$

$$\lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$$

Horizontal asymptote: $y = 0$

$$\frac{dy}{dx} = 2x(-e^{-x}) + 2e^{-x}$$

$$= 2e^{-x}(1 - x) = 0$$

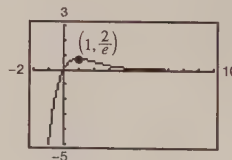
Critical number: $x = 1$

Intervals: $(-\infty, 1)$ $(1, \infty)$

Sign of dy/dx : $+$ $-$

$y = f(x)$: Increasing Decreasing

Relative maximum: $\left(1, \frac{2}{e}\right)$



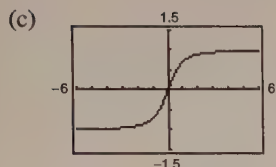
77. $\lim_{x \rightarrow \infty} x \cos \frac{1}{x} = \infty(1) = \infty$

Limit is not of the form $0/0$ or ∞/∞ .
 L'Hôpital's Rule does not apply.

$$\begin{aligned}
 79. (a) \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} &= \lim_{x \rightarrow \infty} \frac{x/x}{\sqrt{x^2 + 1}/x} \\
 &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2 + 1}/\sqrt{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + (1/x^2)}} \\
 &= \frac{1}{\sqrt{1 + 0}} = 1
 \end{aligned}$$

$$\begin{aligned}
 (b) \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} &= \lim_{x \rightarrow \infty} \frac{1}{x/\sqrt{x^2 + 1}} \\
 &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x} = \lim_{x \rightarrow \infty} \frac{x/\sqrt{x^2 + 1}}{1} \\
 &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}}
 \end{aligned}$$

Applying L'Hôpital's rule twice results in the original limit, so L'Hôpital's rule fails.



$$\begin{aligned}
 81. \lim_{k \rightarrow 0} \frac{32 \left(1 - e^{-kt} + \frac{v_0 k e^{-kt}}{32} \right)}{k} &= \lim_{k \rightarrow 0} \frac{32(1 - e^{-kt})}{k} + \lim_{k \rightarrow 0} (v_0 e^{-kt}) \\
 &= \lim_{k \rightarrow 0} \frac{32(0 + t e^{-kt})}{1} + \lim_{k \rightarrow 0} \left(\frac{v_0}{e^{kt}} \right) = 32t + v_0
 \end{aligned}$$

83. Area of triangle: $\frac{1}{2}(2x)(1 - \cos x) = x - x \cos x$

Shaded area: Area of rectangle - Area under curve

$$\begin{aligned}
 2x(1 - \cos x) - 2 \int_0^x (1 - \cos t) dt &= 2x(1 - \cos x) - 2 \left[t - \sin t \right]_0^x \\
 &= 2x(1 - \cos x) - 2(x - \sin x) = 2 \sin x - 2x \cos x
 \end{aligned}$$

$$\begin{aligned}
 \text{Ratio: } \lim_{x \rightarrow 0} \frac{x - x \cos x}{2 \sin x - 2x \cos x} &= \lim_{x \rightarrow 0} \frac{1 + x \sin x - \cos x}{2 \cos x + 2x \sin x - 2 \cos x} \\
 &= \lim_{x \rightarrow 0} \frac{1 + x \sin x - \cos x}{2x \sin x} \\
 &= \lim_{x \rightarrow 0} \frac{x \cos x + \sin x + \sin x}{2x \cos x + 2 \sin x} \\
 &= \lim_{x \rightarrow 0} \frac{x \cos x + 2 \sin x}{2x \cos x + 2 \sin x} \cdot \frac{1/\cos x}{1/\cos x} \\
 &= \lim_{x \rightarrow 0} \frac{x + 2 \tan x}{2x + 2 \tan x} \\
 &= \lim_{x \rightarrow 0} \frac{1 + 2 \sec^2 x}{2 + 2 \sec^2 x} = \frac{3}{4}
 \end{aligned}$$

85. $f(x) = x^3$, $g(x) = x^2 + 1$, $[0, 1]$

$$\begin{aligned}
 \frac{f(b) - f(a)}{g(b) - g(a)} &= \frac{f'(c)}{g'(c)} \\
 \frac{f(1) - f(0)}{g(1) - g(0)} &= \frac{3c^2}{2c} \\
 \frac{1}{1} &= \frac{3c}{2} \\
 c &= \frac{2}{3}
 \end{aligned}$$

87. $f(x) = \sin x$, $g(x) = \cos x$, $\left[0, \frac{\pi}{2}\right]$

$$\begin{aligned}
 \frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)} &= \frac{f'(c)}{g'(c)} \\
 \frac{1}{-1} &= \frac{\cos c}{-\sin c} \\
 -1 &= -\cot c \\
 c &= \frac{\pi}{4}
 \end{aligned}$$

89. False. L'Hôpital's Rule does not apply since

$$\lim_{x \rightarrow 0} (x^2 + x + 1) \neq 0.$$

$$\lim_{x \rightarrow 0^+} \frac{x^2 + x + 1}{x} = \lim_{x \rightarrow 0^+} \left(x + 1 + \frac{1}{x} \right) = 1 + \infty = \infty$$

91. True

93. (a) $\sin \theta = BD$

$$\cos \theta = DO \Rightarrow AD = 1 - \cos \theta$$

$$\text{Area } \triangle ABD = \frac{1}{2}bh = \frac{1}{2}(1 - \cos \theta) \sin \theta = \frac{1}{2} \sin \theta - \frac{1}{2} \sin \theta \cos \theta$$

(b) Area of sector: $\frac{1}{2}\theta$

$$\text{Shaded area: } \frac{1}{2}\theta - \text{Area } \triangle OBD = \frac{1}{2}\theta - \frac{1}{2}(\cos \theta)(\sin \theta) = \frac{1}{2}\theta - \frac{1}{2} \sin \theta \cos \theta$$

$$(c) R = \frac{(1/2) \sin \theta - (1/2) \sin \theta \cos \theta}{(1/2)\theta - (1/2) \sin \theta \cos \theta} = \frac{\sin \theta - \sin \theta \cos \theta}{\theta - \sin \theta \cos \theta}$$

$$(d) \lim_{\theta \rightarrow 0} R = \lim_{\theta \rightarrow 0} \frac{\sin \theta - (1/2) \sin 2\theta}{\theta - (1/2) \sin 2\theta}$$

$$= \lim_{\theta \rightarrow 0} \frac{\cos \theta - \cos 2\theta}{1 - \cos 2\theta} = \lim_{\theta \rightarrow 0} \frac{-\sin \theta + 2 \sin 2\theta}{2 \sin 2\theta} = \lim_{\theta \rightarrow 0} \frac{-\cos \theta + 4 \cos 2\theta}{4 \cos 2\theta} = \frac{3}{4}$$

95. $\lim_{x \rightarrow a} f(x)^{g(x)}$

$$y = f(x)^{g(x)}$$

$$\ln y = g(x) \ln f(x)$$

$$\lim_{x \rightarrow a} g(x) \ln f(x) = (\infty)(-\infty) = -\infty$$

As $x \rightarrow a$, $\ln y \Rightarrow -\infty$, and hence $y = 0$. Thus,

$$\lim_{x \rightarrow a} f(x)^{g(x)} = 0.$$

$$97. f'(a)(b-a) - \int_a^b f''(t)(t-b) dt = f'(a)(b-a) - \left\{ \left[f'(t)(t-b) \right]_a^b - \int_a^b f'(t) dt \right\}$$

$$= f'(a)(b-a) + f'(a)(a-b) + \left[f(t) \right]_a^b = f(b) - f(a)$$

$$dv = f''(t)dt \Rightarrow v = f'(t)$$

$$u = t - b \Rightarrow du = dt$$

Section 7.8 Improper Integrals

1. Infinite discontinuity at $x = 0$.

$$\int_0^4 \frac{1}{\sqrt{x}} dx = \lim_{b \rightarrow 0^+} \int_b^4 \frac{1}{\sqrt{x}} dx$$

$$= \lim_{b \rightarrow 0^+} \left[2\sqrt{x} \right]_b^4$$

$$= \lim_{b \rightarrow 0^+} (4 - 2\sqrt{b}) = 4$$

Converges

3. Infinite discontinuity at $x = 1$.

$$\begin{aligned}\int_0^2 \frac{1}{(x-1)^2} dx &= \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx \\&= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{c \rightarrow 1^+} \int_c^2 \frac{1}{(x-1)^2} dx \\&= \lim_{b \rightarrow 1^-} \left[-\frac{1}{x-1} \right]_0^b + \lim_{c \rightarrow 1^+} \left[-\frac{1}{x-1} \right]_c^2 = (\infty - 1) + (-1 + \infty)\end{aligned}$$

Diverges

5. Infinite limit of integration.

$$\begin{aligned}\int_0^\infty e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \\&= \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b = 0 + 1 = 1\end{aligned}$$

Converges

$$7. \int_{-1}^1 \frac{1}{x^2} dx \neq -2$$

because the integrand is not defined at $x = 0$.

Diverges

$$\begin{aligned}9. \int_1^\infty \frac{1}{x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx \\&= \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b = 1\end{aligned}$$

$$\begin{aligned}11. \int_1^\infty \frac{3}{\sqrt[3]{x}} dx &= \lim_{b \rightarrow \infty} \int_1^b 3x^{-1/3} dx \\&= \lim_{b \rightarrow \infty} \left[\frac{9}{2} x^{2/3} \right]_1^b = \infty\end{aligned}$$

Diverges

$$13. \int_{-\infty}^0 x e^{-2x} dx = \lim_{b \rightarrow -\infty} \int_b^0 x e^{-2x} dx = \lim_{b \rightarrow -\infty} \frac{1}{4} \left[(-2x - 1) e^{-2x} \right]_b^0 = \lim_{b \rightarrow -\infty} \frac{1}{4} [-1 + (2b + 1) e^{-2b}] = -\infty \quad (\text{Integration by parts})$$

Diverges

$$15. \int_0^\infty x^2 e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b x^2 e^{-x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x}(x^2 + 2x + 2) \right]_0^b = \lim_{b \rightarrow \infty} \left(-\frac{b^2 + 2b + 2}{e^b} + 2 \right) = 2$$

Since $\lim_{b \rightarrow \infty} \left(-\frac{b^2 + 2b + 2}{e^b} \right) = 0$ by L'Hôpital's Rule.

$$\begin{aligned}17. \int_0^\infty e^{-x} \cos x dx &= \lim_{b \rightarrow \infty} \frac{1}{2} \left[e^{-x} (-\cos x + \sin x) \right]_0^b \\&= \frac{1}{2} [0 - (-1)] = \frac{1}{2}\end{aligned}$$

$$\begin{aligned}19. \int_4^\infty \frac{1}{x(\ln x)^3} dx &= \lim_{b \rightarrow \infty} \int_4^b (\ln x)^{-3} \frac{1}{x} dx \\&= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} (\ln x)^{-2} \right]_4^b \\&= -\frac{1}{2} (\ln b)^{-2} + \frac{1}{2} (\ln 4)^{-2} \\&= \frac{1}{2} \frac{1}{(\ln 2)^2} = \frac{1}{8(\ln 2)^2}\end{aligned}$$

$$\begin{aligned}21. \int_{-\infty}^\infty \frac{2}{4 + x^2} dx &= \int_{-\infty}^0 \frac{2}{4 + x^2} dx + \int_0^\infty \frac{2}{4 + x^2} dx \\&= \lim_{b \rightarrow -\infty} \int_b^0 \frac{2}{4 + x^2} dx + \lim_{c \rightarrow \infty} \int_0^c \frac{2}{4 + x^2} dx \\&= \lim_{b \rightarrow -\infty} \left[\arctan\left(\frac{x}{2}\right) \right]_b^0 + \lim_{c \rightarrow \infty} \left[\arctan\left(\frac{x}{2}\right) \right]_0^c \\&= \left(0 - \left(-\frac{\pi}{2} \right) \right) + \left(\frac{\pi}{2} - 0 \right) = \pi\end{aligned}$$

$$\begin{aligned}
 23. \int_0^{\infty} \frac{1}{e^x + e^{-x}} dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{e^x}{1 + e^{2x}} dx \\
 &= \lim_{b \rightarrow \infty} \left[\arctan(e^x) \right]_0^b \\
 &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
 \end{aligned}$$

$$25. \int_0^{\infty} \cos \pi x dx = \lim_{b \rightarrow \infty} \left[\frac{1}{\pi} \sin \pi x \right]_0^b$$

Diverges since $\sin \pi x$ does not approach a limit as $x \rightarrow \infty$.

$$27. \int_0^1 \frac{1}{x^2} dx = \lim_{b \rightarrow 0^+} \left[-\frac{1}{x} \right]_b^1 = -1 + \infty$$

Diverges

$$29. \int_0^8 \frac{1}{\sqrt[3]{8-x}} dx = \lim_{b \rightarrow 8^-} \int_0^b \frac{1}{\sqrt[3]{8-x}} dx = \lim_{b \rightarrow 8^-} \left[\frac{-3}{2} (8-x)^{2/3} \right]_0^b = 6$$

$$31. \int_0^1 x \ln x dx = \lim_{b \rightarrow 0^+} \left[\frac{x^2}{2} \ln|x| - \frac{x^2}{4} \right]_b^1 = \lim_{b \rightarrow 0^+} \left[\frac{-1}{4} - \frac{b^2 \ln b}{2} + \frac{b^2}{4} \right] = \frac{-1}{4} \text{ since } \lim_{b \rightarrow 0^+} (b^2 \ln b) = 0 \text{ by L'Hôpital's Rule.}$$

$$33. \int_0^{\pi/2} \tan \theta d\theta = \lim_{b \rightarrow (\pi/2)^-} \left[\ln|\sec \theta| \right]_0^b = \infty$$

Diverges

$$\begin{aligned}
 35. \int_2^4 \frac{2}{x\sqrt{x^2-4}} dx &= \lim_{b \rightarrow 2^+} \int_b^4 \frac{2}{x\sqrt{x^2-4}} dx \\
 &= \lim_{b \rightarrow 2^+} \left[\operatorname{arcsec} \left| \frac{x}{2} \right| \right]_b^4 \\
 &= \lim_{b \rightarrow 2^+} \left(\operatorname{arcsec} 2 - \operatorname{arcsec} \left(\frac{b}{2} \right) \right) \\
 &= \frac{\pi}{3} - 0 = \frac{\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 37. \int_2^4 \frac{1}{\sqrt{x^2-4}} dx &= \lim_{b \rightarrow 2^+} \left[\ln|x + \sqrt{x^2-4}| \right]_b^4 \\
 &= \ln(4 + 2\sqrt{3}) - \ln 2 \\
 &= \ln(2 + \sqrt{3}) \approx 1.317
 \end{aligned}$$

$$\begin{aligned}
 39. \int_0^2 \frac{1}{\sqrt[3]{x-1}} dx &= \int_0^1 \frac{1}{\sqrt[3]{x-1}} dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}} dx \\
 &= \lim_{b \rightarrow 1^-} \left[\frac{3}{2} (x-1)^{2/3} \right]_0^b + \lim_{c \rightarrow 1^+} \left[\frac{3}{2} (x-1)^{2/3} \right]_c^2 = \frac{-3}{2} + \frac{3}{2} = 0
 \end{aligned}$$

$$41. \int_0^{\infty} \frac{4}{\sqrt{x}(x+6)} dx = \int_0^1 \frac{4}{\sqrt{x}(x+6)} dx + \int_1^{\infty} \frac{4}{\sqrt{x}(x+6)} dx$$

Let $u = \sqrt{x}$, $u^2 = x$, $2u du = dx$.

$$\int \frac{4}{\sqrt{x}(x+6)} dx = \int \frac{4(2u du)}{u(u^2+6)} = 8 \int \frac{du}{u^2+6} = \frac{8}{\sqrt{6}} \arctan\left(\frac{u}{\sqrt{6}}\right) + C = \frac{8}{\sqrt{6}} \arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

$$\begin{aligned}
 \text{Thus, } \int_0^{\infty} \frac{4}{\sqrt{x}(x+6)} dx &= \lim_{b \rightarrow 0^+} \left[\frac{8}{\sqrt{6}} \arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) \right]_b^1 + \lim_{c \rightarrow \infty} \left[\frac{8}{\sqrt{6}} \arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) \right]_1^c \\
 &= \left(\frac{8}{\sqrt{6}} \arctan\left(\frac{1}{\sqrt{6}}\right) - \frac{8}{\sqrt{6}} 0 \right) + \left(\frac{8}{\sqrt{6}} \frac{\pi}{2} - \frac{8}{\sqrt{6}} \arctan\left(\frac{1}{\sqrt{6}}\right) \right) \\
 &= \frac{8\pi}{2\sqrt{6}} = \frac{2\pi\sqrt{6}}{3}
 \end{aligned}$$

43. If $p = 1$, $\int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx = \lim_{b \rightarrow \infty} \ln x \Big|_1^b$.

Diverges. For $p \neq 1$,

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \left[\frac{x^{1-p}}{1-p} \right]_1^b = \lim_{b \rightarrow \infty} \left[\frac{b^{1-p}}{1-p} - \frac{1}{1-p} \right].$$

This converges to $\frac{1}{p-1}$ if $1-p < 0$ or $p > 1$.

45. For $n = 1$ we have

$$\begin{aligned} \int_0^{\infty} x e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b x e^{-x} dx \\ &= \lim_{b \rightarrow \infty} \left[-e^{-x} x - e^{-x} \right]_0^b \quad (\text{Parts: } u = x, dv = e^{-x} dx) \\ &= \lim_{b \rightarrow \infty} [-e^{-b} b - e^{-b} + 1] \\ &= \lim_{b \rightarrow \infty} \left[\frac{-b}{e^b} - \frac{1}{e^b} + 1 \right] = 1 \quad (\text{L'Hôpital's Rule}) \end{aligned}$$

Assume that $\int_0^{\infty} x^n e^{-x} dx$ converges. Then for $n+1$ we have

$$\int x^{n+1} e^{-x} dx = -x^{n+1} e^{-x} + (n+1) \int x^n e^{-x} dx$$

by parts ($u = x^{n+1}$, $du = (n+1)x^n dx$, $dv = e^{-x} dx$, $v = -e^{-x}$).

Thus,

$$\int_0^{\infty} x^{n+1} e^{-x} dx = \lim_{b \rightarrow \infty} \left[-x^{n+1} e^{-x} \right]_0^b + (n+1) \int_0^{\infty} x^n e^{-x} dx = 0 + (n+1) \int_0^{\infty} x^n e^{-x} dx, \text{ which converges.}$$

47. $\int_0^1 \frac{1}{x^3} dx$ diverges.

(See Exercise 44, $p = 3 < 1$.)

49. $\int_1^{\infty} \frac{1}{x^3} dx = \frac{1}{3-1} = \frac{1}{2}$ converges.

(See Exercise 43, $p = 3$.)

51. Since $\frac{1}{x^2+5} \leq \frac{1}{x^2}$ on $[1, \infty)$ and $\int_1^{\infty} \frac{1}{x^2} dx$ converges by Exercise 43, $\int_1^{\infty} \frac{1}{x^2+5} dx$ converges.

53. Since $\frac{1}{\sqrt[3]{x(x-1)}} \geq \frac{1}{\sqrt[3]{x^2}}$ on $[2, \infty)$ and $\int_2^{\infty} \frac{1}{\sqrt[3]{x^2}} dx$ diverges by Exercise 43, $\int_2^{\infty} \frac{1}{\sqrt[3]{x(x-1)}} dx$ diverges.

55. Since $e^{-x^2} \leq e^{-x}$ on $[1, \infty)$ and $\int_0^{\infty} e^{-x} dx$ converges (see Exercise 5), $\int_0^{\infty} e^{-x^2} dx$ converges.

57. Answers will vary. See pages 540, 543.

59. $\int_{-1}^1 \frac{1}{x^3} dx = \int_{-1}^0 \frac{1}{x^3} dx + \int_0^1 \frac{1}{x^3} dx$

These two integrals diverge by Exercise 44.

61. $f(t) = 1$

$$F(s) = \int_0^{\infty} e^{-st} dt = \lim_{b \rightarrow \infty} \left[-\frac{1}{s} e^{-st} \right]_0^b = \frac{1}{s}, s > 0$$

63. $f(t) = t^2$

$$\begin{aligned} F(s) &= \int_0^{\infty} t^2 e^{-st} dt = \lim_{b \rightarrow \infty} \left[\frac{1}{s^3} (-s^2 t^2 - 2st - 2) e^{-st} \right]_0^b \\ &= \frac{2}{s^3}, s > 0 \end{aligned}$$

65. $f(t) = \cos at$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} \cos at \, dt \\
 &= \lim_{b \rightarrow \infty} \left[\frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \right]_0^b \\
 &= 0 + \frac{s}{s^2 + a^2} = \frac{s}{s^2 + a^2}, s > 0
 \end{aligned}$$

67. $f(t) = \cosh at$

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} \cosh at \, dt = \int_0^{\infty} e^{-st} \left(\frac{e^{at} + e^{-at}}{2} \right) dt = \frac{1}{2} \int_0^{\infty} \left[e^{t(-s+a)} + e^{t(-s-a)} \right] dt \\
 &= \lim_{b \rightarrow \infty} \frac{1}{2} \left[\frac{1}{(-s+a)} e^{t(-s+a)} + \frac{1}{(-s-a)} e^{t(-s-a)} \right]_0^b = 0 - \frac{1}{2} \left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)} \right] \\
 &= \frac{-1}{2} \left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)} \right] = \frac{s}{s^2 - a^2}, s > |a|
 \end{aligned}$$

69. (a) $A = \int_0^{\infty} e^{-x} dx$

$$= \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b = 0 - (-1) = 1$$

(b) **Disk:**

$$\begin{aligned}
 V &= \pi \int_0^{\infty} (e^{-x})^2 dx \\
 &= \lim_{b \rightarrow \infty} \pi \left[-\frac{1}{2} e^{-2x} \right]_0^b = \frac{\pi}{2}
 \end{aligned}$$

(c) **Shell:**

$$\begin{aligned}
 V &= 2\pi \int_0^{\infty} x e^{-x} dx \\
 &= \lim_{b \rightarrow \infty} \left\{ 2\pi \left[-e^{-x}(x+1) \right]_0^b \right\} = 2\pi
 \end{aligned}$$

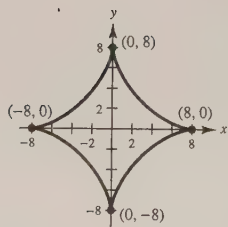
71. $x^{2/3} + y^{2/3} = 4$

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} y' = 0$$

$$y' = \frac{-y^{1/3}}{x^{1/3}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}} = \sqrt{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}} = \sqrt{\frac{4}{x^{2/3}}} = \frac{2}{x^{1/3}}, (x > 0)$$

$$s = 4 \int_0^8 \frac{2}{x^{1/3}} dx = \lim_{b \rightarrow 0^+} \left[8 \cdot \frac{3}{2} x^{2/3} \right]_b^8 = 48$$



$$73. \Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$$

$$(a) \Gamma(1) = \int_0^{\infty} e^{-x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x} \right]_0^b = 1$$

$$\Gamma(2) = \int_0^{\infty} x e^{-x} dx = \lim_{b \rightarrow \infty} \left[-e^{-x}(x+1) \right]_0^b = 1$$

$$\Gamma(3) = \int_0^{\infty} x^2 e^{-x} dx = \lim_{b \rightarrow \infty} \left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \right]_0^b = 2$$

$$(b) \Gamma(n+1) = \int_0^{\infty} x^n e^{-x} dx = \lim_{b \rightarrow \infty} \left[-x^n e^{-x} \right]_0^b + \lim_{b \rightarrow \infty} n \int_0^b x^{n-1} e^{-x} dx = 0 + n\Gamma(n) \quad (u = x^n, dv = e^{-x} dx)$$

$$(c) \Gamma(n) = (n-1)!$$

$$75. (a) \int_{-\infty}^{\infty} \frac{1}{7} e^{-t/7} dt = \int_0^{\infty} \frac{1}{7} e^{-t/7} dt = \lim_{b \rightarrow \infty} \left[-e^{-t/7} \right]_0^b = 1$$

$$(b) \int_0^4 \frac{1}{7} e^{-t/7} dt = \left[-e^{-t/7} \right]_0^4 = -e^{-4/7} + 1 \\ \approx 0.4353 = 43.53\%$$

$$(c) \int_0^{\infty} t \left[\frac{1}{7} e^{-t/7} \right] dt = \lim_{b \rightarrow \infty} \left[-te^{-t/7} - 7e^{-t/7} \right]_0^b \\ = 0 + 7 = 7$$

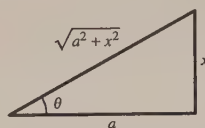
$$77. (a) C = 650,000 + \int_0^5 25,000 e^{-0.06t} dt = 650,000 - \left[\frac{25,000}{0.06} e^{-0.06t} \right]_0^5 \approx \$757,992.41$$

$$(b) C = 650,000 + \int_0^{10} 25,000 e^{-0.06t} dt \approx \$837,995.15$$

$$(c) C = 650,000 + \int_0^{\infty} 25,000 e^{-0.06t} dt = 650,000 - \lim_{b \rightarrow \infty} \left[\frac{25,000}{0.06} e^{-0.06t} \right]_0^b \approx \$1,066,666.67$$

$$79. \text{ Let } x = a \tan \theta, dx = a \sec^2 \theta d\theta, \sqrt{a^2 + x^2} = a \sec \theta.$$

$$\int \frac{1}{(a^2 + x^2)^{3/2}} dx = \int \frac{a \sec^2 \theta d\theta}{a^3 \sec^3 \theta} = \frac{1}{a^2} \int \cos \theta d\theta \\ = \frac{1}{a^2} \sin \theta = \frac{1}{a^2} \frac{x}{\sqrt{a^2 + x^2}}$$



Hence,

$$P = k \int_1^{\infty} \frac{1}{(a^2 + x^2)^{3/2}} dx = \frac{k}{a^2} \lim_{b \rightarrow \infty} \left[\frac{x}{\sqrt{a^2 + x^2}} \right]_1^b \\ = \frac{k}{a^2} \left[1 - \frac{1}{\sqrt{a^2 + 1}} \right] = \frac{k(\sqrt{a^2 + 1} - 1)}{a^2 \sqrt{a^2 + 1}}.$$

$$81. \frac{10}{x^2 - 2x} = \frac{10}{x(x-2)} \Rightarrow x = 0, 2.$$

You must analyze three improper integrals, and each must converge in order for the original integral to converge.

$$\int_0^3 f(x) dx = \int_0^1 f(x) dx + \int_1^2 f(x) dx + \int_2^3 f(x) dx$$

83. For $n = 1$,

$$I_1 = \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \lim_{b \rightarrow \infty} \frac{1}{2} \int_0^b (x^2 + 1)^{-4} (2x dx) = \lim_{b \rightarrow \infty} \left[-\frac{1}{6} \frac{1}{(x^2 + 1)^3} \right]_0^b = \frac{1}{6}.$$

For $n > 1$,

$$I_n = \int_0^\infty \frac{x^{2n-1}}{(x^2 + 1)^{n+3}} dx = \lim_{b \rightarrow \infty} \left[\frac{-x^{2n-2}}{2(n+2)(x^2 + 1)^{n+2}} \right]_0^b + \frac{n-1}{n+2} \int_0^\infty \frac{x^{2n-3}}{(x^2 + 1)^{n+2}} dx = 0 + \frac{n-1}{n+2} (I_{n-1})$$

$$u = x^{2n-2}, du = (2n-2)x^{2n-3} dx, dv = \frac{x}{(x^2 + 1)^{n+3}} dx, v = \frac{-1}{2(n+2)(x^2 + 1)^{n+2}}$$

$$(a) \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{6(x^2 + 1)^3} \right]_0^b = \frac{1}{6}$$

$$(b) \int_0^\infty \frac{x^3}{(x^2 + 1)^5} dx = \frac{1}{4} \int_0^\infty \frac{x}{(x^2 + 1)^4} dx = \frac{1}{4} \left(\frac{1}{6} \right) = \frac{1}{24}$$

$$(c) \int_0^\infty \frac{x^5}{(x^2 + 1)^6} dx = \frac{2}{5} \int_0^\infty \frac{x^3}{(x^2 + 1)^5} dx = \frac{2}{5} \left(\frac{1}{24} \right) = \frac{1}{60}$$

85. False. $f(x) = 1/(x+1)$ is continuous on $[0, \infty)$, $\lim_{x \rightarrow \infty} 1/(x+1) = 0$, but $\int_0^\infty \frac{1}{x+1} dx = \lim_{b \rightarrow \infty} \left[\ln|x+1| \right]_0^b = \infty$.

Diverges

87. True

Review Exercises for Chapter 7

$$\begin{aligned} 1. \int x\sqrt{x^2-1} dx &= \frac{1}{2} \int (x^2-1)^{1/2} (2x) dx \\ &= \frac{1}{2} \frac{(x^2-1)^{3/2}}{3/2} + C \\ &= \frac{1}{3} (x^2-1)^{3/2} + C \end{aligned}$$

$$\begin{aligned} 3. \int \frac{x}{x^2-1} dx &= \frac{1}{2} \int \frac{2x}{x^2-1} dx \\ &= \frac{1}{2} \ln|x^2-1| + C \end{aligned}$$

$$5. \int \frac{\ln(2x)}{x} dx = \frac{(\ln 2x)^2}{2} + C$$

$$7. \int \frac{16}{\sqrt{16-x^2}} dx = 16 \arcsin\left(\frac{x}{4}\right) + C$$

$$\begin{aligned} 9. \int e^{2x} \sin 3x dx &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x dx \\ &= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \left(\frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \int e^{2x} \sin 3x dx \right) \end{aligned}$$

$$\frac{13}{9} \int e^{2x} \sin 3x dx = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x$$

$$\int e^{2x} \sin 3x dx = \frac{e^{2x}}{13} (2 \sin 3x - 3 \cos 3x) + C$$

$$(1) dv = \sin 3x dx \Rightarrow v = -\frac{1}{3} \cos 3x$$

$$(2) dv = \cos 3x dx \Rightarrow v = \frac{1}{3} \sin 3x$$

$$u = e^{2x} \Rightarrow du = 2e^{2x} dx$$

$$u = e^{2x} \Rightarrow du = 2e^{2x} dx$$

$$11. u = x, du = dx, dv = (x-5)^{1/2} dx, v = \frac{2}{3}(x-5)^{3/2}$$

$$\begin{aligned}\int x\sqrt{x-5} dx &= \frac{2}{3}x(x-5)^{3/2} - \int \frac{2}{3}(x-5)^{3/2} dx \\ &= \frac{2}{3}x(x-5)^{3/2} - \frac{4}{15}(x-5)^{5/2} + C \\ &= (x-5)^{3/2} \left[\frac{2}{3}x - \frac{4}{15}(x-5) \right] + C \\ &= (x-5)^{3/2} \left[\frac{6}{15}x + \frac{4}{3} \right] + C \\ &= \frac{2}{15}(x-5)^{3/2}[3x+10] + C\end{aligned}$$

$$\begin{aligned}13. \int x^2 \sin 2x dx &= -\frac{1}{2}x^2 \cos 2x + \int x \cos 2x dx \\ &= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x - \frac{1}{2} \int \sin 2x dx \\ &= -\frac{1}{2}x^2 \cos 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x + C\end{aligned}$$

$$(1) dv = \sin 2x dx \Rightarrow v = -\frac{1}{2} \cos 2x$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$(2) dv = \cos 2x dx \Rightarrow v = \frac{1}{2} \sin 2x$$

$$u = x \Rightarrow du = dx$$

$$\begin{aligned}15. \int x \arcsin 2x dx &= \frac{x^2}{2} \arcsin 2x - \int \frac{x^2}{\sqrt{1-4x^2}} dx \\ &= \frac{x^2}{2} \arcsin 2x - \frac{1}{8} \int \frac{2(2x)^2}{\sqrt{1-(2x)^2}} dx \\ &= \frac{x^2}{2} \arcsin 2x - \frac{1}{8} \left(\frac{1}{2} \right) [-(2x)\sqrt{1-4x^2} + \arcsin 2x] + C \quad (\text{by Formula 43 of Integration Tables}) \\ &= \frac{1}{16} [(8x^2-1)\arcsin 2x + 2x\sqrt{1-4x^2}] + C\end{aligned}$$

$$dv = x dx \Rightarrow v = \frac{x^2}{2}$$

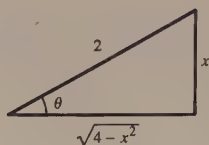
$$u = \arcsin 2x \Rightarrow du = \frac{2}{\sqrt{1-4x^2}} dx$$

$$\begin{aligned}17. \int \cos^3(\pi x - 1) dx &= \int [1 - \sin^2(\pi x - 1)] \cos(\pi x - 1) dx \\ &= \frac{1}{\pi} \left[\sin(\pi x - 1) - \frac{1}{3} \sin^3(\pi x - 1) \right] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [3 - \sin^2(\pi x - 1)] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [3 - (1 - \cos^2(\pi x - 1))] + C \\ &= \frac{1}{3\pi} \sin(\pi x - 1) [2 + \cos^2(\pi x - 1)] + C\end{aligned}$$

$$\begin{aligned}19. \int \sec^4\left(\frac{x}{2}\right) dx &= \int \left[\tan^2\left(\frac{x}{2}\right) + 1 \right] \sec^2\left(\frac{x}{2}\right) dx \\ &= \int \tan^2\left(\frac{x}{2}\right) \sec^2\left(\frac{x}{2}\right) dx + \int \sec^2\left(\frac{x}{2}\right) dx \\ &= \frac{2}{3} \tan^3\left(\frac{x}{2}\right) + 2 \tan\left(\frac{x}{2}\right) + C = \frac{2}{3} \left[\tan^3\left(\frac{x}{2}\right) + 3 \tan\left(\frac{x}{2}\right) \right] + C\end{aligned}$$

$$21. \int \frac{1}{1 - \sin \theta} d\theta = \int \frac{1 + \sin \theta}{\cos^2 \theta} d\theta = \int (\sec^2 \theta + \sec \theta \tan \theta) d\theta = \tan \theta + \sec \theta + C$$

$$\begin{aligned}
 23. \int \frac{-12}{x^2 \sqrt{4-x^2}} dx &= \int \frac{-24 \cos \theta d\theta}{(4 \sin^2 \theta)(2 \cos \theta)} \\
 &= -3 \int \csc^2 \theta d\theta \\
 &= 3 \cot \theta + C \\
 &= \frac{3\sqrt{4-x^2}}{x} + C
 \end{aligned}$$

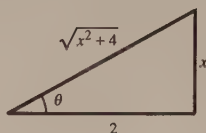


$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

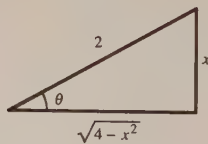
$$\begin{aligned}
 25. \quad x &= 2 \tan \theta \\
 dx &= 2 \sec^2 \theta d\theta
 \end{aligned}$$

$$4 + x^2 = 4 \sec^2 \theta$$

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{4+x^2}} dx &= \int \frac{8 \tan^3 \theta}{2 \sec \theta} 2 \sec^2 \theta d\theta \\
 &= 8 \int \tan^3 \theta \sec \theta d\theta \\
 &= 8 \int (\sec^2 \theta - 1) \tan \theta \sec \theta d\theta \\
 &= 8 \left[\frac{\sec^3 \theta}{3} - \sec \theta \right] + C \\
 &= 8 \left[\frac{(x^2+4)^{3/2}}{24} - \frac{\sqrt{x^2+4}}{2} \right] + C \\
 &= \sqrt{x^2+4} \left[\frac{1}{3}(x^2+4) - 4 \right] + C \\
 &= \frac{1}{3} x^2 \sqrt{x^2+4} - \frac{8}{3} \sqrt{x^2+4} + C \\
 &= \frac{1}{3} (x^2+4)^{1/2} (x^2-8) + C
 \end{aligned}$$



$$\begin{aligned}
 27. \int \sqrt{4-x^2} dx &= \int (2 \cos \theta)(2 \cos \theta) d\theta \\
 &= 2 \int (1 + \cos 2\theta) d\theta \\
 &= 2 \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\
 &= 2(\theta + \sin \theta \cos \theta) + C \\
 &= 2 \left[\arcsin\left(\frac{x}{2}\right) + \frac{x}{2} \left(\frac{\sqrt{4-x^2}}{2} \right) \right] + C \\
 &= \frac{1}{2} \left[4 \arcsin\left(\frac{x}{2}\right) + x \sqrt{4-x^2} \right] + C
 \end{aligned}$$

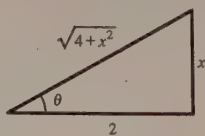


$$x = 2 \sin \theta, dx = 2 \cos \theta d\theta, \sqrt{4-x^2} = 2 \cos \theta$$

29. (a) $x = 2 \tan \theta$

$$dx = 2 \sec^2 \theta d\theta$$

$$\sqrt{4 + x^2} = 2 \sec \theta$$



$$\begin{aligned} \int \frac{x^3}{\sqrt{4 + x^2}} dx &= \int \frac{8 \tan^3 \theta}{2 \sec \theta} 2 \sec^2 \theta d\theta \\ &= 8 \int \frac{\sin^3 \theta}{\cos^4 \theta} d\theta \\ &= 8 \int (1 - \cos^2 \theta) \cos^{-4} \theta \sin \theta d\theta \\ &= 8 \int (\cos^{-4} \theta - \cos^{-2} \theta) \sin \theta d\theta \\ &= 8 \left[\frac{\cos^{-3} \theta}{-3} + \frac{\cos^{-1} \theta}{-1} \right] + C \\ &= \frac{8}{3} \sec \theta (\sec^2 \theta - 3) + C \\ &= \frac{8}{3} \frac{\sqrt{4 + x^2}}{2} \left(\frac{4 + x^2}{4} - 3 \right) + C \\ &= \frac{1}{3} \sqrt{4 + x^2} (x^2 - 8) + C \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int \frac{x^3}{\sqrt{4 + x^2}} dx &= \int \frac{x^2}{\sqrt{4 + x^2}} x dx \\ &= \int \frac{(u^2 - 4)}{u} u du \\ &= \int (u^2 - 4) du \\ &= \frac{1}{3} u^3 - 4u + C \\ &= \frac{u}{3} (u^2 - 12) + C \\ &= \frac{\sqrt{4 + x^2}}{3} (x^2 - 8) + C \end{aligned}$$

$$u^2 = 4 + x^2, 2u du = 2x dx$$

$$\begin{aligned} \text{(c)} \quad \int \frac{x^3}{\sqrt{4 + x^2}} dx &= x^2 \sqrt{4 + x^2} - \int 2x \sqrt{4 + x^2} dx \\ &= x^2 \sqrt{4 + x^2} - \frac{2}{3} (4 + x^2)^{3/2} + C \\ &= \frac{\sqrt{4 + x^2}}{3} (x^2 - 8) + C \end{aligned}$$

$$dv = \frac{x}{\sqrt{4 + x^2}} dx \Rightarrow v = \sqrt{4 + x^2}$$

$$u = x^2 \Rightarrow du = 2x dx$$

31. $\frac{x - 28}{x^2 - x - 6} = \frac{A}{x - 3} + \frac{B}{x + 2}$

$$x - 28 = A(x + 2) + B(x - 3)$$

$$x = -2 \Rightarrow -30 = B(-5) \Rightarrow B = 6$$

$$x = 3 \Rightarrow -25 = A(5) \Rightarrow A = -5$$

$$\int \frac{x - 28}{x^2 - x - 6} dx = \int \left(\frac{-5}{x - 3} + \frac{6}{x + 2} \right) dx = -5 \ln|x - 3| + 6 \ln|x + 2| + C$$

33. $\frac{x^2 + 2x}{(x - 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1}$

$$x^2 + 2x = A(x^2 + 1) + (Bx + C)(x - 1)$$

$$\text{Let } x = 1: 3 = 2A \Rightarrow A = \frac{3}{2}$$

$$\text{Let } x = 0: 0 = A - C \Rightarrow C = \frac{3}{2}$$

$$\text{Let } x = 2: 8 = 5A + 2B + C \Rightarrow B = -\frac{1}{2}$$

$$\begin{aligned} \int \frac{x^2 + 2x}{x^3 - x^2 + x - 1} dx &= \frac{3}{2} \int \frac{1}{x - 1} dx - \frac{1}{2} \int \frac{x - 3}{x^2 + 1} dx \\ &= \frac{3}{2} \int \frac{1}{x - 1} dx - \frac{1}{4} \int \frac{2x}{x^2 + 1} dx + \frac{3}{2} \int \frac{1}{x^2 + 1} dx \\ &= \frac{3}{2} \ln|x - 1| - \frac{1}{4} \ln|x^2 + 1| + \frac{3}{2} \arctan x + C \\ &= \frac{1}{4} [6 \ln|x - 1| - \ln(x^2 + 1) + 6 \arctan x] + C \end{aligned}$$

$$35. \frac{x^2}{x^2 + 2x - 15} = 1 + \frac{15 - 2x}{x^2 + 2x - 15}$$

$$\frac{15 - 2x}{(x - 3)(x + 5)} = \frac{A}{x - 3} + \frac{B}{x + 5}$$

$$15 - 2x = A(x + 5) + B(x - 3)$$

$$\text{Let } x = 3: 9 = 8A \Rightarrow A = \frac{9}{8}$$

$$\text{Let } x = -5: 25 = -8B \Rightarrow B = -\frac{25}{8}$$

$$\begin{aligned} \int \frac{x^2}{x^2 + 2x - 15} dx &= \int dx + \frac{9}{8} \int \frac{1}{x - 3} dx - \frac{25}{8} \int \frac{1}{x + 5} dx \\ &= x + \frac{9}{8} \ln|x - 3| - \frac{25}{8} \ln|x + 5| + C \end{aligned}$$

$$37. \int \frac{x}{(2 + 3x)^2} dx = \frac{1}{9} \left[\frac{2}{2 + 3x} + \ln|2 + 3x| \right] + C$$

(Formula 4)

$$39. \int \frac{x}{1 + \sin x^2} dx = \frac{1}{2} \int \frac{1}{1 + \sin u} du \quad (u = x^2)$$

$$= \frac{1}{2} [\tan u - \sec u] + C \quad (\text{Formula 56})$$

$$= \frac{1}{2} [\tan x^2 - \sec x^2] + C$$

$$41. \int \frac{x}{x^2 + 4x + 8} dx = \frac{1}{2} \left[\ln|x^2 + 4x + 8| - 4 \int \frac{1}{x^2 + 4x + 8} dx \right] \quad (\text{Formula 15})$$

$$= \frac{1}{2} [\ln|x^2 + 4x + 8|] - 2 \left[\frac{2}{\sqrt{32 - 16}} \arctan \left(\frac{2x + 4}{\sqrt{32 - 16}} \right) \right] + C \quad (\text{Formula 14})$$

$$= \frac{1}{2} \ln|x^2 + 4x + 8| - \arctan \left(1 + \frac{x}{2} \right) + C$$

$$43. \int \frac{1}{\sin \pi x \cos \pi x} dx = \frac{1}{\pi} \int \frac{1}{\sin \pi x \cos \pi x} (\pi) dx \quad (u = \pi x)$$

$$= \frac{1}{\pi} \ln|\tan \pi x| + C \quad (\text{Formula 58})$$

$$45. dv = dx \Rightarrow v = x$$

$$u = (\ln x)^n \Rightarrow du = n(\ln x)^{n-1} \frac{1}{x} dx$$

$$\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$$

$$47. \int \theta \sin \theta \cos \theta d\theta = \frac{1}{2} \int \theta \sin 2\theta d\theta$$

$$= -\frac{1}{4} \theta \cos 2\theta + \frac{1}{4} \int \cos 2\theta d\theta = -\frac{1}{4} \theta \cos 2\theta + \frac{1}{8} \sin 2\theta + C = \frac{1}{8} (\sin 2\theta - 2\theta \cos 2\theta) + C$$

$$dv = \sin 2\theta d\theta \Rightarrow v = -\frac{1}{2} \cos 2\theta$$

$$u = \theta \Rightarrow du = d\theta$$

$$\begin{aligned}
 49. \int \frac{x^{1/4}}{1+x^{1/2}} dx &= 4 \int \frac{u(u^3)}{1+u^2} du \\
 &= 4 \int \left(u^2 - 1 + \frac{1}{u^2+1} \right) du \\
 &= 4 \left(\frac{1}{3} u^3 - u + \arctan u \right) + C \\
 &= \frac{4}{3} [x^{3/4} - 3x^{1/4} + 3 \arctan(x^{1/4})] + C
 \end{aligned}$$

$$u = \sqrt[4]{x}, x = u^4, dx = 4u^3 du$$

$$\begin{aligned}
 53. \int \cos x \ln(\sin x) dx &= \sin x \ln(\sin x) - \int \cos x dx \\
 &= \sin x \ln(\sin x) - \sin x + C
 \end{aligned}$$

$$dv = \cos x dx \Rightarrow v = \sin x$$

$$u = \ln(\sin x) \Rightarrow du = \frac{\cos x}{\sin x} dx$$

$$\begin{aligned}
 57. y = \int \ln(x^2 + x) dx &= x \ln|x^2 + x| - \int \frac{2x^2 + x}{x^2 + x} dx \\
 &= x \ln|x^2 + x| - \int \frac{2x + 1}{x + 1} dx \\
 &= x \ln|x^2 + x| - \int 2 dx + \int \frac{1}{x + 1} dx \\
 &= x \ln|x^2 + x| - 2x + \ln|x + 1| + C
 \end{aligned}$$

$$dv = dx \Rightarrow v = x$$

$$u = \ln(x^2 + x) \Rightarrow du = \frac{2x + 1}{x^2 + x} dx$$

$$61. \int_1^4 \frac{\ln x}{x} dx = \left[\frac{1}{2} (\ln x)^2 \right]_1^4 = \frac{1}{2} (\ln 4)^2 = 2(\ln 2)^2 \approx 0.961$$

$$\begin{aligned}
 65. A &= \int_0^4 x \sqrt{4-x} dx = \int_2^0 (4-u^2)u(-2u) du \\
 &= \int_2^0 2(u^4 - 4u^2) du \\
 &= \left[2 \left(\frac{u^5}{5} - \frac{4u^3}{3} \right) \right]_2^0 = \frac{128}{15}
 \end{aligned}$$

$$u = \sqrt{4-x}, x = 4-u^2, dx = -2u du$$

$$69. s = \int_0^\pi \sqrt{1 + \cos^2 x} dx \approx 3.82$$

$$73. \lim_{x \rightarrow \infty} \frac{e^{2x}}{x^2} = \lim_{x \rightarrow \infty} \frac{2e^{2x}}{2x} = \lim_{x \rightarrow \infty} \frac{4e^{2x}}{2} = \infty$$

$$\begin{aligned}
 51. \int \sqrt{1 + \cos x} dx &= \int \frac{\sin x}{\sqrt{1 - \cos x}} dx \\
 &= \int (1 - \cos x)^{-1/2} (\sin x) dx \\
 &= 2\sqrt{1 - \cos x} + C
 \end{aligned}$$

$$u = 1 - \cos x, du = \sin x dx$$

$$\begin{aligned}
 55. y &= \int \frac{9}{x^2 - 9} dx = \frac{3}{2} \ln \left| \frac{x-3}{x+3} \right| + C \\
 &\text{(by Formula 24 of Integration Tables)}
 \end{aligned}$$

$$59. \int_2^{\sqrt{5}} x(x^2 - 4)^{3/2} dx = \left[\frac{1}{5} (x^2 - 4)^{5/2} \right]_2^{\sqrt{5}} = \frac{1}{5}$$

$$63. \int_0^\pi x \sin x dx = \left[-x \cos x + \sin x \right]_0^\pi = \pi$$

$$67. \text{By symmetry, } \bar{x} = 0, A = \frac{1}{2} \pi.$$

$$\bar{y} = \frac{2}{\pi} \left(\frac{1}{2} \right) \int_{-1}^1 (\sqrt{1-x^2})^2 dx = \frac{1}{\pi} \left[x - \frac{1}{3} x^3 \right]_{-1}^1 = \frac{4}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4}{3\pi} \right)$$

$$71. \lim_{x \rightarrow 1} \left[\frac{(\ln x)^2}{x-1} \right] = \lim_{x \rightarrow 1} \left[\frac{2(1/x) \ln x}{1} \right] = 0$$

$$75. y = \lim_{x \rightarrow \infty} (\ln x)^{2/x}$$

$$\ln y = \lim_{x \rightarrow \infty} \frac{2 \ln(\ln x)}{x} = \lim_{x \rightarrow \infty} \left[\frac{2/(x \ln x)}{1} \right] = 0$$

$$\text{Since } \ln y = 0, y = 1.$$

$$77. \lim_{n \rightarrow \infty} 1000 \left(1 + \frac{0.09}{n} \right)^n = 1000 \lim_{n \rightarrow \infty} \left(1 + \frac{0.09}{n} \right)^n$$

$$\text{Let } y = \lim_{n \rightarrow \infty} \left(1 + \frac{0.09}{n} \right)^n.$$

$$\ln y = \lim_{n \rightarrow \infty} n \ln \left(1 + \frac{0.09}{n} \right) = \lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{0.09}{n} \right)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(\frac{\frac{-0.09/n^2}{1 + (0.09/n)}}{-\frac{1}{n^2}} \right) = \lim_{n \rightarrow \infty} \frac{0.09}{1 + \left(\frac{0.09}{n} \right)} = 0.09$$

$$\text{Thus, } \ln y = 0.09 \Rightarrow y = e^{0.09} \text{ and } \lim_{n \rightarrow \infty} 1000 \left(1 + \frac{0.09}{n} \right)^n = 1000e^{0.09} \approx 1094.17.$$

$$79. \int_0^{16} \frac{1}{\sqrt[4]{x}} dx = \lim_{b \rightarrow 0^+} \left[\frac{4}{3} x^{3/4} \right]_b^{16} = \frac{32}{3}$$

Converges

$$81. \int_1^{\infty} x^2 \ln x dx = \lim_{b \rightarrow \infty} \left[\frac{x^3}{9} (-1 + 3 \ln x) \right]_1^b = \infty$$

Diverges

$$\begin{aligned} 83. \int_0^{t_0} 500,000 e^{-0.05t} dt &= \left[\frac{500,000}{-0.05} e^{-0.05t} \right]_0^{t_0} \\ &= \frac{-500,000}{0.05} (e^{-0.05t_0} - 1) \\ &= 10,000,000 (1 - e^{-0.05t_0}) \end{aligned}$$

$$(a) t_0 = 20: \$6,321,205.59$$

$$(b) t_0 \rightarrow \infty: \$10,000,000$$

$$85. (a) P(13 \leq x < \infty) = \frac{1}{0.95\sqrt{2\pi}} \int_{13}^{\infty} e^{-(x-12.9)^2/2(0.95)^2} dx \approx 0.4581$$

$$(b) P(15 \leq x < \infty) = \frac{1}{0.95\sqrt{2\pi}} \int_{15}^{\infty} e^{-(x-12.9)^2/2(0.95)^2} dx \approx 0.0135$$

Problem Solving for Chapter 7

$$1. (a) \int_{-1}^1 (1 - x^2) dx = \left[x - \frac{x^3}{3} \right]_{-1}^1 = 2 \left(1 - \frac{1}{3} \right) = \frac{4}{3}$$

$$\int_{-1}^1 (1 - x^2)^2 dx = \int_{-1}^1 (1 - 2x^2 + x^4) dx = \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^1 = 2 \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{16}{15}$$

$$(b) \text{ Let } x = \sin u, dx = \cos u du, 1 - x^2 = 1 - \sin^2 u = \cos^2 u.$$

$$\begin{aligned} \int_{-1}^1 (1 - x^2)^n dx &= \int_{-\pi/2}^{\pi/2} (\cos^2 u)^n \cos u du \\ &= \int_{-\pi/2}^{\pi/2} \cos^{2n+1} u du \\ &= 2 \left[\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{(2n)}{(2n+1)} \right] \quad (\text{Wallis's Formula}) \\ &= 2 \left[\frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n)(2n+1)} \right] \\ &= \frac{2(2^{2n})(n!)^2}{(2n+1)!} = \frac{2^{2n+1}(n!)^2}{(2n+1)!} \end{aligned}$$

$$3. \quad \lim_{x \rightarrow \infty} \left(\frac{x+c}{x-c} \right)^x = 9$$

$$\lim_{x \rightarrow \infty} x \ln \left(\frac{x+c}{x-c} \right) = \ln 9$$

$$\lim_{x \rightarrow \infty} \frac{\ln(x+c) - \ln(x-c)}{1/x} = \ln 9$$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x+c} - \frac{1}{x-c}}{-\frac{1}{x^2}} = \ln 9$$

$$\lim_{x \rightarrow \infty} \frac{-2c}{(x+c)(x-c)} (-x^2) = \ln 9$$

$$\lim_{x \rightarrow \infty} \left(\frac{2cx^2}{x^2 - c^2} \right) = \ln 9$$

$$2c = \ln 9$$

$$2c = 2 \ln 3$$

$$c = \ln 3$$

$$5. \quad \sin \theta = \frac{PB}{OP} = PB, \cos \theta = OB$$

$$AQ = \widehat{AP} = \theta$$

$$BR = OR + OB = OR + \cos \theta$$

The triangles $\triangle AQR$ and $\triangle BPR$ are similar:

$$\frac{AR}{AQ} = \frac{BR}{BP} \Rightarrow \frac{OR+1}{\theta} = \frac{OR+\cos \theta}{\sin \theta}$$

$$\sin \theta (OR) + \sin \theta = (OR)\theta + \theta \cos \theta$$

$$OR = \frac{\theta \cos \theta - \sin \theta}{\sin \theta - \theta}$$

$$\lim_{\theta \rightarrow 0^+} OR = \lim_{\theta \rightarrow 0^+} \frac{\theta \cos \theta - \sin \theta}{\sin \theta - \theta}$$

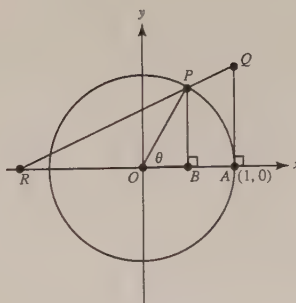
$$= \lim_{\theta \rightarrow 0^+} \frac{-\theta \sin \theta + \cos \theta - \cos \theta}{\cos \theta - 1}$$

$$= \lim_{\theta \rightarrow 0^+} \frac{-\theta \sin \theta}{\cos \theta - 1}$$

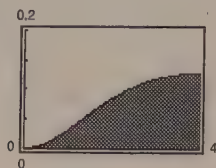
$$= \lim_{\theta \rightarrow 0^+} \frac{-\sin \theta - \theta \cos \theta}{-\sin \theta}$$

$$= \lim_{\theta \rightarrow 0^+} \frac{\cos \theta + \cos \theta - \theta \sin \theta}{\cos \theta}$$

$$= 2$$



7. (a) Area
- ≈ 0.2986



- (b) Let
- $x = 3 \tan \theta$
- ,
- $dx = 3 \sec^2 \theta d\theta$
- ,
- $x^2 + 9 = 9 \sec^2 \theta$
- .

$$\int \frac{x^2}{(x^2 + 9)^{3/2}} dx = \int \frac{9 \tan^2 \theta}{(9 \sec^2 \theta)^{3/2}} (3 \sec^2 \theta d\theta)$$

$$= \int \frac{\tan^2 \theta}{\sec \theta} d\theta$$

$$= \int \frac{\sin^2 \theta}{\cos \theta} d\theta$$

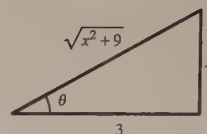
$$= \int \frac{1 - \cos^2 \theta}{\cos \theta} d\theta$$

$$= \ln |\sec \theta + \tan \theta| - \sin \theta + C$$

$$\text{Area} = \int_0^4 \frac{x^2}{(x^2 + 9)^{3/2}} dx = \left[\ln |\sec \theta + \tan \theta| - \sin \theta \right]_0^{\tan^{-1}(4/3)}$$

$$= \left[\ln \left(\frac{\sqrt{x^2 + 9}}{3} + \frac{x}{3} \right) - \frac{x}{\sqrt{x^2 + 9}} \right]_0^4$$

$$= \ln \left(\frac{5}{3} + \frac{4}{3} \right) - \frac{4}{5} = \ln 3 - \frac{4}{5}$$



- (c)
- $x = 3 \sinh u$
- ,
- $dx = 3 \cosh u du$
- ,
- $x^2 + 9 = 9 \sinh^2 u + 9 = 9 \cosh^2 u$

$$A = \int_0^4 \frac{x^2}{(x^2 + 9)^{3/2}} dx = \int_0^{\sinh^{-1}(4/3)} \frac{9 \sinh^2 u}{(9 \cosh^2 u)^{3/2}} (3 \cosh u du)$$

$$= \int_0^{\sinh^{-1}(4/3)} \tanh^2 u du$$

$$= \int_0^{\sinh^{-1}(4/3)} (1 - \operatorname{sech}^2 u) du$$

$$= \left[u - \tanh u \right]_0^{\sinh^{-1}(4/3)}$$

$$= \sinh^{-1} \left(\frac{4}{3} \right) - \tanh \left(\sinh^{-1} \left(\frac{4}{3} \right) \right)$$

$$= \ln \left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1} \right) - \tanh \left[\ln \left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1} \right) \right]$$

$$= \ln \left(\frac{4}{3} + \frac{5}{3} \right) - \tanh \left(\ln \left(\frac{4}{3} + \frac{5}{3} \right) \right)$$

$$= \ln 3 - \tanh(\ln 3)$$

$$= \ln 3 - \frac{3 - (1/3)}{3 + (1/3)}$$

$$= \ln 3 - \frac{4}{5}$$

$$9. y = \ln(1 - x^2), y' = \frac{-2x}{1 - x^2}$$

$$1 + (y')^2 = 1 + \frac{4x^2}{(1 - x^2)^2} = \frac{1 - 2x^2 + x^4 + 4x^2}{(1 - x^2)^2} = \left(\frac{1 + x^2}{1 - x^2}\right)^2$$

$$\begin{aligned} \text{Arc length} &= \int_0^{1/2} \sqrt{1 + (y')^2} dx \\ &= \int_0^{1/2} \left(\frac{1 + x^2}{1 - x^2}\right) dx \\ &= \int_0^{1/2} \left(-1 + \frac{2}{1 - x^2}\right) dx \\ &= \int_0^{1/2} \left(-1 + \frac{1}{x + 1} + \frac{1}{1 - x}\right) dx \\ &= \left[-x + \ln(1 + x) - \ln(1 - x)\right]_0^{1/2} \\ &= \left(-\frac{1}{2} + \ln \frac{3}{2} - \ln \frac{1}{2}\right) \\ &= -\frac{1}{2} + \ln 3 - \ln 2 + \ln 2 \\ &= \ln 3 - \frac{1}{2} \approx 0.5986 \end{aligned}$$

$$11. \text{ Consider } \int \frac{1}{\ln x} dx.$$

$$\text{Let } u = \ln x, du = \frac{1}{x} dx, x = e^u. \text{ Then } \int \frac{1}{\ln x} dx = \int \frac{1}{u} e^u du = \int \frac{e^u}{u} du.$$

$$\text{If } \int \frac{1}{\ln x} dx \text{ were elementary, then } \int \frac{e^u}{u} du \text{ would be too, which is false.}$$

$$\text{Hence, } \int \frac{1}{\ln x} dx \text{ is not elementary.}$$

$$13. x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d)$$

$$= x^4 + (a + c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd$$

$$a = -c, b = d = 1, a = \sqrt{2}$$

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

$$\begin{aligned} \int_0^1 \frac{1}{x^4 + 1} dx &= \int_0^1 \frac{Ax + B}{x^2 + \sqrt{2}x + 1} dx + \int_0^1 \frac{Cx + D}{x^2 - \sqrt{2}x + 1} dx \\ &= \int_0^1 \frac{\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1} dx - \int_0^1 \frac{-\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1} dx \\ &= \frac{\sqrt{2}}{4} \left[\arctan(\sqrt{2}x + 1) + \arctan(\sqrt{2}x - 1) \right]_0^1 + \frac{\sqrt{2}}{8} \left[\ln(x^2 + \sqrt{2}x + 1) - \ln(x^2 - \sqrt{2}x + 1) \right]_0^1 \\ &= \frac{\sqrt{2}}{4} [\arctan(\sqrt{2} + 1) + \arctan(\sqrt{2} - 1)] + \frac{\sqrt{2}}{8} [\ln(2 + \sqrt{2}) - \ln(2 - \sqrt{2})] - \frac{\sqrt{2}}{4} \left[\frac{\pi}{4} - \frac{\pi}{4} \right] - \frac{\sqrt{2}}{8} [0] \\ &\approx 0.5554 + 0.3116 \\ &\approx 0.8670 \end{aligned}$$

15. Using a graphing utility,

$$(a) \lim_{x \rightarrow 0^+} \left(\cot x + \frac{1}{x} \right) = \infty$$

$$(b) \lim_{x \rightarrow 0^+} \left(\cot x - \frac{1}{x} \right) = 0$$

$$(c) \lim_{x \rightarrow 0^+} \left(\cot x + \frac{1}{x} \right) \left(\cot x - \frac{1}{x} \right) \approx -\frac{2}{3}.$$

Analytically,

$$(a) \lim_{x \rightarrow 0^+} \left(\cot x + \frac{1}{x} \right) = \infty + \infty = \infty$$

$$\begin{aligned} (b) \lim_{x \rightarrow 0^+} \left(\cot x - \frac{1}{x} \right) &= \lim_{x \rightarrow 0^+} \frac{x \cot x - 1}{x} = \lim_{x \rightarrow 0^+} \frac{x \cos x - \sin x}{x \sin x} \\ &= \lim_{x \rightarrow 0^+} \frac{\cos x - x \sin x - \cos x}{\sin x + x \cos x} = \lim_{x \rightarrow 0^+} \frac{-x \sin x}{\sin x + x \cos x} \\ &= \lim_{x \rightarrow 0^+} \frac{-\sin x - x \cos x}{\cos x + \cos x - x \sin x} = 0. \end{aligned}$$

$$\begin{aligned} (c) \left(\cot x + \frac{1}{x} \right) \left(\cot x - \frac{1}{x} \right) &= \cot^2 x - \frac{1}{x^2} \\ &= \frac{x^2 \cot^2 x - 1}{x^2} \\ \lim_{x \rightarrow 0^+} \frac{x^2 \cot^2 x - 1}{x^2} &= \lim_{x \rightarrow 0^+} \frac{2x \cot^2 x - 2x^2 \cot x \csc^2 x}{2x} \\ &= \lim_{x \rightarrow 0^+} \frac{\cot^2 x - x \cot x \csc^2 x}{1} \\ &= \lim_{x \rightarrow 0^+} \frac{\cos^2 x \sin x - x \cos x}{\sin^3 x} \\ &= \lim_{x \rightarrow 0^+} \frac{(1 - \sin^2 x) \sin x - x \cos x}{\sin^3 x} \\ &= \lim_{x \rightarrow 0^+} \frac{\sin x - x \cos x}{\sin^3 x} - 1 \end{aligned}$$

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow 0^+} \frac{\sin x - x \cos x}{\sin^3 x} &= \lim_{x \rightarrow 0^+} \frac{\cos x - \cos x + x \sin x}{3 \sin^2 x \cos x} \\ &= \lim_{x \rightarrow 0^+} \frac{x}{3 \sin x \cdot \cos x} \\ &= \lim_{x \rightarrow 0^+} \left(\frac{x}{\sin x} \right) \frac{1}{3 \cos x} = \frac{1}{3}. \end{aligned}$$

$$\text{Thus, } \lim_{x \rightarrow 0^+} \left(\cot x + \frac{1}{x} \right) \left(\cot x - \frac{1}{x} \right) = \frac{1}{3} - 1 = -\frac{2}{3}.$$

The form $0 \cdot \infty$ is indeterminate.

$$17. \frac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \frac{P_1}{x} + \frac{P_2}{x-1} + \frac{P_3}{x+4} + \frac{P_4}{x-3} \Rightarrow c_1 = 0, c_2 = 1, c_3 = -4, c_4 = 3$$

$$N(x) = x^3 - 3x^2 + 1$$

$$D'(x) = 4x^3 - 26x + 12$$

$$P_1 = \frac{N(0)}{D'(0)} = \frac{1}{12}$$

$$P_2 = \frac{N(1)}{D'(1)} = \frac{-1}{-10} = \frac{1}{10}$$

$$P_3 = \frac{N(-4)}{D'(-4)} = \frac{-111}{-140} = \frac{111}{140}$$

$$P_4 = \frac{N(3)}{D'(3)} = \frac{1}{42}$$

$$\text{Thus, } \frac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \frac{1/12}{x} + \frac{1/10}{x-1} + \frac{111/140}{x+4} + \frac{1/42}{x-3}.$$

19. By parts,

$$\begin{aligned} \int_a^b f(x)g''(x) dx &= \left[f(x)g'(x) \right]_a^b - \int_a^b f'(x)g'(x) dx \\ &= - \int_a^b f'(x)g'(x) dx \quad 8 \\ &= \left[-f'(x)g(x) \right]_a^b + \int_a^b g(x)f''(x) dx \\ &= \int_a^b f''(x)g(x) dx. \end{aligned}$$

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Infinite Series

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CHAPTER 8

Infinite Series

Section 8.1 Sequences

Solutions to Odd-Numbered Exercises

1. $a_n = 2^n$

$$a_1 = 2^1 = 2$$

$$a_2 = 2^2 = 4$$

$$a_3 = 2^3 = 8$$

$$a_4 = 2^4 = 16$$

$$a_5 = 2^5 = 32$$

3. $a_n = \left(-\frac{1}{2}\right)^n$

$$a_1 = \left(-\frac{1}{2}\right)^1 = -\frac{1}{2}$$

$$a_2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$a_3 = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$$

$$a_4 = \left(-\frac{1}{2}\right)^4 = \frac{1}{16}$$

$$a_5 = \left(-\frac{1}{2}\right)^5 = -\frac{1}{32}$$

5. $a_n = \sin \frac{n\pi}{2}$

$$a_1 = \sin \frac{\pi}{2} = 1$$

$$a_2 = \sin \pi = 0$$

$$a_3 = \sin \frac{3\pi}{2} = -1$$

$$a_4 = \sin 2\pi = 0$$

$$a_5 = \sin \frac{5\pi}{2} = 1$$

7. $a_n = \frac{(-1)^{n(n+1)/2}}{n^2}$

$$a_1 = \frac{(-1)^1}{1^2} = -1$$

$$a_2 = \frac{(-1)^3}{2^2} = -\frac{1}{4}$$

$$a_3 = \frac{(-1)^6}{3^2} = \frac{1}{9}$$

$$a_4 = \frac{(-1)^{10}}{4^2} = \frac{1}{16}$$

$$a_5 = \frac{(-1)^{15}}{5^2} = -\frac{1}{25}$$

9. $a_n = 5 - \frac{1}{n} + \frac{1}{n^2}$

$$a_1 = 5 - 1 + 1 = 5$$

$$a_2 = 5 - \frac{1}{2} + \frac{1}{4} = \frac{19}{4}$$

$$a_3 = 5 - \frac{1}{3} + \frac{1}{9} = \frac{43}{9}$$

$$a_4 = 5 - \frac{1}{4} + \frac{1}{16} = \frac{77}{16}$$

$$a_5 = 5 - \frac{1}{5} + \frac{1}{25} = \frac{121}{25}$$

11. $a_n = \frac{3^n}{n!}$

$$a_1 = \frac{3}{1!} = 3$$

$$a_2 = \frac{3^2}{2!} = \frac{9}{2}$$

$$a_3 = \frac{3^3}{3!} = \frac{27}{6}$$

$$a_4 = \frac{3^4}{4!} = \frac{81}{24}$$

$$a_5 = \frac{3^5}{5!} = \frac{243}{120}$$

13. $a_1 = 3, a_{k+1} = 2(a_k - 1)$

$$\begin{aligned} a_2 &= 2(a_1 - 1) \\ &= 2(3 - 1) = 4 \end{aligned}$$

$$\begin{aligned} a_3 &= 2(a_2 - 1) \\ &= 2(4 - 1) = 6 \end{aligned}$$

$$\begin{aligned} a_4 &= 2(a_3 - 1) \\ &= 2(6 - 1) = 10 \end{aligned}$$

$$\begin{aligned} a_5 &= 2(a_4 - 1) \\ &= 2(10 - 1) = 18 \end{aligned}$$

15. $a_1 = 32, a_{k+1} = \frac{1}{2}a_k$

$$a_2 = \frac{1}{2}a_1 = \frac{1}{2}(32) = 16$$

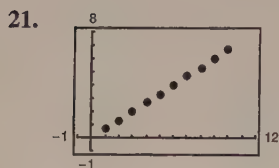
$$a_3 = \frac{1}{2}a_2 = \frac{1}{2}(16) = 8$$

$$a_4 = \frac{1}{2}a_3 = \frac{1}{2}(8) = 4$$

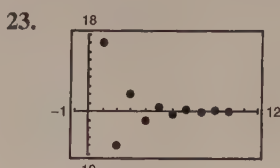
$$a_5 = \frac{1}{2}a_4 = \frac{1}{2}(4) = 2$$

17. Because $a_1 = 8/(1+1) = 4$ and $a_2 = 8/(2+1) = \frac{8}{3}$, the sequence matches graph (d).

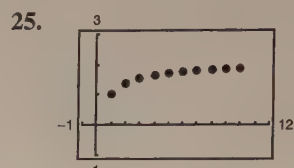
19. This sequence decreases and $a_1 = 4$, $a_2 = 4(0.5) = 2$. Matches (c).



$$a_n = \frac{2}{3}n, n = 1, \dots, 10$$



$$a_n = 16(-0.5)^{n-1}, n = 1, \dots, 10$$



$$a_n = \frac{2n}{n+1}, n = 1, 2, \dots, 10$$

27. $a_n = 3n - 1$

$$a_5 = 3(5) - 1 = 14$$

$$a_6 = 3(6) - 1 = 17$$

Add 3 to preceding term.

29. $a_n = \frac{3}{(-2)^{n-1}}$

$$a_5 = \frac{3}{(-2)^4} = \frac{3}{16}$$

$$a_6 = \frac{3}{(-2)^5} = -\frac{3}{32}$$

Multiply the preceding term by $-\frac{1}{2}$.

31. $\frac{10!}{8!} = \frac{8!(9)(10)}{8!}$
 $= (9)(10) = 90$

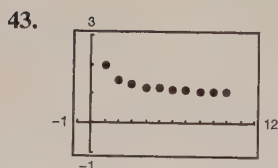
33. $\frac{(n+1)!}{n!} = \frac{n!(n+1)}{n!}$
 $= n+1$

35. $\frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)!}{(2n-1)!(2n)(2n+1)}$
 $= \frac{1}{2n(2n+1)}$

37. $\lim_{n \rightarrow \infty} \frac{5n^2}{n^2 + 2} = 5$

39. $\lim_{n \rightarrow \infty} \frac{2n}{\sqrt{n^2 + 1}} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt{1 + (1/n^2)}}$
 $= \frac{2}{1} = 2$

41. $\lim_{n \rightarrow \infty} \sin\left(\frac{1}{n}\right) = 0$

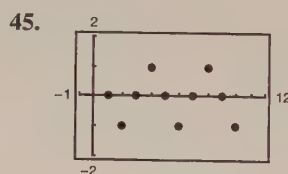


The graph seems to indicate that the sequence converges to 1. Analytically,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+1}{n} = \lim_{x \rightarrow \infty} \frac{x+1}{x} = \lim_{x \rightarrow \infty} 1 = 1.$$

47. $\lim_{n \rightarrow \infty} (-1)^n \left(\frac{n}{n+1} \right)$

does not exist (oscillates between -1 and 1), diverges.



The graph seems to indicate that the sequence diverges. Analytically, the sequence is

$$\{a_n\} = \{0, -1, 0, 1, 0, -1, \dots\}.$$

Hence, $\lim_{n \rightarrow \infty} a_n$ does not exist.

49. $\lim_{n \rightarrow \infty} \frac{3n^2 - n + 4}{2n^2 + 1} = \frac{3}{2}$, converges

51. $\lim_{n \rightarrow \infty} \frac{1 + (-1)^n}{n} = 0$, converges

53. $\lim_{n \rightarrow \infty} \frac{\ln(n^3)}{2n} = \lim_{n \rightarrow \infty} \frac{3 \ln(n)}{2n}$
 $= \lim_{n \rightarrow \infty} \frac{3}{2} \left(\frac{1}{n} \right) = 0$, converges

(L'Hôpital's Rule)

$$55. \lim_{n \rightarrow \infty} \left(\frac{3}{4}\right)^n = 0, \text{ converges}$$

$$57. \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \lim_{n \rightarrow \infty} (n+1) = \infty, \text{ diverges}$$

$$59. \lim_{n \rightarrow \infty} \left(\frac{n-1}{n} - \frac{n}{n-1}\right) = \lim_{n \rightarrow \infty} \frac{(n-1)^2 - n^2}{n(n-1)} \\ = \lim_{n \rightarrow \infty} \frac{1-2n}{n^2-n} = 0, \text{ converges}$$

$$61. \lim_{n \rightarrow \infty} \frac{n^p}{e^n} = 0, \text{ converges} \\ (p > 0, n \geq 2)$$

$$63. a_n = \left(1 + \frac{k}{n}\right)^n \\ \lim_{n \rightarrow \infty} \left(1 + \frac{k}{n}\right)^n = \lim_{u \rightarrow 0} [(1+u)^{1/u}]^k = e^k$$

$$65. \lim_{n \rightarrow \infty} \frac{\sin n}{n} = \lim_{n \rightarrow \infty} (\sin n) \frac{1}{n} = 0, \text{ converges}$$

where $u = \frac{k}{n}$, converges

$$67. a_n = 3n - 2$$

$$69. a_n = n^2 - 2$$

$$71. a_n = \frac{n+1}{n+2}$$

$$73. a_n = \frac{(-1)^{n-1}}{2^{n-2}}$$

$$75. a_n = 1 + \frac{1}{n} = \frac{n+1}{n}$$

$$77. a_n = \frac{n}{(n+1)(n+2)}$$

$$79. a_n = \frac{(-1)^{n-1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{(-1)^{n-1} 2^n n!}{(2n)!}$$

$$81. a_n = 4 - \frac{1}{n} < 4 - \frac{1}{n+1} = a_{n+1}, \\ \text{monotonic; } |a_n| < 4 \text{ bounded.}$$

$$83. \frac{n}{2^{n+2}} \stackrel{?}{\geq} \frac{n+1}{2^{(n+1)+2}} \\ 2^{n+3}n \stackrel{?}{\geq} 2^{n+2}(n+1) \\ 2n \stackrel{?}{\geq} n+1 \\ n \geq 1$$

$$85. a_n = (-1)^n \left(\frac{1}{n}\right)$$

$$a_1 = -1$$

$$a_2 = \frac{1}{2}$$

$$a_3 = -\frac{1}{3}$$

Not monotonic; $|a_n| \leq 1$, bounded

Hence, $n \geq 1$

$$2n \geq n+1$$

$$2^{n+3}n \geq 2^{n+2}(n+1)$$

$$\frac{n}{2^{n+2}} \geq \frac{n+1}{2^{(n+1)+2}}$$

$$a_n \geq a_{n+1}$$

True; monotonic; $|a_n| \leq \frac{1}{8}$, bounded

$$87. a_n = \left(\frac{2}{3}\right)^n > \left(\frac{2}{3}\right)^{n+1} = a_{n+1}$$

Monotonic; $|a_n| \leq \frac{2}{3}$, bounded

$$89. a_n = \sin\left(\frac{n\pi}{6}\right)$$

$$a_1 = 0.500$$

$$a_2 = 0.8660$$

$$a_3 = 1.000$$

$$a_4 = 0.8660$$

Not monotonic; $|a_n| \leq 1$, bounded

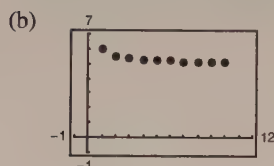
91. (a) $a_n = 5 + \frac{1}{n}$

$$\left| 5 + \frac{1}{n} \right| \leq 6 \Rightarrow \{a_n\} \text{ bounded}$$

$$a_n = 5 + \frac{1}{n} > 5 + \frac{1}{n+1}$$

$$= a_{n+1} \Rightarrow \{a_n\} \text{ monotonic}$$

Therefore, $\{a_n\}$ converges.



$$\lim_{n \rightarrow \infty} \left(5 + \frac{1}{n} \right) = 5$$

95. $A_n = P \left[1 + \frac{r}{12} \right]^n$

(a) $\lim_{n \rightarrow \infty} A_n = \infty$, divergent. The amount will grow arbitrarily large over time.

(b) $A_n = 9000 \left[1 + \frac{0.115}{12} \right]^n$

$$A_1 = \$9086.25 \quad A_6 = \$9530.06$$

$$A_2 = \$9173.33 \quad A_7 = \$9621.39$$

$$A_3 = \$9261.24 \quad A_8 = \$9713.59$$

$$A_4 = \$9349.99 \quad A_9 = \$9806.68$$

$$A_5 = \$9439.60 \quad A_{10} = \$9900.66$$

99. $a_n = 10 - \frac{1}{n}$

103. (a) $A_n = (0.8)^n (2.5)$ billion

(b) $A_1 = \$2$ billion

$$A_2 = \$1.6 \text{ billion}$$

$$A_3 = \$1.28 \text{ billion}$$

$$A_4 = \$1.024 \text{ billion}$$

(c) $\lim_{n \rightarrow \infty} (0.8)^n (2.5) = 0$

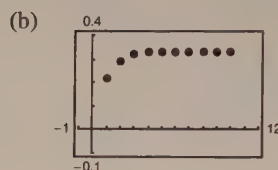
93. (a) $a_n = \frac{1}{3} \left(1 - \frac{1}{3^n} \right)$

$$\left| \frac{1}{3} \left(1 - \frac{1}{3^n} \right) \right| < \frac{1}{3} \Rightarrow \{a_n\} \text{ bounded}$$

$$a_n = \frac{1}{3} \left(1 - \frac{1}{3^n} \right) < \frac{1}{3} \left(1 - \frac{1}{3^{n+1}} \right)$$

$$= a_{n+1} \Rightarrow \{a_n\} \text{ monotonic}$$

Therefore, $\{a_n\}$ converges.



$$\lim_{n \rightarrow \infty} \left[\frac{1}{3} \left(1 - \frac{1}{3^n} \right) \right] = \frac{1}{3}$$

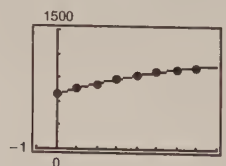
97. (a) A sequence is a function whose domain is the set of positive integers.

(b) A sequence converges if it has a limit.

(c) A bounded monotonic sequence is a sequence that has nondecreasing or nonincreasing terms, and an upper and lower bound.

101. $a_n = \frac{3n}{4n+1}$

105. (a) $a_n = -3.7262n^2 + 75.9167n + 684.25$



(b) For 2004, $n = 14$ and $a_{14} \approx 1017$, or \$1017.

107. $a_n = \frac{10^n}{n!}$

$$\begin{aligned} \text{(a) } a_9 &= a_{10} = \frac{10^9}{9!} \\ &= \frac{1,000,000,000}{362,880} \\ &= \frac{1,562,500}{567} \end{aligned}$$

(b) Decreasing

(c) Factorials increase more rapidly than exponentials.

109. $\{a_n\} = \{\sqrt[n]{n}\} = \{n^{1/n}\}$

$$\begin{aligned} a_1 &= 1^{1/1} = 1 \\ a_2 &= \sqrt{2} \approx 1.4142 \\ a_3 &= \sqrt[3]{3} \approx 1.4422 \\ a_4 &= \sqrt[4]{4} \approx 1.4142 \\ a_5 &= \sqrt[5]{5} \approx 1.3797 \\ a_6 &= \sqrt[6]{6} \approx 1.3480 \end{aligned}$$

Let $y = \lim_{n \rightarrow \infty} n^{1/n}$.

$$\begin{aligned} \ln y &= \lim_{n \rightarrow \infty} \left(\frac{1}{n} \ln n \right) \\ &= \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0 \end{aligned}$$

Since $\ln y = 0$, we have $y = e^0 = 1$. Therefore,
 $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

111. $a_{n+2} = a_n + a_{n+1}$

$$\begin{aligned} \text{(a) } a_1 &= 1 & a_7 &= 8 + 5 = 13 \\ a_2 &= 1 & a_8 &= 13 + 8 = 21 \\ a_3 &= 1 + 1 = 2 & a_9 &= 21 + 13 = 34 \\ a_4 &= 2 + 1 = 3 & a_{10} &= 34 + 21 = 55 \\ a_5 &= 3 + 2 = 5 & a_{11} &= 55 + 34 = 89 \\ a_6 &= 5 + 3 = 8 & a_{12} &= 89 + 55 = 144 \end{aligned}$$

(b) $b_n = \frac{a_{n+1}}{a_n}, n \geq 1$

$$\begin{aligned} b_1 &= \frac{1}{1} = 1 & b_6 &= \frac{13}{8} \\ b_2 &= \frac{2}{1} = 2 & b_7 &= \frac{21}{13} \\ b_3 &= \frac{3}{2} & b_8 &= \frac{34}{21} \\ b_4 &= \frac{5}{3} & b_9 &= \frac{55}{34} \\ b_5 &= \frac{8}{5} & b_{10} &= \frac{89}{55} \end{aligned}$$

113. True

117. $a_1 = \sqrt{2} \approx 1.4142$

$$a_2 = \sqrt{2 + \sqrt{2}} \approx 1.8478$$

$$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.9616$$

$$a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \approx 1.9904$$

$$a_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \approx 1.9976$$

$\{a_n\}$ is increasing and bounded by 2, and hence converges to L . Letting $\lim_{n \rightarrow \infty} a_n = L$ implies that $\sqrt{2 + L} = L \Rightarrow L = 2$.
Hence, $\lim_{n \rightarrow \infty} a_n = 2$.

$$\begin{aligned} \text{(c) } 1 + \frac{1}{b_{n-1}} &= 1 + \frac{1}{a_n/a_{n-1}} \\ &= 1 + \frac{a_{n-1}}{a_n} \\ &= \frac{a_n + a_{n-1}}{a_n} = \frac{a_{n+1}}{a_n} = b_n \end{aligned}$$

(d) If $\lim_{n \rightarrow \infty} b_n = \rho$, then $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{b_{n-1}} \right) = \rho$.

Since $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} b_{n-1}$ we have,

$$1 + (1/\rho) = \rho.$$

$$\rho + 1 = \rho^2$$

$$0 = \rho^2 - \rho - 1$$

$$\rho = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Since a_n , and thus b_n , is positive,

$$\rho = (1 + \sqrt{5})/2 \approx 1.6180.$$

115. True

Section 8.2 Series and Convergence

1. $S_1 = 1$

$$S_2 = 1 + \frac{1}{4} = 1.2500$$

$$S_3 = 1 + \frac{1}{4} + \frac{1}{9} \approx 1.3611$$

$$S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.4236$$

$$S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} \approx 1.4636$$

3. $S_1 = 3$

$$S_2 = 3 - \frac{9}{2} = -1.5$$

$$S_3 = 3 - \frac{9}{2} + \frac{27}{4} = 5.25$$

$$S_4 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} = -4.875$$

$$S_5 = 3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} = 10.3125$$

5. $S_1 = 3$

$$S_2 = 3 + \frac{3}{2} = 4.5$$

$$S_3 = 3 + \frac{3}{2} + \frac{3}{4} = 5.250$$

$$S_4 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} = 5.625$$

$$S_5 = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} = 5.8125$$

7. $\sum_{n=0}^{\infty} 3\left(\frac{3}{2}\right)^n$ Geometric series

$$r = \frac{3}{2} > 1$$

Diverges by Theorem 8.6

9. $\sum_{n=0}^{\infty} 1000(1.055)^n$ Geometric series

$$r = 1.055 > 1$$

Diverges by Theorem 8.6

11. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$$

Diverges by Theorem 8.9

13. $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = 1 \neq 0$$

Diverges by Theorem 8.9

15. $\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$

$$\lim_{n \rightarrow \infty} \frac{2^n+1}{2^{n+1}} = \lim_{n \rightarrow \infty} \frac{1+2^{-n}}{2} = \frac{1}{2} \neq 0$$

Diverges by Theorem 8.9

17. $\sum_{n=0}^{\infty} \frac{9}{4}\left(\frac{1}{4}\right)^n = \frac{9}{4}\left[1 + \frac{1}{4} + \frac{1}{16} + \cdots\right]$

$$S_0 = \frac{9}{4}, S_1 = \frac{9}{4} \cdot \frac{5}{4} = \frac{45}{16}, S_2 = \frac{9}{4} \cdot \frac{21}{16} \approx 2.95, \dots$$

Matches graph (c).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \left(\frac{9}{4}\right)\left(\frac{1}{4}\right)^n = \frac{9/4}{1-1/4} = \frac{9/4}{3/4} = 3$$

19. $\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n = \frac{15}{4}\left[1 - \frac{1}{4} + \frac{1}{16} - \cdots\right]$

$$S_0 = \frac{15}{4}, S_1 = \frac{45}{16}, S_2 \approx 3.05, \dots$$

Matches graph (a).

Analytically, the series is geometric:

$$\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n = \frac{15/4}{1-(-1/4)} = \frac{15/4}{5/4} = 3$$

21. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$ $S_n = 1 - \frac{1}{n+1}$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

23. $\sum_{n=0}^{\infty} 2\left(\frac{3}{4}\right)^n$

Geometric series with $r = \frac{3}{4} < 1$.

Converges by Theorem 8.6

25. $\sum_{n=0}^{\infty} (0.9)^n$

Geometric series with $r = 0.9 < 1$.

Converges by Theorem 8.6

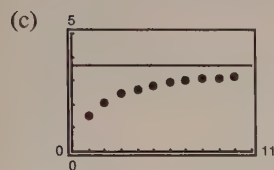
$$27. (a) \sum_{n=1}^{\infty} \frac{6}{n(n+3)} = 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3} \right)$$

$$= 2 \left[\left(1 - \frac{1}{4} \right) + \left(\frac{1}{2} - \frac{1}{5} \right) + \left(\frac{1}{3} - \frac{1}{6} \right) + \left(\frac{1}{4} - \frac{1}{7} \right) + \cdots \right]$$

$$= 2 \left[1 + \frac{1}{2} + \frac{1}{3} \right] = \frac{11}{3} \approx 3.667$$

(b)

n	5	10	20	50	100
S_n	2.7976	3.1643	3.3936	3.5513	3.6078

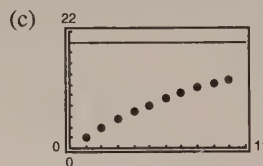


(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

$$29. (a) \sum_{n=1}^{\infty} 2(0.9)^{n-1} = \sum_{n=0}^{\infty} 2(0.9)^n = \frac{2}{1-0.9} = 20$$

(b)

n	5	10	20	50	100
S_n	8.1902	13.0264	17.5685	19.8969	19.9995

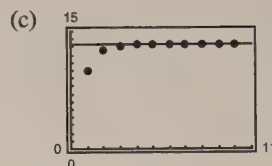


(d) The terms of the series decrease in magnitude slowly. Thus, the sequence of partial sums approaches the sum slowly.

$$31. (a) \sum_{n=1}^{\infty} 10(0.25)^{n-1} = \frac{10}{1-0.25} = \frac{40}{3} \approx 13.3333$$

(b)

n	5	10	20	50	100
S_n	13.3203	13.3333	13.3333	13.3333	13.3333



(d) The terms of the series decrease in magnitude rapidly. Thus, the sequence of partial sums approaches the sum rapidly.

$$33. \sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \left(\frac{1/2}{n-1} - \frac{1/2}{n+1} \right) = \frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$= \frac{1}{2} \left[\left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \cdots \right]$$

$$= \frac{1}{2} \left(1 + \frac{1}{2} \right) = \frac{3}{4}$$

$$35. \sum_{n=1}^{\infty} \frac{8}{(n+1)(n+2)} = 8 \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = 8 \left[\left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \cdots \right] = 8 \left(\frac{1}{2} \right) = 4$$

$$37. \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n = \frac{1}{1-(1/2)} = 2$$

$$39. \sum_{n=0}^{\infty} \left(-\frac{1}{2} \right)^n = \frac{1}{1-(-1/2)} = \frac{2}{3}$$

$$41. \sum_{n=0}^{\infty} \left(\frac{1}{10} \right)^n = \frac{1}{1-(1/10)} = \frac{10}{9}$$

$$43. \sum_{n=0}^{\infty} 3 \left(-\frac{1}{3} \right)^n = \frac{3}{1-(-1/3)} = \frac{9}{4}$$

$$\begin{aligned}
 45. \sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n} \right) &= \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^n \\
 &= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/3)} \\
 &= 2 - \frac{3}{2} = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 49. 0.075\overline{75} &= \sum_{n=0}^{\infty} \frac{3}{40} \left(\frac{1}{100} \right)^n \\
 \text{Geometric series with } a &= \frac{3}{40} \text{ and } r = \frac{1}{100} \\
 S &= \frac{a}{1 - r} = \frac{3/40}{99/100} = \frac{5}{66}
 \end{aligned}$$

$$53. \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right) = \left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \cdots = 1 + \frac{1}{2} = \frac{3}{2}, \text{ converges}$$

$$\begin{aligned}
 55. \sum_{n=1}^{\infty} \frac{3n-1}{2n+1} \\
 \lim_{n \rightarrow \infty} \frac{3n-1}{2n+1} &= \frac{3}{2} \neq 0 \\
 \text{Diverges by Theorem 8.9}
 \end{aligned}$$

$$\begin{aligned}
 57. \sum_{n=0}^{\infty} \frac{4}{2^n} &= 4 \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n \\
 \text{Geometric series with } r &= \frac{1}{2} \\
 \text{Converges by Theorem 8.6}
 \end{aligned}$$

$$\begin{aligned}
 59. \sum_{n=0}^{\infty} (1.075)^n \\
 \text{Geometric series with } r &= 1.075 \\
 \text{Diverges by Theorem 8.6}
 \end{aligned}$$

$$\begin{aligned}
 61. \sum_{n=2}^{\infty} \frac{n}{\ln n} \\
 \lim_{n \rightarrow \infty} \frac{n}{\ln n} &= \lim_{n \rightarrow \infty} \frac{1}{1/n} = \infty \\
 \text{(by L'Hôpital's Rule) Diverges by Theorem 8.9}
 \end{aligned}$$

63. See definition, page 567.

65. The series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, a \neq 0$$

is a geometric series with ratio r . When $0 < |r| < 1$, the series converges to $\frac{a}{1-r}$. The series diverges if $|r| \geq 1$.

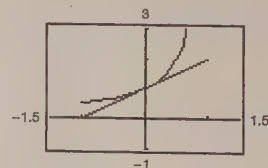
67. (a) x is the common ratio.

$$(b) 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1$$

Geometric series: $a = 1, r = x, |x| < 1$

$$(c) y_1 = \frac{1}{1-x}, |x| < 1$$

$$y_2 = s_2 = 1 + x$$

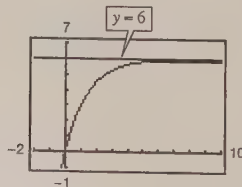


$$69. f(x) = 3 \left[\frac{1 - 0.5^x}{1 - 0.5} \right]$$

Horizontal asymptote: $y = 6$

$$\sum_{n=0}^{\infty} 3 \left(\frac{1}{2} \right)^n$$

$$S = \frac{3}{1 - (1/2)} = 6$$



The horizontal asymptote is the sum of the series. $f(n)$ is the n^{th} partial sum.

$$71. \frac{1}{n(n+1)} < 0.001$$

$$10,000 < n^2 + n$$

$$0 < n^2 + n - 10,000$$

$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10,000)}}{2}$$

Choosing the positive value for n we have $n \approx 99.5012$. The first term that is less than 0.001 is $n = 100$.

$$\left(\frac{1}{8}\right)^n < 0.001$$

$$10,000 < 8^n$$

This inequality is true when $n = 5$. This series converges at a faster rate.

$$73. \sum_{i=0}^{n-1} 8000(0.9)^i = \frac{8000[1 - (0.9)^{(n-1)+1}]}{1 - 0.9}$$

$$= 80,000(1 - 0.9^n), \quad n > 0$$

$$75. \sum_{i=0}^{n-1} 100(0.75)^i = \frac{100[1 - 0.75^{(n-1)+1}]}{1 - 0.75}$$

$$= 400(1 - 0.75^n) \text{ million dollars.}$$

Sum = 400 million dollars

$$77. D_1 = 16$$

$$D_2 = \underbrace{0.81(16)}_{\text{up}} + \underbrace{0.81(16)}_{\text{down}} = 32(0.81)$$

$$D_3 = 16(0.81)^2 + 16(0.81)^2 = 32(0.81)^2$$

\vdots

$$D = 16 + 32(0.81) + 32(0.81)^2 + \cdots = -16 + \sum_{n=0}^{\infty} 32(0.81)^n = -16 + \frac{32}{1 - 0.81} \approx 152.42 \text{ feet}$$

$$79. P(n) = \frac{1}{2} \left(\frac{1}{2}\right)^n$$

$$P(2) = \frac{1}{2} \left(\frac{1}{2}\right)^2 = \frac{1}{8}$$

$$\sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)^n = \frac{1/2}{1 - (1/2)} = 1$$

$$81. (a) \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)^n = \frac{1}{2} \frac{1}{1 - (1/2)} = 1$$

(b) No, the series is not geometric.

$$(c) \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = 2$$

83. Assuming that the payments are made at the beginning of each year,

$$\text{Present Value} = \sum_{n=0}^{19} 50,000 \left(\frac{1}{1.06}\right)^n$$

$$= 50,000 \left(\frac{1 - 1.06^{-20}}{1 - 1.06^{-1}}\right)$$

$$\approx \$607,905.82$$

The present value is less than \$1,000,000. After accruing interest over 20 years, it attains its full value.

$$85. w = \sum_{i=0}^{n-1} 0.01(2)^i = \frac{0.01(1 - 2^n)}{1 - 2} = 0.01(2^n - 1)$$

(a) When $n = 29$: $w = \$5,368,709.11$

(b) When $n = 30$: $w = \$10,737,418.23$

(c) When $n = 31$: $w = \$21,474,836.47$

87. $P = 50, r = 0.03, t = 20$

(a) $A = 50 \left(\frac{12}{0.03} \right) \left[\left(1 + \frac{0.03}{12} \right)^{12(20)} - 1 \right] \approx \$16,415.10$

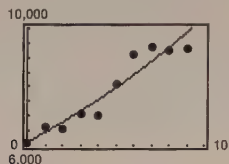
(b) $A = \frac{50 - (e^{0.03(20)} - 1)}{e^{0.03/12} - 1} \approx \$16,421.83$

89. $P = 100, r = 0.04, t = 40$

(a) $A = 100 \left(\frac{12}{0.04} \right) \left[\left(1 + \frac{0.04}{12} \right)^{12(40)} - 1 \right] \approx \$118,196.13$

(b) $A = \frac{100(e^{0.04(40)} - 1)}{e^{0.04/12} - 1} \approx \$118,393.43$

91. (a) $a_n = 6110.1832(1.0544)^x = 6110.1832e^{0.05297n}$

(b) Adding the 10 values a_0, a_1, \dots, a_9 , you obtain 78,530 or \$78,530,000,000

(c) Total = $\sum_{n=0}^9 6110.1832e^{-0.05297n} \approx 78,449$ or \$78,449,000,000

93. $x = 0.749999 \dots = 0.74 + \sum_{n=0}^{\infty} 0.009(0.1)^n$

$$= 0.74 + \frac{0.009}{1 - 0.1}$$

$$= 0.74 + 0.01 = 0.75$$

95. By letting $S_0 = 0$, we have $a_n = \sum_{k=1}^n a_k - \sum_{k=1}^{n-1} a_k = S_n - S_{n-1}$. Thus,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (S_n - S_{n-1}) = \sum_{n=1}^{\infty} (S_n - S_{n-1} + c - c) = \sum_{n=1}^{\infty} [(c - S_{n-1}) - (c - S_n)].$$

97. Let $\sum a_n = \sum_{n=0}^{\infty} 1$ and $\sum b_n = \sum_{n=0}^{\infty} (-1)$.

Both are divergent series.

$$\sum (a_n + b_n) = \sum_{n=0}^{\infty} [1 + (-1)] = \sum_{n=0}^{\infty} [1 - 1] = 0$$

99. False. $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

101. False

$$\sum_{n=1}^{\infty} ar^n = \left(\frac{a}{1-r} \right) - a$$

The formula requires that the geometric series begins with $n = 0$.103. Let H represent the half-life of the drug. If a patient receives n equal doses of P units each of this drug, administered at equal time interval of length t , the total amount of the drug in the patient's system at the time the last dose is administered is given by

$$T_n = P + Pe^{kt} + Pe^{2kt} + \dots + Pe^{(n-1)kt}$$

where $k = -(\ln 2)/H$. One time interval *after* the last dose is administered is given by

$$T_{n+1} = Pe^{kt} + Pe^{2kt} + Pe^{3kt} + \dots + Pe^{nkt}.$$

Two time intervals *after* the last dose is administered is given by

$$T_{n+2} = Pe^{2kt} + Pe^{3kt} + Pe^{4kt} + \dots + Pe^{(n+1)kt}$$

and so on. Since $k < 0$, $T_{n+s} \rightarrow 0$ as $s \rightarrow \infty$, where s is an integer.

Section 8.3 The Integral Test and p -Series

1. $\sum_{n=1}^{\infty} \frac{1}{n+1}$

Let $f(x) = \frac{1}{x+1}$.

 f is positive, continuous and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x+1} dx = \left[\ln(x+1) \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

5. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Let $f(x) = \frac{1}{x^2+1}$.

 f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^2+1} dx = \left[\arctan x \right]_1^{\infty} = \frac{\pi}{4}$$

Converges by Theorem 8.10

9. $\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+c}$

Let $f(x) = \frac{x^{k-1}}{x^k+c}$.

 f is positive, continuous, and decreasing for $x > \sqrt[k]{c(k-1)}$ since

$$f'(x) = \frac{x^{k-2}[c(k-1) - x^k]}{(x^k+c)^2} < 0$$

for $x > \sqrt[k]{c(k-1)}$.

$$\int_1^{\infty} \frac{x^{k-1}}{x^k+c} dx = \left[\frac{1}{k} \ln(x^k+c) \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

13. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$

Divergent p -series with $p = \frac{1}{5} < 1$

17. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Convergent p -series with $p = \frac{3}{2} > 1$

3. $\sum_{n=1}^{\infty} e^{-n}$

Let $f(x) = e^{-x}$.

 f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} e^{-x} dx = \left[-e^{-x} \right]_1^{\infty} = \frac{1}{e}$$

Converges by Theorem 8.10

7. $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$

Let $f(x) = \frac{\ln(x+1)}{x+1}$.

 f is positive, continuous, and decreasing for $x \geq 2$ since

$$f'(x) = \frac{1 - \ln(x+1)}{(x+1)^2} < 0 \text{ for } x \geq 2.$$

$$\int_1^{\infty} \frac{\ln(x+1)}{x+1} dx = \left[\frac{\ln^2(x+1)}{2} \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

11. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Let $f(x) = \frac{1}{x^3}$.

 f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^3} dx = \left[-\frac{1}{2x^2} \right]_1^{\infty} = \frac{1}{2}$$

Converges by Theorem 8.10

15. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

Divergent p -series with $p = \frac{1}{2} < 1$

19. $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

Convergent p -series with $p = 1.04 > 1$

$$21. \sum_{n=1}^{\infty} \frac{2}{4\sqrt[n]{n^3}} = \frac{2}{1} + \frac{2}{2^{3/4}} + \frac{2}{3^{3/4}} + \cdots$$

$$S_1 = 2$$

$$S_2 \approx 3.189$$

$$S_3 \approx 4.067$$

Matches (a)

Diverges— p -series with $p = \frac{3}{4} < 1$

$$23. \sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}} = 2 + 2/2^{3/2} + 2/3^{3/2} + \cdots$$

$$S_1 = 2$$

$$S_2 \approx 2.707$$

$$S_3 \approx 3.092$$

Matches (b)

Converges— p -series with $p = 3/2 > 1$

25. No. Theorem 8.9 says that if the series converges, then the terms a_n tend to zero. Some of the series in Exercises 21–24 converge because the terms tend to 0 very rapidly.

$$27. \sum_{n=1}^N \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} > M$$

(a)

M	2	4	6	8
N	4	31	227	1674

(b) No. Since the terms are decreasing (approaching zero), more and more terms are required to increase the partial sum by 2.

$$29. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

If $p = 1$, then the series diverges by the Integral Test. If $p \neq 1$,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \int_2^{\infty} (\ln x)^{-p} \frac{1}{x} dx = \left[\frac{(\ln x)^{-p+1}}{-p+1} \right]_2^{\infty}$$

Converges for $-p+1 < 0$ or $p > 1$.

31. Let f be positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$. Then,

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x) dx$$

either both converge or both diverge (Theorem 8.10).
See Example 1, page 578.

33. Your friend is not correct. The series

$$\sum_{n=10,000}^{\infty} \frac{1}{n} = \frac{1}{10,000} + \frac{1}{10,001} + \cdots$$

is the harmonic series, starting with the 10,000th term, and hence diverges.

35. Since f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, we have,

$$R_N = S - S_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^N a_n = \sum_{n=N+1}^{\infty} a_n > 0.$$

$$\text{Also, } R_N = S - S_N = \sum_{n=N+1}^{\infty} a_n \leq a_{N+1} + \int_{N+1}^{\infty} f(x) dx \leq \int_N^{\infty} f(x) dx. \text{ Thus,}$$

$$0 \leq R_N \leq \int_N^{\infty} f(x) dx.$$

$$37. S_6 = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} \approx 1.0811$$

$$R_6 \leq \int_6^{\infty} \frac{1}{x^4} dx = \left[-\frac{1}{3x^3} \right]_6^{\infty} \approx 0.0015$$

$$1.0811 \leq \sum_{n=1}^{\infty} \frac{1}{n^4} \leq 1.0811 + 0.0015 = 1.0826$$

$$39. S_{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \frac{1}{82} + \frac{1}{101} \approx 0.9818$$

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^2 + 1} dx = \left[\arctan x \right]_{10}^{\infty} = \frac{\pi}{2} - \arctan 10 \approx 0.0997$$

$$0.9818 \leq \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \leq 0.9818 + 0.0997 = 1.0815$$

$$41. S_4 = \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}} \approx 0.4049$$

$$R_4 \leq \int_4^{\infty} x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_4^{\infty} = \frac{e^{-16}}{2} \approx 5.6 \times 10^{-8}$$

$$0.4049 \leq \sum_{n=1}^{\infty} n e^{-n^2} \leq 0.4049 + 5.6 \times 10^{-8}$$

$$43. 0 \leq R_N \leq \int_N^{\infty} \frac{1}{x^4} dx = \left[-\frac{1}{3x^3} \right]_N^{\infty} = \frac{1}{3N^3} < 0.001$$

$$\frac{1}{N^3} < 0.003$$

$$N^3 > 333.33$$

$$N > 6.93$$

$$N \geq 7$$

$$45. R_N \leq \int_N^{\infty} e^{-5x} dx = \left[-\frac{1}{5} e^{-5x} \right]_N^{\infty} = \frac{e^{-5N}}{5} < 0.001$$

$$\frac{1}{e^{5N}} < 0.005$$

$$e^{5N} > 200$$

$$5N > \ln 200$$

$$N > \frac{\ln 200}{5}$$

$$N > 1.0597$$

$$N \geq 2$$

$$47. R_N \leq \int_N^{\infty} \frac{1}{x^2 + 1} dx = \left[\arctan x \right]_N^{\infty}$$

$$= \frac{\pi}{2} - \arctan N < 0.001$$

$$-\arctan N < -1.5698$$

$$\arctan N > 1.5698$$

$$N > \tan 1.5698$$

$$N \geq 1004$$

$$49. (a) \sum_{n=2}^{\infty} \frac{1}{n^{1.1}}. \text{ This is a convergent } p\text{-series with } p = 1.1 > 1.$$

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ is a divergent series. Use the Integral Test.}$$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \left[\ln |\ln x| \right]_2^{\infty} = \infty$$

$$(b) \sum_{n=2}^6 \frac{1}{n^{1.1}} = \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \frac{1}{4^{1.1}} + \frac{1}{5^{1.1}} + \frac{1}{6^{1.1}} \approx 0.4665 + 0.2987 + 0.2176 + 0.1703 + 0.1393$$

$$\sum_{n=2}^6 \frac{1}{n \ln n} = \frac{1}{2 \ln 2} + \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} + \frac{1}{5 \ln 5} + \frac{1}{6 \ln 6} \approx 0.7213 + 0.3034 + 0.1803 + 0.1243 + 0.0930$$

For $n \geq 4$ the terms of the convergent series seem to be larger than those of the divergent series!

$$(c) \frac{1}{n^{1.1}} < \frac{1}{n \ln n}$$

$$n \ln n < n^{1.1}$$

$$\ln n < n^{0.1}$$

This inequality holds when $n \geq 3.5 \times 10^{15}$. Or, $n > e^{40}$. Then $\ln e^{40} = 40 < (e^{40})^{0.1} = e^4 \approx 55$.

51. (a) Let
- $f(x) = 1/x$
- .
- f
- is positive, continuous, and decreasing on
- $[1, \infty)$
- .

$$S_n - 1 \leq \int_1^n \frac{1}{x} dx$$

$$S_n - 1 \leq \ln n$$

Hence, $S_n \leq 1 + \ln n$. Similarly,

$$S_n \geq \int_1^{n+1} \frac{1}{x} dx = \ln(n+1).$$

Thus, $\ln(n+1) \leq S_n \leq 1 + \ln n$.

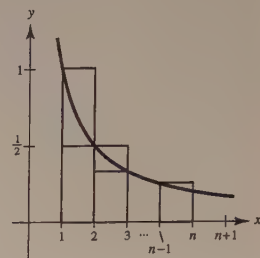
- (b) Since $\ln(n+1) \leq S_n \leq 1 + \ln n$, we have $\ln(n+1) - \ln n \leq S_n - \ln n \leq 1$. Also, since $\ln x$ is an increasing function, $\ln(n+1) - \ln n > 0$ for $n \geq 1$. Thus, $0 \leq S_n - \ln n \leq 1$ and the sequence $\{a_n\}$ is bounded.

$$(c) a_n - a_{n+1} = [S_n - \ln n] - [S_{n+1} - \ln(n+1)] = \int_n^{n+1} \frac{1}{x} dx - \frac{1}{n+1} \geq 0$$

Thus, $a_n \geq a_{n+1}$ and the sequence is decreasing.

- (d) Since the sequence is bounded and monotonic, it converges to a limit, γ .

$$(e) a_{100} = S_{100} - \ln 100 \approx 0.5822 \text{ (Actually } \gamma \approx 0.577216).$$



$$53. \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

$$\text{Let } f(x) = \frac{1}{2x-1}.$$

f is positive, continuous, and decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{2x-1} dx = \left[\ln \sqrt{2x-1} \right]_1^{\infty} = \infty$$

Diverges by Theorem 8.10

$$55. \sum_{n=1}^{\infty} \frac{1}{n^{4/5}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$$

p -series with $p = \frac{5}{4}$

Converges by Theorem 8.11

$$57. \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$$

Geometric series with $r = \frac{2}{3}$

Converges by Theorem 8.6

$$59. \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$$

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 \neq 0$$

Diverges by Theorem 8.9

$$61. \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$$

Fails n th Term Test

Diverges by Theorem 8.9

$$63. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

$$\text{Let } f(x) = \frac{1}{x(\ln x)^3}.$$

f is positive, continuous and decreasing for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x(\ln x)^3} dx = \int_2^{\infty} (\ln x)^{-3} \frac{1}{x} dx = \left[\frac{(\ln x)^{-2}}{-2} \right]_2^{\infty} = \left[-\frac{1}{2(\ln x)^2} \right]_2^{\infty} = \frac{1}{2(\ln 2)^2}$$

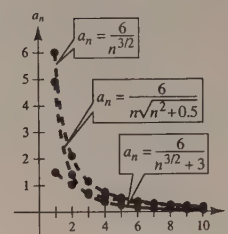
Converges by Theorem 8.10. See Exercise 29.

Section 8.4 Comparisons of Series

1. (a) $\sum_{n=1}^{\infty} \frac{6}{n^{3/2}} = \frac{6}{1} + \frac{6}{2^{3/2}} + \cdots \quad S_1 = 6$

$$\sum_{n=1}^{\infty} \frac{6}{n^{3/2} + 3} = \frac{6}{4} + \frac{6}{2^{3/2} + 3} + \cdots \quad S_1 = \frac{3}{2}$$

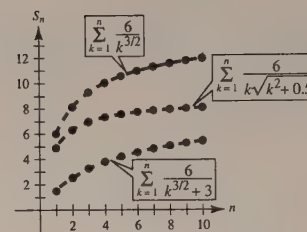
$$\sum_{n=1}^{\infty} \frac{6}{n\sqrt{n^2 + 0.5}} = \frac{6}{1\sqrt{1.5}} + \frac{6}{2\sqrt{4.5}} + \cdots \quad S_1 = \frac{6}{\sqrt{1.5}} \approx 4.9$$



(b) The first series is a p -series. It converges ($p = 3/2 > 1$).

(c) The magnitude of the terms of the other two series are less than the corresponding terms at the convergent p -series. Hence, the other two series converge.

(d) The smaller the magnitude of the terms, the smaller the magnitude of the terms of the sequence of partial sums.



3. $\frac{1}{n^2 + 1} < \frac{1}{n^2}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

converges by comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

5. $\frac{1}{n-1} > \frac{1}{n}$ for $n \geq 2$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{n-1}$$

diverges by comparison with the divergent p -series

$$\sum_{n=2}^{\infty} \frac{1}{n}$$

7. $\frac{1}{3^n + 1} < \frac{1}{3^n}$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{3^n + 1}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

9. For $n \geq 3$, $\frac{\ln n}{n+1} > \frac{1}{n+1}$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n+1}$$

diverges by comparison with the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n+1}$$

Note: $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges by the integral test.

11. For $n > 3$, $\frac{1}{n^2} > \frac{1}{n!}$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges by comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

13. $\frac{1}{e^{n^2}} \leq \frac{1}{e^n}$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{e^{n^2}}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$$

$$15. \lim_{n \rightarrow \infty} \frac{n/(n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$19. \lim_{n \rightarrow \infty} \frac{\frac{2n^2 - 1}{3n^5 + 2n + 1}}{1/n^3} = \lim_{n \rightarrow \infty} \frac{2n^5 - n^3}{3n^5 + 2n + 1} = \frac{2}{3}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1}$$

converges by a limit comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

$$23. \lim_{n \rightarrow \infty} \frac{1/(n\sqrt{n^2 + 1})}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n\sqrt{n^2 + 1}} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2 + 1}}$$

converges by a limit comparison with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$27. \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{n \rightarrow \infty} \frac{(-1/n^2)\cos(1/n)}{-1/n^2} \\ = \lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$31. \sum_{n=1}^{\infty} \frac{1}{3^n + 2}$$

Converges

Direct comparison with $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

$$33. \sum_{n=1}^{\infty} \frac{n}{2n + 3}$$

Diverges; n th Term Test

$$\lim_{n \rightarrow \infty} \frac{n}{2n + 3} = \frac{1}{2} \neq 0$$

$$17. \lim_{n \rightarrow \infty} \frac{1/\sqrt{n^2 + 1}}{1/n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + 1}} = 1$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$21. \lim_{n \rightarrow \infty} \frac{\frac{n + 3}{n(n + 2)}}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2 + 3n}{n^2 + 2n} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n + 3}{n(n + 2)}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$25. \lim_{n \rightarrow \infty} \frac{(n^{k-1})/(n^k + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^k}{n^k + 1} = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + 1}$$

diverges by a limit comparison with the divergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$29. \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Diverges

p -series with $p = \frac{1}{2}$

$$35. \sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$$

Converges; integral test

37. $\lim_{n \rightarrow \infty} \frac{a_n}{1/n} = \lim_{n \rightarrow \infty} na_n$. By given conditions $\lim_{n \rightarrow \infty} na_n$ is finite and nonzero.

Therefore,

$$\sum_{n=1}^{\infty} a_n$$

diverges by a limit comparison with the p -series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

41. $\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$

converges since the degree of the numerator is three less than the degree of the denominator.

45. See Theorem 8.12, page 583. One example is $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ converges because

$$\frac{1}{n^2 + 1} < \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges (p -series).

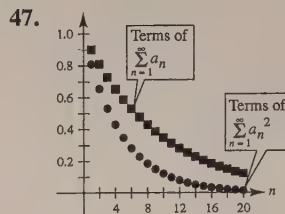
39. $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \frac{5}{26} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n^2 + 1},$

which diverges since the degree of the numerator is only one less than the degree of the denominator.

43. $\lim_{n \rightarrow \infty} n \left(\frac{n^3}{5n^4 + 3} \right) = \lim_{n \rightarrow \infty} \frac{n^4}{5n^4 + 3} = \frac{1}{5} \neq 0$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^3}{5n^4 + 3} \text{ diverges.}$$



For $0 < a_n < 1$, $0 < a_n^2 < a_n < 1$. Hence, the lower terms are those of $\sum a_n^2$.

49. $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \cdots = \sum_{n=1}^{\infty} \frac{1}{200n}$, diverges (harmonic)

51. $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} = \sum_{n=1}^{\infty} \frac{1}{200 + n^2}$, converges

53. Some series diverge or converge very slowly. You cannot decide convergence or divergence of a series by comparing the first few terms.

55. False. Let $a_n = 1/n^3$ and $b_n = 1/n^2$. $0 < a_n \leq b_n$ and both

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converge.

57. True

59. Since $\sum_{n=1}^{\infty} b_n$ converges, $\lim_{n \rightarrow \infty} b_n = 0$. There exists N such that $b_n < 1$ for $n > N$. Thus,

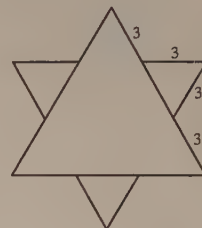
$$a_n b_n < a_n \text{ for } n > N \text{ and } \sum_{n=1}^{\infty} a_n b_n$$

converges by comparison to the convergent series $\sum_{n=1}^{\infty} a_n$.

61. $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^3}$ both converge, and hence so does $\sum \left(\frac{1}{n^2} \right) \left(\frac{1}{n^3} \right) = \sum \frac{1}{n^5}$.

63. (a) Suppose $\sum b_n$ converges and $\sum a_n$ diverges. Then there exists N such that $0 < b_n < a_n$ for $n \geq N$. This means that $1 < a_n/b_n$ for $n \geq N$. Therefore, $\lim_{n \rightarrow \infty} a_n/b_n \neq 0$. Thus, $\sum a_n$ must also converge.
- (b) Suppose $\sum b_n$ diverges and $\sum a_n$ converges. Then there exists N such that $0 < a_n < b_n$ for $n \geq N$. This means that $0 < a_n/b_n < 1$ for $n \geq N$. Therefore, $\lim_{n \rightarrow \infty} a_n/b_n \neq \infty$. Thus, $\sum a_n$ must also diverge.
65. Start with one triangle whose sides have length 9. At the n th step, each side is replaced by four smaller line segments each having $\frac{1}{3}$ the length of the original side.

#Sides	Length of sides
3	9
$3 \cdot 4$	$9(\frac{1}{3})$
$3 \cdot 4^2$	$9(\frac{1}{3})^2$
\vdots	
$3 \cdot 4^n$	$9(\frac{1}{3})^n$



At the n th step there are $3 \cdot 4^n$ sides, each of length $9(\frac{1}{3})^n$. At the next step, there are $3 \cdot 4^{n+1}$ new triangles of side $9(\frac{1}{3})^{n+1}$. The area of an equilateral triangle of side x is $\frac{1}{4}\sqrt{3}x^2$. Thus, the new triangles each have area

$$\frac{\sqrt{3}}{4} \left[9 \left(\frac{1}{3^{n+1}} \right) \right]^2 = \frac{9\sqrt{3}}{4} \cdot \frac{1}{3^{2n}} = \frac{9}{4} \sqrt{3} \left(\frac{1}{9} \right)^n$$

The area of the $3 \cdot 4^n$ new triangles is

$$(3 \cdot 4^n) \left[\frac{9\sqrt{3}}{4} \left(\frac{1}{9} \right)^n \right] = \frac{27\sqrt{3}}{4} \left(\frac{4}{9} \right)^n$$

The total area is the infinite sum

$$\frac{81\sqrt{3}}{4} + \sum_{n=1}^{\infty} \frac{27\sqrt{3}}{4} \left(\frac{4}{9} \right)^n = \frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4} \left(\frac{1}{1 - 4/9} \right) = \frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4} \left(\frac{9}{5} \right) = \frac{162\sqrt{3}}{5}.$$

The perimeter is infinite, since at step n there are $3 \cdot 4^n$ sides of length $9(\frac{1}{3})^n$. Thus, the perimeter at step n is $27(\frac{4}{3})^n \rightarrow \infty$.

Section 8.5 Alternating Series

1. $\sum_{n=1}^{\infty} \frac{6}{n^2} = \frac{6}{1} + \frac{6}{4} + \frac{6}{9} + \dots$

$$S_1 = 6, S_2 = 7.5$$

Matches (b)

3. $\sum_{n=1}^{\infty} \frac{10}{n2^n} = \frac{10}{2} + \frac{10}{8} + \dots$

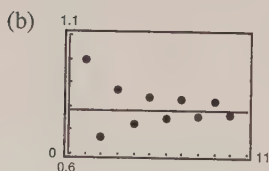
$$S_1 = 5, S_2 = 6.25$$

Matches (c)

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \approx 0.7854$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.6667	0.8667	0.7238	0.8349	0.7440	0.8209	0.7543	0.8131	0.7605

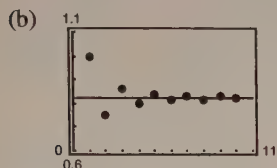


- (c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.
- (d) The distance in part (c) is always less than the magnitude of the next term of the series.

$$7. \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \approx 0.8225$$

(a)

n	1	2	3	4	5	6	7	8	9	10
S_n	1	0.75	0.8611	0.7986	0.8386	0.8108	0.8312	0.8156	0.8280	0.8180



(c) The points alternate sides of the horizontal line that represents the sum of the series. The distance between successive points and the line decreases.

(d) The distance in part (c) is always less than the magnitude of the next term in the series.

$$9. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Converges by Theorem 8.14.

$$13. \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2 + 1}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1$$

Diverges by the n th Term Test

$$17. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$$

$$\lim_{n \rightarrow \infty} \frac{n+1}{\ln(n+1)} = \lim_{n \rightarrow \infty} \frac{1}{1/(n+1)} = \lim_{n \rightarrow \infty} (n+1) = \infty$$

Diverges by the n th Term Test

$$21. \sum_{n=1}^{\infty} \cos n\pi = \sum_{n=1}^{\infty} (-1)^n$$

Diverges by the n th Term Test

$$11. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$a_{n+1} = \frac{1}{2(n+1)-1} < \frac{1}{2n-1} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{2n-1} = 0$$

Converges by Theorem 8.14

$$15. \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$$a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

Converges by Theorem 8.14

$$19. \sum_{n=1}^{\infty} \sin\left[\frac{(2n-1)\pi}{2}\right] = \sum_{n=1}^{\infty} (-1)^{n+1}$$

Diverges by the n th Term Test

$$23. \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

$$a_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n!} = 0$$

Converges by Theorem 8.14

$$25. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{n+2}$$

$$a_{n+1} = \frac{\sqrt{n+1}}{(n+1)+2} < \frac{\sqrt{n}}{n+2} \text{ for } n \geq 2$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+2} = 0$$

Converges by Theorem 8.14

$$27. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2)}{e^n - e^{-n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2e^n)}{e^{2n} - 1}$$

$$\text{Let } f(x) = \frac{2e^x}{e^{2x} - 1}. \text{ Then}$$

$$f'(x) = \frac{-2e^x(e^{2x} + 1)}{(e^{2x} - 1)^2} < 0.$$

Thus, $f(x)$ is decreasing. Therefore, $a_{n+1} < a_n$, and

$$\lim_{n \rightarrow \infty} \frac{2e^n}{e^{2n} - 1} = \lim_{n \rightarrow \infty} \frac{2e^n}{2e^{2n}} = \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0.$$

The series converges by Theorem 8.14.

$$29. S_6 = \sum_{n=1}^6 \frac{3(-1)^{n+1}}{n^2} = 2.4325$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{3}{49} \approx 0.0612; 2.3713 \leq S \leq 2.4937$$

$$31. S_6 = \sum_{n=0}^5 \frac{2(-1)^n}{n!} \approx 0.7333$$

$$|R_6| = |S - S_6| \leq a_7 = \frac{2}{6!} = 0.002778; 0.7305 \leq S \leq 0.7361$$

$$33. \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{(N+1)!} < 0.001.$$

This inequality is valid when $N = 6$.

(b) We may approximate the series by

$$\sum_{n=0}^6 \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368.$$

(7 terms. Note that the sum begins with $n = 0$.)

$$35. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{[2(N+1)+1]!} < 0.001.$$

This inequality is valid when $N = 2$.

(b) We may approximate the series by

$$\sum_{n=0}^2 \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{6} + \frac{1}{120} \approx 0.842.$$

(3 terms. Note that the sum begins with $n = 0$.)

$$37. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

(a) By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{N+1} < 0.001.$$

This inequality is valid when $N = 1000$.

(b) We may approximate the series by

$$\sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1000} \approx 0.693.$$

(1000 terms)

$$39. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 1}$$

By Theorem 8.15,

$$|R_N| \leq a_{N+1} = \frac{1}{2(N+1)^3 - 1} < 0.001.$$

This inequality is valid when $N = 7$.

$$41. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$$

$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges by comparison to the p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Therefore, the given series converge absolutely.

$$45. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+1)^2}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = 1$$

Therefore, the series diverges by the n th Term Test.

$$49. \sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3 - 1}$$

$$\sum_{n=2}^{\infty} \frac{n}{n^3 - 1}$$

converges by a limit comparison to the convergent p -series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Therefore, the given series converges absolutely.

$$53. \sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

The given series converges by the Alternating Series Test, but

$$\sum_{n=0}^{\infty} \frac{|\cos n\pi|}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

diverges by a limit comparison to the divergent harmonic series,

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

$$\lim_{n \rightarrow \infty} \frac{|\cos n\pi|/(n+1)}{1/n} = 1, \text{ therefore the series}$$

converges conditionally.

$$43. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

The given series converges by the Alternating Series Test, but does not converge absolutely since

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a divergent p -series. Therefore, the series converges conditionally.

$$47. \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$$

The given series converges by the Alternating Series Test, but does not converge absolutely since the series

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

diverges by comparison to the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Therefore, the series converges conditionally.

$$51. \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$$

is convergent by comparison to the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

since

$$\frac{1}{(2n+1)!} < \frac{1}{2^n} \text{ for } n > 0.$$

Therefore, the given series converges absolutely.

$$55. \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p -series. Therefore, the given series converges absolutely.

57. An alternating series is a series whose terms alternate in sign. See Theorem 8.14.

59. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

$\sum a_n$ is conditionally convergent if $\sum |a_n|$ diverges, but $\sum a_n$ converges.

61. (b). The partial sums alternate above and below the horizontal line representing the sum.

63. Since $\sum_{n=1}^{\infty} |a_n|$ converges we have

$$\lim_{n \rightarrow \infty} |a_n| = 0.$$

Thus, there must exist an $N > 0$ such that $|a_n| < 1$ for all $n > N$ and it follows that $a_n^2 \leq |a_n|$ for all $n > N$. Hence, by the Comparison Test,

$$\sum_{n=1}^{\infty} a_n^2$$

converges. Let $a_n = 1/n$ to see that the converse is false.

65. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, hence so does $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

67. False

$$\text{Let } a_n = \frac{(-1)^n}{n}.$$

69. $\sum_{n=1}^{\infty} \frac{10}{n^{3/2}} = 10 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ convergent p -series

71. Diverges by n th Term Test. $\lim_{n \rightarrow \infty} a_n = \infty$

73. Convergent Geometric Series ($r = \frac{7}{8} < 1$)

75. Convergent Geometric Series ($r = \frac{1}{\sqrt{e}}$) or Integral Test

77. Converges (absolutely) by Alternating Series Test

79. The first term of the series is zero, not one. You cannot regroup series terms arbitrarily.

Section 8.6 The Ratio and Root Tests

$$\begin{aligned} 1. \frac{(n+1)!}{(n-2)!} &= \frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!} \\ &= (n+1)(n)(n-1) \end{aligned}$$

3. Use the Principle of Mathematical Induction. When $k = 1$, the formula is valid since $1 = \frac{(2(1))!}{2^1 \cdot 1!}$. Assume that

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2^n n!}$$

and show that

$$1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) = \frac{(2n+2)!}{2^{n+1}(n+1)!}.$$

—CONTINUED—

3. —CONTINUED—

To do this, note that:

$$\begin{aligned}
 1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) &= [1 \cdot 3 \cdot 5 \cdots (2n-1)](2n+1) \\
 &= \frac{(2n)!}{2^n n!} \cdot (2n+1) \quad (\text{Induction hypothesis}) \\
 &= \frac{(2n)!(2n+1)}{2^n n!} \cdot \frac{(2n+2)}{2(n+1)} \\
 &= \frac{(2n)!(2n+1)(2n+2)}{2^{n+1} n! (n+1)} \\
 &= \frac{(2n+2)!}{2^{n+1} (n+1)!}
 \end{aligned}$$

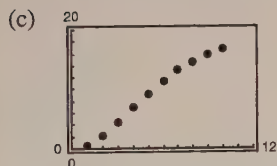
The formula is valid for all $n \geq 1$.

$$\begin{aligned}
 5. \sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^n &= 1\left(\frac{3}{4}\right) + 2\left(\frac{9}{16}\right) + \cdots & 7. \sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!} &= 9 - \frac{3^3}{2} + \cdots & 9. \sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n &= \frac{4}{2} + \left(\frac{8}{7}\right)^2 + \cdots \\
 S_1 &= \frac{3}{4}, S_2 \approx 1.875 & S_1 &= 9 & S_1 &= 2 \\
 \text{Matches (d)} & & \text{Matches (f)} & & \text{Matches (a)} &
 \end{aligned}$$

11. (a) Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2 (5/8)^{n+1}}{n^2 (5/8)^n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^2 \frac{5}{8} = \frac{5}{8} < 1$. Converges

(b)

n	5	10	15	20	25
S_n	9.2104	16.7598	18.8016	19.1878	19.2491



(d) The sum is approximately 19.26.

(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of the partial sums approaches the sum of the series.

13. $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{n+1}{3} = \infty
 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

15. $\sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^n$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(3/4)^{n+1}}{n(3/4)^n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{3(n+1)}{4n} \right| = \frac{3}{4}
 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

17. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{n+1}{2n} = \frac{1}{2}
 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

19. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} \right| \\
 &= \lim_{n \rightarrow \infty} \frac{2n^2}{(n+1)^2} = 2
 \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$21. \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$23. \sum_{n=1}^{\infty} \frac{n!}{n3^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{(n+1)3^{n+1}} \cdot \frac{n3^n}{n!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n}{3} = \infty \end{aligned}$$

Therefore, by the Ratio Test, the series diverges.

$$25. \sum_{n=0}^{\infty} \frac{4^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}}{(n+1)!} \cdot \frac{n!}{4^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{4}{n+1} = 0 \end{aligned}$$

Therefore, by the Ratio Test, the series converges.

$$27. \sum_{n=0}^{\infty} \frac{3^n}{(n+1)^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+2)^{n+1}} \cdot \frac{(n+1)^n}{3^n} \right| = \lim_{n \rightarrow \infty} \frac{3(n+1)^n}{(n+2)^{n+1}} = \lim_{n \rightarrow \infty} \frac{3}{n+2} \left(\frac{n+1}{n+2} \right)^n = (0) \left(\frac{1}{e} \right) = 0$$

To find $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right)^n$, let $y = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2} \right)^n$. Then,

$$\ln y = \lim_{n \rightarrow \infty} n \ln \left(\frac{n+1}{n+2} \right) = \lim_{n \rightarrow \infty} \frac{\ln[(n+1)/(n+2)]}{1/n} = \frac{0}{0}$$

$$\ln y = \lim_{n \rightarrow \infty} \frac{[(1)/(n+1)] - [(1)/(n+2)]}{-(1/n^2)} = -1 \text{ by L'Hôpital's Rule}$$

$$y = e^{-1} = \frac{1}{e}.$$

Therefore, by the Ratio Test, the series converges.

$$29. \sum_{n=0}^{\infty} \frac{4^n}{3^n + 1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{4^{n+1}}{3^{n+1} + 1} \cdot \frac{3^n + 1}{4^n} \right| = \lim_{n \rightarrow \infty} \frac{4(3^n + 1)}{3^{n+1} + 1} = \lim_{n \rightarrow \infty} \frac{4(1 + 1/3^n)}{3 + 1/3^n} = \frac{4}{3}$$

Therefore, by the Ratio Test, the series diverges.

$$31. \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n+1)(2n+3)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{n!} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{2n+3} = \frac{1}{2}$$

Therefore, by the Ratio Test, the series converges.

Note: The first few terms of this series are $-1 + \frac{1}{1 \cdot 3} - \frac{2!}{1 \cdot 3 \cdot 5} + \frac{3!}{1 \cdot 3 \cdot 5 \cdot 7} - \cdots$

$$33. (a) \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{1} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{3/2} = 1 \quad \text{Ratio Test is inconclusive.}$$

$$(b) \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^{1/2}} \cdot \frac{n^{1/2}}{1} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^{1/2} = 1 \quad \text{Ratio Test is inconclusive.}$$

$$35. \sum_{n=1}^{\infty} \left(\frac{n}{2n+1} \right)^n$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n}{2n+1} \right)^n} \\ &= \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2} \end{aligned}$$

Therefore, by the Root Test, the series converges.

$$37. \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{(\ln n)^n}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{|\ln n|} = 0 \end{aligned}$$

Therefore, by the Root Test, the series converges.

$$39. \sum_{n=1}^{\infty} (2\sqrt[n]{n} + 1)^n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{(2\sqrt[n]{n} + 1)^n} = \lim_{n \rightarrow \infty} (2\sqrt[n]{n} + 1)$$

To find $\lim_{n \rightarrow \infty} \sqrt[n]{n}$, let $y = \lim_{n \rightarrow \infty} \sqrt[n]{n}$. Then

$$\ln y = \lim_{n \rightarrow \infty} (\ln \sqrt[n]{n}) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0.$$

Thus, $\ln y = 0$, so $y = e^0 = 1$ and $\lim_{n \rightarrow \infty} (2\sqrt[n]{n} + 1) = 2(1) + 1 = 3$. Therefore, by the Root Test, the series diverges.

$$41. \sum_{n=3}^{\infty} \frac{1}{(\ln n)^n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$$

Therefore, by the Root Test, the series converges.

$$43. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 5}{n}$$

$$a_{n+1} = \frac{5}{n+1} < \frac{5}{n} = a_n$$

$$\lim_{n \rightarrow \infty} \frac{5}{n} = 0$$

Therefore, by the Alternating Series Test, the series converges (conditional convergence).

$$45. \sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}} = 3 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

This is convergent p -series.

$$47. \sum_{n=1}^{\infty} \frac{2n}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{2n}{n+1} = 2 \neq 0$$

This diverges by the n th Term Test for Divergence.

$$49. \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n 3^{-2}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{9} \left(-\frac{3}{2} \right)^n$$

Since $|r| = \frac{3}{2} > 1$, this is a divergent geometric series.

$$51. \sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$$

$$\lim_{n \rightarrow \infty} \frac{(10n+3)/n2^n}{1/2^n} = \lim_{n \rightarrow \infty} \frac{10n+3}{n} = 10$$

Therefore, the series converges by a limit comparison test with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n.$$

$$53. \sum_{n=1}^{\infty} \frac{\cos(n)}{2^n}$$

$$\left| \frac{\cos(n)}{2^n} \right| \leq \frac{1}{2^n}$$

Therefore, the series

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n)}{2^n} \right|$$

converges by comparison with the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n.$$

$$57. \sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^n}{(n+1)!} \cdot \frac{n!}{3^{n-1}} \right| = \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0$$

Therefore, by the Ratio Test, the series converges.

$$59. \sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+1}}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{(-3)^n} \right| = \lim_{n \rightarrow \infty} \frac{3}{2n+3} = 0$$

Therefore, by the Ratio Test, the series converges.

61. (a) and (c)

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n5^n}{n!} &= \sum_{n=0}^{\infty} \frac{(n+1)5^{n+1}}{(n+1)!} \\ &= 5 + \frac{(2)(5)^2}{2!} + \frac{(3)(5)^3}{3!} + \frac{(4)(5)^4}{4!} + \cdots \end{aligned}$$

$$55. \sum_{n=1}^{\infty} \frac{n7^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)7^{n+1}}{(n+1)!} \cdot \frac{n!}{n7^n} \right| = \lim_{n \rightarrow \infty} \frac{7}{n} = 0$$

Therefore, by the Ratio Test, the series converges.

63. (a) and (b) are the same.

65. Replace n with $n+1$.

$$\sum_{n=1}^{\infty} \frac{n}{4^n} = \sum_{n=0}^{\infty} \frac{n+1}{4^{n+1}}$$

67. Since

$$\frac{3^{10}}{2^{10}10!} \approx 1.59 \times 10^{-5},$$

use 9 terms.

$$\sum_{k=1}^9 \frac{(-3)^k}{2^k k!} \approx -0.7769$$

69. See Theorem 8.17, page 597.

$$71. \text{ No. Let } a_n = \frac{1}{n+10,000}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n+10,000}$ diverges.

73. The series converges absolutely. See Theorem 8.17.

75. First, let

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = r < 1$$

and choose R such that $0 \leq r < R < 1$. There must exist some $N > 0$ such that $\sqrt[n]{|a_n|} < R$ for all $n > N$. Thus, for $n > N$, we $|a_n| < R^n$ and since the geometric series

$$\sum_{n=0}^{\infty} R^n$$

converges, we can apply the Comparison Test to conclude that

$$\sum_{n=1}^{\infty} |a_n|$$

converges which in turn implies that $\sum_{n=1}^{\infty} a_n$ converges.

Second, let

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = r > R > 1.$$

Then there must exist some $M > 0$ such that $\sqrt[n]{|a_n|} > R$ for infinitely many $n > M$. Thus, for infinitely many $n > M$, we have $|a_n| > R^n > 1$ which implies that $\lim_{n \rightarrow \infty} a_n \neq 0$ which in turn implies that

$$\sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Section 8.7 Taylor Polynomials and Approximations

1. $y = -\frac{1}{2}x^2 + 1$

Parabola

Matches (d)

3. $y = e^{-1/2}[(x+1) + 1]$

Linear

Matches (a)

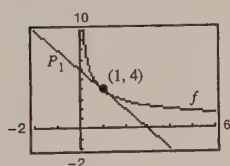
5. $f(x) = \frac{4}{\sqrt{x}} = 4x^{-1/2} \quad f(1) = 4$

$$f'(x) = -2x^{-3/2} \quad f'(1) = -2$$

$$P_1(x) = f(1) + f'(1)(x-1)$$

$$= 4 + (-2)(x-1)$$

$$P_1(x) = -2x + 6$$



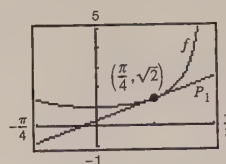
P_1 is called the first degree Taylor polynomial for f at c .

7. $f(x) = \sec x \quad f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$$f'(x) = \sec x \tan x \quad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$P_1(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)$$

$$P_1(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right)$$



P_1 is called the first degree Taylor polynomial for f at c .

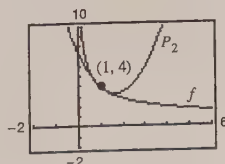
9. $f(x) = \frac{4}{\sqrt{x}} = 4x^{-1/2} \quad f(1) = 4$

$$f'(x) = -2x^{-3/2} \quad f'(1) = -2$$

$$f''(x) = 3x^{-5/2} \quad f''(1) = 3$$

$$P_2 = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$$

$$= 4 - 2(x-1) + \frac{3}{2}(x-1)^2$$



x	0	0.8	0.9	1.0	1.1	1.2	2
$f(x)$	Error	4.4721	4.2164	4.0	3.8139	3.6515	2.8284
$P_2(x)$	7.5	4.46	4.215	4.0	3.815	3.66	3.5

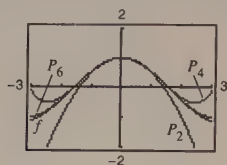
11. $f(x) = \cos x$

$$P_2(x) = 1 - \frac{1}{2}x^2$$

$$P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$$

(a)



13. $f(x) = e^{-x}$ $f(0) = 1$

$$f'(x) = -e^{-x} \quad f'(0) = -1$$

$$f''(x) = e^{-x} \quad f''(0) = 1$$

$$f'''(x) = -e^{-x} \quad f'''(0) = -1$$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

17. $f(x) = \sin x$ $f(0) = 0$

$$f'(x) = \cos x \quad f'(0) = 1$$

$$f''(x) = -\sin x \quad f''(0) = 0$$

$$f'''(x) = -\cos x \quad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \quad f^{(5)}(0) = 1$$

$$P_5(x) = 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

21. $f(x) = \frac{1}{x+1}$ $f(0) = 1$

$$f'(x) = -\frac{1}{(x+1)^2} \quad f'(0) = -1$$

$$f''(x) = \frac{2}{(x+1)^3} \quad f''(0) = 2$$

$$f'''(x) = \frac{-6}{(x+1)^4} \quad f'''(0) = -6$$

$$f^{(4)}(x) = \frac{24}{(x+1)^5} \quad f^{(4)}(0) = 24$$

$$P_4(x) = 1 - x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{24}{4!}x^4$$

$$= 1 - x + x^2 - x^3 + x^4$$

(b) $f'(x) = -\sin x$ $P_2'(x) = -x$

$$f''(x) = -\cos x \quad P_2''(x) = -1$$

$$f''(0) = P_2''(0) = -1$$

$$f'''(x) = \sin x \quad P_4'''(x) = x$$

$$f^{(4)}(x) = \cos x \quad P_4^{(4)}(x) = 1$$

$$f^{(4)}(0) = 1 = P_4^{(4)}(0)$$

$$f^{(5)}(x) = -\sin x \quad P_6^{(5)}(x) = -x$$

$$f^{(6)}(x) = -\cos x \quad P_6^{(6)}(x) = -1$$

$$f^{(6)}(0) = -1 = P_6^{(6)}(0)$$

(c) In general, $f^{(n)}(0) = P_n^{(n)}(0)$ for all n .

15. $f(x) = e^{2x}$ $f(0) = 1$

$$f'(x) = 2e^{2x} \quad f'(0) = 2$$

$$f''(x) = 4e^{2x} \quad f''(0) = 4$$

$$f'''(x) = 8e^{2x} \quad f'''(0) = 8$$

$$f^{(4)}(x) = 16e^{2x} \quad f^{(4)}(0) = 16$$

$$P_4(x) = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$$

19. $f(x) = xe^x$ $f(0) = 0$

$$f'(x) = xe^x + e^x \quad f'(0) = 1$$

$$f''(x) = xe^x + 2e^x \quad f''(0) = 2$$

$$f'''(x) = xe^x + 3e^x \quad f'''(0) = 3$$

$$f^{(4)}(x) = xe^x + 4e^x \quad f^{(4)}(0) = 4$$

$$P_4(x) = 0 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4$$

$$= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

23. $f(x) = \sec x$ $f(0) = 1$

$$f'(x) = \sec x \tan x \quad f'(0) = 0$$

$$f''(x) = \sec^3 x + \sec x \tan^2 x \quad f''(0) = 1$$

$$P_2(x) = 1 + 0x + \frac{1}{2!}x^2 = 1 + \frac{1}{2}x^2$$

$$25. \quad f(x) = \frac{1}{x} \quad f(1) = 1$$

$$f'(x) = -\frac{1}{x^2} \quad f'(1) = -1$$

$$f''(x) = \frac{2}{x^3} \quad f''(1) = 2$$

$$f'''(x) = -\frac{6}{x^4} \quad f'''(1) = -6$$

$$f^{(4)}(x) = \frac{24}{x^5} \quad f^{(4)}(1) = 24$$

$$P_4(x) = 1 - (x-1) + \frac{2}{2!}(x-1)^2 + \frac{-6}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4$$

$$27. \quad f(x) = \sqrt{x} \quad f(1) = 1$$

$$f'(x) = \frac{1}{2\sqrt{x}} \quad f'(1) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4x\sqrt{x}} \quad f''(1) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8x^2\sqrt{x}} \quad f'''(1) = \frac{3}{8}$$

$$f^{(4)}(x) = -\frac{15}{16x^3\sqrt{x}} \quad f^{(4)}(1) = -\frac{15}{16}$$

$$P_4(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$$

$$+ \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$$

$$29. \quad f(x) = \ln x \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \quad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \quad f^{(4)}(1) = -6$$

$$P_4(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2$$

$$+ \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

$$31. \quad f(x) = \tan x$$

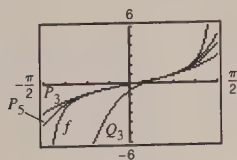
$$f'(x) = \sec^2 x$$

$$f''(x) = 2 \sec^2 x \tan x$$

$$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$$

$$f^{(4)}(x) = 8 \sec^2 x \tan^3 x + 16 \sec^4 x \tan x$$

$$f^{(5)}(x) = 16 \sec^2 x \tan^4 x + 88 \sec^4 x \tan^2 x + 16 \sec^6 x$$



$$(a) \quad n = 3, c = 0$$

$$P_3(x) = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 = x + \frac{1}{3}x^3$$

$$(b) \quad n = 5, c = 0$$

$$P_5(x) = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{16}{5!}x^5$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5$$

$$(c) \quad n = 3, c = \frac{\pi}{4}$$

$$Q_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x - \frac{\pi}{4}\right)^3$$

$$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

33. $f(x) = \sin x$

$P_1(x) = x$

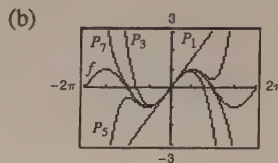
$P_3(x) = x - \frac{1}{6}x^3$

$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

$P_7(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$

(a)

x	0.00	0.25	0.50	0.75	1.00
$\sin x$	0.0000	0.2474	0.4794	0.6816	0.8415
$P_1(x)$	0.0000	0.2500	0.5000	0.7500	1.0000
$P_3(x)$	0.0000	0.2474	0.4792	0.6797	0.8333
$P_5(x)$	0.0000	0.2474	0.4794	0.6817	0.8417
$P_7(x)$	0.0000	0.2474	0.4794	0.6816	0.8415



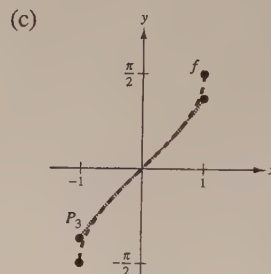
(c) As the distance increases, the accuracy decreases

35. $f(x) = \arcsin x$

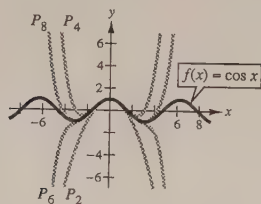
(a) $P_3(x) = x + \frac{x^3}{6}$

(b)

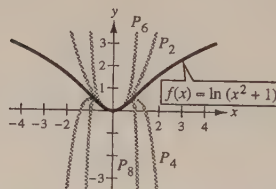
x	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
$f(x)$	-0.848	-0.524	-0.253	0	0.253	0.524	0.848
$P_3(x)$	-0.820	-0.521	-0.253	0	0.253	0.521	0.820



37. $f(x) = \cos x$



39. $f(x) = \ln(x^2 + 1)$



41. $f(x) = e^{-x} \approx 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$

$f\left(\frac{1}{2}\right) \approx 0.6042$

43. $f(x) = \ln x \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$
 $f(1.2) \approx 0.1823$

45. $f(x) = \cos x$; $f^{(5)}(x) = -\sin x \Rightarrow \text{Max on } [0, 0.3] \text{ is } 1.$

$R_4(x) \leq \frac{1}{5!}(0.3)^5 = 2.025 \times 10^{-5}$

Note: You could use $R_5(x)$: $f^{(6)}(x) = -\cos x$, max on $[0, 0.3]$ is 1. $R_5(x) \leq \frac{1}{6!}(0.3)^6 = 1.0125 \times 10^{-6}$

Exact Error: $0.000001 = 1.0 \times 10^{-6}$

$$47. f(x) = \arcsin x; f_4^{(4)}(x) = \frac{x(6x^2 + 9)}{(1 - x^2)^{7/2}} \Rightarrow \text{Max on } [0, 0.4] \text{ is } f_4^{(4)}(0.4) \approx 7.3340.$$

$$R_3(x) \leq \frac{7.3340}{4!}(0.4)^4 \approx 0.00782 = 7.82 \times 10^{-3}$$

$$49. g(x) = \sin x$$

$$g^{(n+1)}(x) \leq 1 \text{ for all } x$$

$$R_n(x) \leq \frac{1}{(n+1)!}(0.3)^{n+1} < 0.001$$

By trial and error, $n = 3$.

$$51. f(x) = \ln(x+1)$$

$$f^{(n+1)}(x) = \frac{(-1)^n n!}{(x+1)^{n+1}} \Rightarrow \text{Max on } [0, 0.5] \text{ is } n!.$$

$$R_n \leq \frac{n!}{(n+1)!}(0.5)^{n+1} = \frac{(0.5)^{n+1}}{n+1} < 0.0001$$

By trial and error, $n = 9$. (See Example 9.) Using 9 terms, $\ln(1.5) \approx 0.4055$.

$$53. f(x) = e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}, x < 0$$

$$R_3(x) = \frac{e^z}{4!}x^4 < 0.001$$

$$e^z x^4 < 0.024$$

$$xe^{z/4} < 0.3936$$

$$x < \frac{0.3936}{e^{z/4}} < 0.3936, z < 0$$

$$-0.3936 < x < 0$$

55. The graph of the approximating polynomial P and the elementary function f both pass through the point $(c, f(c))$ and the slopes of P and f agree at $(c, f(c))$. Depending on the degree of P , the n th derivatives of P and f agree at $(c, f(c))$.

57. See definition on page 607.

59. The accuracy increases as the degree increases (for values within the interval of convergence).

$$61. (a) f(x) = e^x$$

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$g(x) = xe^x$$

$$Q_5(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$$

$$Q_5(x) = xP_4(x)$$

$$(b) f(x) = \sin x$$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$g(x) = x \sin x$$

$$Q_6(x) = xP_5(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}$$

$$(c) g(x) = \frac{\sin x}{x} = \frac{1}{x}P_5(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!}$$

$$63. (a) Q_2(x) = -1 + \frac{\pi^2(x+2)^2}{32}$$

$$(b) R_2(x) = -1 + \frac{\pi^2(x-6)^2}{32}$$

(c) No. The polynomial will be linear.
Translations are possible at $x = -2 + 8n$.

65. Let f be an even function and P_n be the n th Maclaurin polynomial for f . Since f is even, f' is odd, f'' is even, f''' is odd, etc. All of the odd derivatives of f are odd and thus, all of the odd powers of x will have coefficients of zero. P_n will only have terms with even powers of x .

67. As you move away from $x = c$, the Taylor Polynomial becomes less and less accurate.

Section 8.8 Power Series

1. Centered at 0

$$5. \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$$

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} \right| |x| = |x| \\ |x| < 1 &\Rightarrow R = 1 \end{aligned}$$

$$9. \sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!}$$

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{2n+2}/(2n+2)!}{(2x)^{2n}/(2n)!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(2x)^2}{(2n+2)(2n+1)} \right| = 0 \end{aligned}$$

Thus, the series converges for all x . $R = \infty$.

3. Centered at 2

$$7. \sum_{n=1}^{\infty} \frac{(2x)^n}{n^2}$$

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(2x)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{2n^2 x}{(n+1)^2} \right| = 2|x| \\ 2|x| < 1 &\Rightarrow R = \frac{1}{2} \end{aligned}$$

$$11. \sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n$$

Since the series is geometric, it converges only if $|x/2| < 1$ or $-2 < x < 2$.

$$13. \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{nx}{n+1} \right| = |x|. \end{aligned}$$

Interval: $-1 < x < 1$

When $x = 1$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

When $x = -1$, the p -series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Therefore, the interval of convergence is $-1 < x \leq 1$.

$$15. \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = 0 \end{aligned}$$

The series converges for all x . Therefore, the interval of convergence is $-\infty < x < \infty$.

$$17. \sum_{n=0}^{\infty} (2n)! \left(\frac{x}{2} \right)^n$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)! x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(2n)! x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)x}{2} \right| = \infty$$

Therefore, the series converges only for $x = 0$.

$$19. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4^n}$$

Since the series is geometric, it converges only if $|x/4| < 1$ or $-4 < x < 4$.

$$21. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{(-1)^{n+1}(x-5)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n(x-5)}{5(n+1)} \right| = \frac{1}{5}|x-5|$$

$$R = 5$$

$$\text{Center: } x = 5$$

$$\text{Interval: } -5 < x - 5 < 5 \text{ or } 0 < x < 10$$

When $x = 0$, the p -series $\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.

When $x = 10$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Therefore, the interval of convergence is $0 < x \leq 10$.

$$23. \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(-1)^{n+1}(x-1)^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-1)}{n+2} \right| = |x-1|$$

$$R = 1$$

$$\text{Center: } x = 1$$

$$\text{Interval: } -1 < x - 1 < 1 \text{ or } 0 < x < 2$$

When $x = 0$, the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges by the integral test.

When $x = 2$, the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$ converges.

Therefore, the interval of convergence is $0 < x \leq 2$.

$$25. \sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-c)^n}{c^n} \cdot \frac{c^{n-1}}{(x-c)^{n-1}} \right| = \frac{1}{c}|x-c|$$

$$R = c$$

$$\text{Center: } x = c$$

$$\text{Interval: } -c < x - c < c \text{ or } 0 < x < 2c$$

When $x = 0$, the series $\sum_{n=1}^{\infty} (-1)^{n-1}$ diverges.

When $x = 2c$, the series $\sum_{n=1}^{\infty} 1$ diverges.

Therefore, the interval of convergence is $0 < x < 2c$.

$$27. \sum_{n=1}^{\infty} \frac{n}{n+1} (-2x)^{n-1}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(-2x)^n}{n+2} \cdot \frac{n+1}{n(-2x)^{n-1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-2x)(n+1)^2}{n(n+2)} \right| = 2|x| \end{aligned}$$

$$R = \frac{1}{2}$$

$$\text{Interval: } -\frac{1}{2} < x < \frac{1}{2}$$

When $x = -\frac{1}{2}$, the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges by the n th Term Test.

When $x = \frac{1}{2}$, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$ diverges.

Therefore, the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$.

$$29. \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x^2}{(2n+2)(2n+3)} \right| = 0 \end{aligned}$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

$$31. \sum_{n=1}^{\infty} \frac{k(k+1) \cdots (k+n-1)x^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{k(k+1) \cdots (k+n-1)(k+n)x^{n+1}}{(n+1)!} \cdot \frac{n!}{k(k+1) \cdots (k+n-1)x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(k+n)x}{n+1} \right| = |x|$$

$$R = 1$$

When $x = \pm 1$, the series diverges and the interval of convergence is $-1 < x < 1$.

$$\left[\frac{k(k+1) \cdots (k+n-1)}{1 \cdot 2 \cdots n} \geq 1 \right]$$

$$33. \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n}{4^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)(4n+3)(x-3)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(-1)^{n+1} \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(4n+3)(x-3)}{4} \right| = \infty \end{aligned}$$

$$R = 0$$

Center: $x = 3$

Therefore, the series converges only for $x = 3$.

$$35. (a) f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n, -2 < x < 2 \quad (\text{Geometric})$$

$$(b) f'(x) = \sum_{n=1}^{\infty} \left(\frac{n}{2}\right) \left(\frac{x}{2}\right)^{n-1}, -2 < x < 2$$

$$(c) f''(x) = \sum_{n=2}^{\infty} \left(\frac{n}{2}\right) \left(\frac{n-1}{2}\right) \left(\frac{x}{2}\right)^{n-2}, -2 < x < 2$$

$$(d) \int f(x) dx = \sum_{n=0}^{\infty} \frac{2}{n+1} \left(\frac{x}{2}\right)^{n+1}, -2 \leq x < 2$$

$$37. (a) f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}, 0 < x \leq 2$$

$$(b) f'(x) = \sum_{n=0}^{\infty} (-1)^{n+1}(x-1)^n, 0 < x < 2$$

$$(c) f''(x) = \sum_{n=1}^{\infty} (-1)^{n+1}n(x-1)^{n-1}, 0 < x < 2$$

$$(d) \int f(x) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+2}}{(n+1)(n+2)}, 0 \leq x \leq 2$$

$$39. g(1) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \frac{1}{9} + \cdots$$

$$S_1 = 1, S_2 = 1.33. \text{ Matches (c)}$$

$$41. g(3.1) = \sum_{n=0}^{\infty} \left(\frac{3.1}{3}\right)^n \text{ diverges. Matches (b)}$$

43. A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

is called a power series centered at c .

45. A single point, an interval, or the entire real line.

47. (a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, -\infty < x < \infty$ (See Exercise 29.)

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, -\infty < x < \infty$$

(b) $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = g(x)$

(c) $g'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = -f(x)$

(d) $f(x) = \sin x$ and $g(x) = \cos x$

49. $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$

$$y' = \sum_{n=1}^{\infty} \frac{2nx^{2n-1}}{2^n n!}$$

$$y'' = \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!}$$

$$\begin{aligned} y'' - xy' - y &= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=1}^{\infty} \frac{2nx^{2n}}{2^n n!} - \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \\ &= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{2^n n!} \\ &= \sum_{n=0}^{\infty} \left[\frac{(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} - \frac{(2n+1)x^{2n}}{2^n n!} \cdot \frac{2(n+1)}{2(n+1)} \right] \\ &= \sum_{n=0}^{\infty} \frac{2(n+1)x^{2n} [(2n+1) - (2n+1)]}{2^{n+1}(n+1)!} = 0 \end{aligned}$$

51. $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!)^2}$

(a) $\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} x^{2k+2}}{2^{2k+2} [(k+1)!]^2} \cdot \frac{2^{2k} (k!)^2}{(-1)^k x^{2k}} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)x^2}{2^2(k+1)^2} \right| = 0$

Therefore, the interval of convergence is $-\infty < x < \infty$.

(b) $J_0 = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k (k!)^2}$

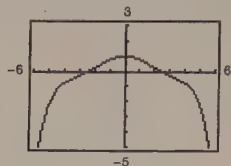
$$J_0' = \sum_{k=1}^{\infty} (-1)^k \frac{2kx^{2k-1}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)x^{2k+1}}{4^{k+1} [(k+1)!]^2}$$

$$J_0'' = \sum_{k=1}^{\infty} (-1)^k \frac{2k(2k-1)x^{2k-2}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)(2k+1)x^{2k}}{4^{k+1} [(k+1)!]^2}$$

$$\begin{aligned} x^2 J_0'' + x J_0' + x^2 J_0 &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2(2k+1)x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+2}}{4^k (k!)^2} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2} \left[(-1) \frac{2(2k+1)}{4(k+1)} + (-1) \frac{2}{4(k+1)} + 1 \right] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2} \left[\frac{-4k-2}{4k+4} - \frac{2}{4k+4} + \frac{4k+4}{4k+4} \right] = 0 \end{aligned}$$

51. —CONTINUED—

$$(c) P_6(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$

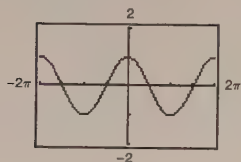


$$\begin{aligned} (d) \int_0^1 J_0 dx &= \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{4^k (k!)^2} dx \\ &= \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{4^k (k!)^2 (2k+1)} \right]_0^1 \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2 (2k+1)} \\ &= 1 - \frac{1}{12} + \frac{1}{320} \approx 0.92 \end{aligned}$$

(integral is approximately 0.9197304101)

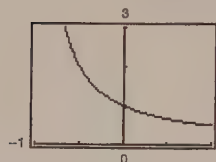
$$53. f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

(See Exercise 47.)



$$55. f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n$$

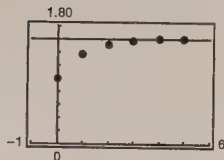
$$= \frac{1}{1 - (-x)} = \frac{1}{1+x} \text{ for } -1 < x < 1$$



$$57. \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

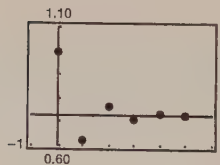
$$(a) \sum_{n=0}^{\infty} \left(\frac{3/4}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{3}{8}\right)^n$$

$$= \frac{1}{1 - (3/8)} = \frac{8}{5} = 1.6$$



$$(b) \sum_{n=0}^{\infty} \left(\frac{-3/4}{2}\right)^n = \sum_{n=0}^{\infty} \left(-\frac{3}{8}\right)^n$$

$$= \frac{1}{1 - (-3/8)} = \frac{8}{11} \approx 0.7272$$



(c) The alternating series converges more rapidly. The partial sums of the series of positive terms approach the sum from below. The partial sums of the alternating series alternate sides of the horizontal line representing the sum.

$$(d) \sum_{n=0}^N \left(\frac{3}{2}\right)^n > M$$

M	10	100	1000	10,000
N	4	9	15	21

59. False;

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n2^n}$$

converges for $x = 2$ but diverges for $x = -2$.

61. True; the radius of convergence is $R = 1$ for both series.

Section 8.9 Representation of Functions by Power Series

$$1. (a) \frac{1}{2-x} = \frac{1/2}{1-(x/2)} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

This series converges on $(-2, 2)$.

$$\frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \cdots$$

$$(b) \begin{array}{r} 2-x \overline{) 1} \end{array}$$

$$\begin{array}{r} 1 - \frac{x}{2} \\ \underline{\phantom{1 - \frac{x}{2}} x} \\ \frac{x}{2} - \frac{x^2}{4} \\ \underline{\phantom{\frac{x}{2} - \frac{x^2}{4}} x^2} \\ \frac{x^2}{4} - \frac{x^3}{8} \\ \underline{\phantom{\frac{x^2}{4} - \frac{x^3}{8}} x^3} \\ \frac{x^3}{8} - \frac{x^4}{16} \\ \vdots \end{array}$$

5. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{1}{2-x} = \frac{1}{-3-(x-5)} = \frac{-1/3}{1+(1/3)(x-5)}$$

which implies that $a = -1/3$ and $r = (1/3)(x-5)$.

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned} \frac{1}{2-x} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} -\frac{1}{3} \left[-\frac{1}{3}(x-5) \right]^n \\ &= \sum_{n=0}^{\infty} \frac{(x-5)^n}{(-3)^{n+1}}, \quad |x-5| < 3 \text{ or } 2 < x < 8. \end{aligned}$$

9. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\begin{aligned} \frac{1}{2x-5} &= \frac{-1}{11-2(x+3)} \\ &= \frac{-1/11}{1-(2/11)(x+3)} = \frac{a}{1-r} \end{aligned}$$

which implies that $a = -1/11$ and $r = (2/11)(x+3)$.

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned} \frac{1}{2x-5} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left(-\frac{1}{11} \right) \left[\frac{2}{11}(x+3) \right]^n \\ &= -\sum_{n=0}^{\infty} \frac{2^n(x+3)^n}{11^{n+1}}, \end{aligned}$$

$$|x+3| < \frac{11}{2} \text{ or } -\frac{17}{2} < x < \frac{5}{2}.$$

$$\begin{aligned} 3. (a) \frac{1}{2+x} &= \frac{1/2}{1-(-x/2)} = \frac{a}{1-r} \\ &= \sum_{n=0}^{\infty} \frac{1}{2} \left(-\frac{x}{2} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} \end{aligned}$$

This series converges on $(-2, 2)$.

$$\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots$$

$$(b) \begin{array}{r} 2+x \overline{) 1} \end{array}$$

$$\begin{array}{r} 1 + \frac{x}{2} \\ \underline{\phantom{1 + \frac{x}{2}} x} \\ -\frac{x}{2} - \frac{x^2}{4} \\ \underline{\phantom{-\frac{x}{2} - \frac{x^2}{4}} x^2} \\ \frac{x^2}{4} + \frac{x^3}{8} \\ \underline{\phantom{\frac{x^2}{4} + \frac{x^3}{8}} x^3} \\ -\frac{x^3}{8} - \frac{x^4}{16} \\ \vdots \end{array}$$

7. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{2x-1} = \frac{-3}{1-2x} = \frac{a}{1-r}$$

which implies that $a = -3$ and $r = 2x$.

Therefore, the power series for $f(x)$ is given by

$$\begin{aligned} \frac{3}{2x-1} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (-3)(2x)^n \\ &= -3 \sum_{n=0}^{\infty} (2x)^n, \quad |2x| < 1 \text{ or } -\frac{1}{2} < x < \frac{1}{2}. \end{aligned}$$

11. Writing $f(x)$ in the form $a/(1-r)$, we have

$$\frac{3}{x+2} = \frac{3}{2+x} = \frac{3/2}{1+(1/2)x} = \frac{a}{1-r}$$

which implies that $a = 3/2$ and $r = (-1/2)x$. Therefore, the power series for $f(x)$ is given by

$$\begin{aligned} \frac{3}{x+2} &= \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \frac{3}{2} \left(-\frac{1}{2}x \right)^n \\ &= 3 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}} = \frac{3}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2} \right)^n, \end{aligned}$$

$$|x| < 2 \text{ or } -2 < x < 2.$$

$$13. \frac{3x}{x^2 + x - 2} = \frac{2}{x + 2} + \frac{1}{x - 1} = \frac{2}{2 + x} + \frac{1}{-1 + x} = \frac{1}{1 + (1/2)x} + \frac{-1}{1 - x}$$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{3x}{x^2 + x - 2} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty} (-1)(x)^n = \sum_{n=0}^{\infty} \left[\frac{1}{(-2)^n} - 1\right]x^n.$$

The interval of convergence is $-1 < x < 1$ since

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(1 - (-2)^{n+1})x^{n+1}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{(1 - (-2)^n)x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(1 - (-2)^{n+1})x}{-2 - (-2)^{n+1}} \right| = |x|.$$

$$15. \frac{2}{1 - x^2} = \frac{1}{1 - x} + \frac{1}{1 + x}$$

Writing $f(x)$ as a sum of two geometric series, we have

$$\frac{2}{1 - x^2} = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (1 + (-1)^n)x^n = \sum_{n=0}^{\infty} 2x^{2n}.$$

The interval of convergence is $|x^2| < 1$ or $-1 < x < 1$ since $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2x^{2n+2}}{2x^{2n}} \right| = |x^2|.$

$$17. \frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} (-1)^n (-x)^n = \sum_{n=0}^{\infty} (-1)^{2n} x^n = \sum_{n=0}^{\infty} x^n$$

$$\begin{aligned} h(x) &= \frac{-2}{x^2 - 1} = \frac{1}{1 + x} + \frac{1}{1 - x} = \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} [(-1)^n + 1]x^n \\ &= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots = \sum_{n=0}^{\infty} 2x^{2n}, \quad -1 < x < 1 \text{ (See Exercise 15.)} \end{aligned}$$

19. By taking the first derivative, we have $\frac{d}{dx} \left[\frac{1}{x + 1} \right] = \frac{-1}{(x + 1)^2}$. Therefore,

$$\begin{aligned} \frac{-1}{(x + 1)^2} &= \frac{d}{dx} \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] = \sum_{n=1}^{\infty} (-1)^n n x^{n-1} \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} (n + 1) x^n, \quad -1 < x < 1. \end{aligned}$$

21. By integrating, we have $\int \frac{1}{x + 1} dx = \ln(x + 1)$. Therefore,

$$\ln(x + 1) = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n + 1}, \quad -1 < x \leq 1.$$

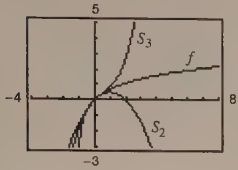
To solve for C , let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x + 1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n + 1}, \quad -1 < x \leq 1.$$

$$23. \frac{1}{x^2 + 1} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad -1 < x < 1$$

$$25. \text{ Since } \frac{1}{x + 1} = \sum_{n=0}^{\infty} (-1)^n x^n, \text{ we have } \frac{1}{4x^2 + 1} = \sum_{n=0}^{\infty} (-1)^n (4x^2)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n (2x)^{2n}, \quad -\frac{1}{2} < x < \frac{1}{2}.$$

$$27. x - \frac{x^2}{2} \leq \ln(x+1) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$$



x	0.0	0.2	0.4	0.6	0.8	1.0
$x - \frac{x^2}{2}$	0.000	0.180	0.320	0.420	0.480	0.500
$\ln(x+1)$	0.000	0.182	0.336	0.470	0.588	0.693
$x - \frac{x^2}{2} + \frac{x^3}{3}$	0.000	0.183	0.341	0.492	0.651	0.833

$$29. g(x) = x, \text{ line, Matches (c)}$$

$$31. g(x) = x - \frac{x^3}{3} + \frac{x^5}{5}, \text{ Matches (a)}$$

$$33. f(x) = \arctan x \text{ is an odd function (symmetric to the origin)}$$

$$\text{In Exercises 35 and 37, } \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

$$35. \arctan \frac{1}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^{2n+1}} = \frac{1}{4} - \frac{1}{192} + \frac{1}{5120} + \dots$$

Since $\frac{1}{5120} < 0.001$, we can approximate the series by its first two terms: $\arctan \frac{1}{4} \approx \frac{1}{4} - \frac{1}{192} \approx 0.245$.

$$37. \frac{\arctan x^2}{x} = \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2n+1}$$

$$\int \frac{\arctan x^2}{x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(4n+2)(2n+1)} + C \quad (\text{Note: } C = 0)$$

$$\int_0^{1/2} \frac{\arctan x^2}{x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+2)(2n+1)2^{4n+2}} = \frac{1}{8} - \frac{1}{1152} + \dots$$

Since $\frac{1}{1152} < 0.001$, we can approximate the series by its first term: $\int_0^{1/2} \frac{\arctan x^2}{x} dx \approx 0.125$

$$\text{In Exercises 39 and 41, use } \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1.$$

$$39. (a) \frac{1}{(1-x)^2} = \frac{d}{dx} \left[\frac{1}{1-x} \right] = \frac{d}{dx} \left[\sum_{n=0}^{\infty} x^n \right] = \sum_{n=1}^{\infty} nx^{n-1}, |x| < 1$$

$$(b) \frac{x}{(1-x)^2} = x \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^n, |x| < 1$$

$$(c) \frac{1+x}{(1-x)^2} = \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n(x^{n-1} + x^n), |x| < 1$$

$$= \sum_{n=0}^{\infty} (2n+1)x^n, |x| < 1$$

$$(d) \frac{x(1+x)}{(1-x)^2} = x \sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+1)x^{n+1}, |x| < 1$$

$$41. P(n) = \left(\frac{1}{2}\right)^n$$

$$E(n) = \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = \frac{1}{2} \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{1}{2} \frac{1}{[1 - (1/2)]^2} = 2$$

Since the probability of obtaining a head on a single toss is $\frac{1}{2}$, it is expected that, on average, a head will be obtained in two tosses.

43. Replace x with $(-x)$.45. Replace x with $(-x)$ and multiply the series by 5.47. Let $\arctan x + \arctan y = \theta$. Then,

$$\tan(\arctan x + \arctan y) = \tan \theta$$

$$\frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x) \tan(\arctan y)} = \tan \theta$$

$$\frac{x + y}{1 - xy} = \tan \theta$$

$$\arctan\left(\frac{x + y}{1 - xy}\right) = \theta. \text{ Therefore, } \arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right) \text{ for } xy \neq 1.$$

$$49. (a) 2 \arctan \frac{1}{2} = \arctan \frac{1}{2} + \arctan \frac{1}{2} = \arctan \left[\frac{(1/2) + (1/2)}{1 - (1/2)^2} \right] = \arctan \frac{4}{3}$$

$$2 \arctan \frac{1}{2} - \arctan \frac{1}{7} = \arctan \frac{4}{3} + \arctan \left(-\frac{1}{7} \right) = \arctan \left[\frac{(4/3) - (1/7)}{1 + (4/3)(1/7)} \right] = \arctan \frac{25}{25} = \arctan 1 = \frac{\pi}{4}$$

$$(b) \pi = 8 \arctan \frac{1}{2} - 4 \arctan \frac{1}{7} \approx 8 \left[\frac{1}{2} - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5} - \frac{(0.5)^7}{7} \right] - 4 \left[\frac{1}{7} - \frac{(1/7)^3}{3} + \frac{(1/7)^5}{5} - \frac{(1/7)^7}{7} \right] \approx 3.14$$

51. From Exercise 21, we have

$$\begin{aligned} \ln(x + 1) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \end{aligned}$$

$$\begin{aligned} \text{Thus, } \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2^n n} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (1/2)^n}{n} \\ &= \ln\left(\frac{1}{2} + 1\right) = \ln \frac{3}{2} \approx 0.4055 \end{aligned}$$

53. From Exercise 51, we have

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{5^n n} &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2/5)^n}{n} \\ &= \ln\left(\frac{2}{5} + 1\right) = \ln \frac{7}{5} \approx 0.3365. \end{aligned}$$

55. From Exercise 54, we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n+1}(2n+1)} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)^{2n+1}}{2n+1} = \arctan \frac{1}{2} \approx 0.4636.$$

57. The series in Exercise 54 converges to its sum at a slower rate because its terms approach 0 at a much slower rate.

$$59. f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$$

$$f(0.5) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-0.5)^n}{n} = \sum_{n=1}^{\infty} -\frac{(1/2)^n}{n}$$

$$\sum_{n=1}^{50} -\frac{(1/2)^n}{n} \approx -0.693147$$

$$\ln(0.5) \approx -0.693147$$

Section 8.10 Taylor and Maclaurin Series

1. For $c = 0$, we have:

$$f(x) = e^{2x}$$

$$f^{(n)}(x) = 2^n e^{2x} \implies f^{(n)}(0) = 2^n$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

3. For $c = \pi/4$, we have:

$$f(x) = \cos(x) \quad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = -\sin(x) \quad f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f''(x) = -\cos(x) \quad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(x) = \sin(x) \quad f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f^{(4)}(x) = \cos(x) \quad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

and so on. Therefore, we have:

$$\begin{aligned} \cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi/4)[x - (\pi/4)]^n}{n!} \\ &= \frac{\sqrt{2}}{2} \left[1 - \left(x - \frac{\pi}{4}\right) - \frac{[x - (\pi/4)]^2}{2!} + \frac{[x - (\pi/4)]^3}{3!} + \frac{[x - (\pi/4)]^4}{4!} - \dots \right] \\ &= \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2} [x - (\pi/4)]^n}{n!}. \end{aligned}$$

[Note: $(-1)^{n(n+1)/2} = 1, -1, -1, 1, 1, -1, -1, 1, \dots$]

5. For $c = 1$, we have,

$$f(x) = \ln x \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \quad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \quad f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \quad f^{(4)}(1) = -6$$

$$f^{(5)}(x) = \frac{24}{x^5} \quad f^{(5)}(1) = 24$$

and so on. Therefore, we have:

$$\begin{aligned} \ln x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} \\ &= 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} + \frac{24(x-1)^5}{5!} - \dots \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \end{aligned}$$

7. For $c = 0$, we have:

$$\begin{aligned}
 f(x) &= \sin 2x & f(0) &= 0 \\
 f'(x) &= 2 \cos 2x & f'(0) &= 2 \\
 f''(x) &= -4 \sin 2x & f''(0) &= 0 \\
 f'''(x) &= -8 \cos 2x & f'''(0) &= -8 \\
 f^{(4)}(x) &= 16 \sin 2x & f^{(4)}(0) &= 0 \\
 f^{(5)}(x) &= 32 \cos 2x & f^{(5)}(0) &= 32 \\
 f^{(6)}(x) &= -64 \sin 2x & f^{(6)}(0) &= 0 \\
 f^{(7)}(x) &= -128 \cos 2x & f^{(7)}(0) &= -128
 \end{aligned}$$

and so on. Therefore, we have:

$$\begin{aligned}
 \sin 2x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 0 + 2x + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \frac{32x^5}{5!} + \frac{0x^6}{6!} - \frac{128x^7}{7!} + \cdots \\
 &= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}
 \end{aligned}$$

9. For $c = 0$, we have:

$$\begin{aligned}
 f(x) &= \sec(x) & f(0) &= 1 \\
 f'(x) &= \sec(x)\tan(x) & f'(0) &= 0 \\
 f''(x) &= \sec^3(x) + \sec(x)\tan^2(x) & f''(0) &= 1 \\
 f'''(x) &= 5 \sec^3(x)\tan(x) + \sec(x)\tan^3(x) & f'''(0) &= 0 \\
 f^{(4)}(x) &= 5 \sec^5(x) + 18 \sec^3(x)\tan^2(x) + \sec(x)\tan^4(x) & f^{(4)}(0) &= 5 \\
 \sec(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots
 \end{aligned}$$

11. The Maclaurin series for $f(x) = \cos x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

Because $f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$, we have $|f^{(n+1)}(z)| \leq 1$ for all z . Hence by Taylor's Theorem,

$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Since $\lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, it follows that $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$. Hence, the Maclaurin series for $\cos x$ converges to $\cos x$ for all x .

13. Since $(1+x)^{-k} = 1 - kx + \frac{k(k+1)x^2}{2!} - \frac{k(k+1)(k+2)x^3}{3!} + \cdots$, we have

$$\begin{aligned}
 (1+x)^{-2} &= 1 - 2x + \frac{2(3)x^2}{2!} - \frac{2(3)(4)x^3}{3!} + \frac{2(3)(4)(5)x^4}{5!} - \cdots = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots \\
 &= \sum_{n=0}^{\infty} (-1)^n (n+1)x^n.
 \end{aligned}$$

15. $\frac{1}{\sqrt{4+x^2}} = \left(\frac{1}{2}\right) \left[1 + \left(\frac{x}{2}\right)^2\right]^{-1/2}$ and since $(1+x)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^n}{2^n n!}$, we have

$$\frac{1}{\sqrt{4+x^2}} = \frac{1}{2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)(x/2)^{2n}}{2^n n!} \right] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)x^{2n}}{2^{3n+1} n!}.$$

17. Since $(1+x)^{1/2} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^n}{2^n n!}$ (Exercise 14)

we have $(1+x^2)^{1/2} = 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)x^{2n}}{2^n n!}.$

19. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$e^{x^2/2} = \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = 1 + \frac{x^2}{2} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} + \cdots$$

21. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!} = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots$$

23. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$$\cos x^{3/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{3/2})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!} = 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \cdots$$

25. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \cdots$$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

27. $\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$

$$= \frac{1}{2} \left[1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \right]$$

$$= \frac{1}{2} \left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right]$$

29. $x \sin x = x \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)$

$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$$

31. $\frac{\sin x}{x} = \frac{x - (x^3/3!) + (x^5/5!) - \cdots}{x}$

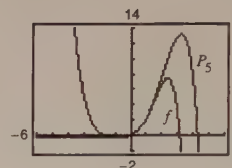
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}, x \neq 0$$

$$\begin{aligned}
 33. \quad e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \cdots \\
 e^{-ix} &= 1 - ix + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \cdots = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots \\
 e^{ix} - e^{-ix} &= 2ix - \frac{2ix^3}{3!} + \frac{2ix^5}{5!} - \frac{2ix^7}{7!} + \cdots \\
 \frac{e^{ix} - e^{-ix}}{2i} &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)
 \end{aligned}$$

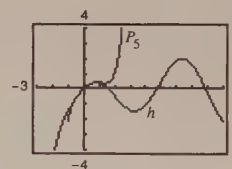
$$35. f(x) = e^x \sin x$$

$$\begin{aligned}
 &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\
 &= x + x^2 + \left(\frac{x^3}{2} - \frac{x^3}{6}\right) + \left(\frac{x^4}{6} - \frac{x^4}{6}\right) + \left(\frac{x^5}{120} - \frac{x^5}{12} + \frac{x^5}{24}\right) + \cdots \\
 &= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots
 \end{aligned}$$



$$37. h(x) = \cos x \ln(1+x)$$

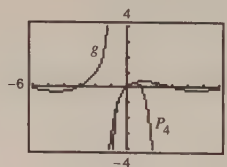
$$\begin{aligned}
 &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) \\
 &= x - \frac{x^2}{2} + \left(\frac{x^3}{3} - \frac{x^3}{2}\right) + \left(\frac{x^4}{4} - \frac{x^4}{4}\right) + \left(\frac{x^5}{5} - \frac{x^5}{6} + \frac{x^5}{24}\right) + \cdots \\
 &= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{3x^5}{40} + \cdots
 \end{aligned}$$



$$39. g(x) = \frac{\sin x}{1+x}. \text{ Divide the series for } \sin x \text{ by } (1+x).$$

$$\begin{array}{r}
 x - x^2 + \frac{5x^2}{6} - \frac{5x^4}{6} + \\
 1+x \overline{) x + 0x^2 - \frac{x^3}{6} + 0x^4 + \frac{x^5}{120} + \cdots} \\
 \underline{x + x^2} \\
 -x^2 - \frac{x^3}{6} \\
 \underline{-x^2 - \frac{x^3}{6}} \\
 \frac{5x^3}{6} + 0x^4 \\
 \underline{\frac{5x^3}{6} + \frac{5x^4}{6}} \\
 -\frac{5x^4}{6} + \frac{x^5}{120} \\
 \underline{-\frac{5x^4}{6} - \frac{5x^5}{6}} \\
 \vdots
 \end{array}$$

$$g(x) = x - x^2 + \frac{5x^3}{6} - \frac{5x^4}{6} + \cdots$$



$$41. y = x^2 - \frac{x^4}{3!} = x \left(x - \frac{x^3}{3!} \right) \approx x \sin x.$$

Matches (a)

$$43. y = x + x^2 + \frac{x^3}{2!} = x \left(1 + x + \frac{x^2}{2!} \right) \approx x e^x.$$

Matches (c)

$$\begin{aligned}
 45. \int_0^x (e^{-t^2} - 1) dt &= \int_0^x \left[\left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \right) - 1 \right] dt \\
 &= \int_0^x \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+2}}{(n+1)!} \right] dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+3}}{(2n+3)(n+1)!} \right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)(n+1)!}
 \end{aligned}$$

$$47. \text{ Since } \ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots \quad (0 < x \leq 2)$$

$$\text{we have } \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \approx 0.6931. \quad (10,001 \text{ terms})$$

$$49. \text{ Since } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,$$

$$\text{we have } e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{2^n}{n!} \approx 7.3891. \quad (12 \text{ terms})$$

51. Since

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!}$$

$$\frac{1 - \cos x}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!}$$

$$\text{we have } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+2)!} = 0.$$

$$53. \int_0^1 \frac{\sin x}{x} dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$$

Since $1/(7 \cdot 7!) < 0.0001$, we need three terms:

$$\int_0^1 \frac{\sin x}{x} dx = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \dots \approx 0.9461. \quad (\text{Using three non-zero terms})$$

Note: We are using $\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$.

$$55. \int_0^{\pi/2} \sqrt{x} \cos x dx = \int_0^{\pi/2} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+1)/2}}{(2n)!} \right] dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{(4n+3)/2}}{\left(\frac{4n+3}{2}\right)(2n)!} \right]_0^{\pi/2} = \left[\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{(4n+3)/2}}{(4n+3)(2n)!} \right]_0^{\pi/2}$$

Since $2(\pi/2)^{23/2}/23 \cdot 10! < 0.0001$, we need five terms

$$\int_0^1 \sqrt{x} \cos x dx = 2 \left[\frac{(\pi/2)^{3/2}}{3} - \frac{(\pi/2)^{7/2}}{14} + \frac{(\pi/2)^{11/2}}{264} - \frac{(\pi/2)^{15/2}}{10,800} + \frac{(\pi/2)^{19/2}}{766,080} \right] \approx 0.7040.$$

$$57. \int_{0.1}^{0.3} \sqrt{1+x^3} dx = \int_{0.1}^{0.3} \left(1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} - \frac{5x^{12}}{128} + \dots \right) dx = \left[x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} - \frac{5x^{13}}{1664} + \dots \right]_{0.1}^{0.3}$$

Since $\frac{1}{56}(0.3^7 - 0.1^7) < 0.0001$, we need two terms

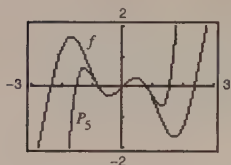
$$\int_{0.1}^{0.3} \sqrt{1+x^3} dx = \left[(0.3 - 0.1) + \frac{1}{8}(0.3^4 - 0.1^4) \right] \approx 0.201.$$

59. From Exercise 19, we have

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_0^1 e^{-x^2/2} dx &= \frac{1}{\sqrt{2\pi}} \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!} dx = \frac{1}{\sqrt{2\pi}} \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n! (2n+1)} \right]_0^1 = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n! (2n+1)} \\ &\approx \frac{1}{\sqrt{2\pi}} \left[1 - \frac{1}{2 \cdot 1 \cdot 3} + \frac{1}{2^2 \cdot 2! \cdot 5} - \frac{1}{2^3 \cdot 3! \cdot 7} \right] \approx 0.3414.\end{aligned}$$

61. $f(x) = x \cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{(2n)!}$

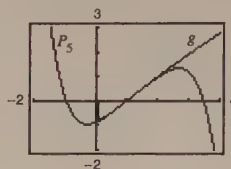
$$P_5(x) = x - 2x^3 + \frac{2x^5}{3}$$



The polynomial is a reasonable approximation on the interval $[-\frac{3}{4}, \frac{3}{4}]$.

63. $f(x) = \sqrt{x} \ln x, c = 1$

$$P_5(x) = (x-1) - \frac{(x-1)^3}{24} + \frac{(x-1)^4}{24} - \frac{71(x-1)^5}{1920}$$



The polynomial is a reasonable approximation on the interval $[\frac{1}{4}, 2]$.

65. See Guidelines, page 636.

67. (a) Replace x with $(-x)$.

(b) Replace x with $3x$.

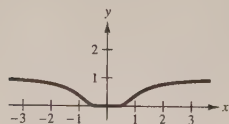
(c) Multiply series by x .

(d) Replace x with $2x$, then replace x with $-2x$, and add the two together.

$$\begin{aligned}69. y &= \left(\tan \theta - \frac{g}{kv_0 \cos \theta} \right) x - \frac{g}{k^2} \ln \left(1 - \frac{kx}{v_0 \cos \theta} \right) \\ &= (\tan \theta)x - \frac{gx}{kv_0 \cos \theta} - \frac{g}{k^2} \left[-\frac{kx}{v_0 \cos \theta} - \frac{1}{2} \left(\frac{kx}{v_0 \cos \theta} \right)^2 - \frac{1}{3} \left(\frac{kx}{v_0 \cos \theta} \right)^3 - \frac{1}{4} \left(\frac{kx}{v_0 \cos \theta} \right)^4 - \dots \right] \\ &= (\tan \theta)x - \frac{gx}{kv_0 \cos \theta} + \frac{gx}{kv_0 \cos \theta} + \frac{gx^2}{2v_0^2 \cos^2 \theta} + \frac{gkx^3}{3v_0^3 \cos^3 \theta} + \frac{gk^2x^4}{4v_0^4 \cos^4 \theta} + \dots \\ &= (\tan \theta)x + \frac{gx^2}{2v_0^2 \cos^2 \theta} + \frac{gkx^3}{3v_0^3 \cos^3 \theta} + \frac{k^2gx^4}{4v_0^4 \cos^4 \theta} + \dots\end{aligned}$$

71. $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(a)



(b) $f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-1/x^2} - 0}{x}$

Let $y = \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x}$. Then

$$\ln y = \lim_{x \rightarrow 0} \ln \left(\frac{e^{-1/x^2}}{x} \right) = \lim_{x \rightarrow 0^+} \left[-\frac{1}{x^2} - \ln x \right] = \lim_{x \rightarrow 0^+} \left[\frac{-1 - x^2 \ln x}{x^2} \right] = -\infty.$$

Thus, $y = e^{-\infty} = 0$ and we have $f'(0) = 0$.

(c) $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \dots = 0 \neq f(x)$

This series converges to f at $x = 0$ only.

73. By the Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0$ which shows that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x .

Review Exercises for Chapter 8

1. $a_n = \frac{1}{n!}$

3. $a_n = 4 + \frac{2}{n}$: 6, 5, 4.67, ...

Matches (a)

5. $a_n = 10(0.3)^{n-1}$: 10, 3, ...

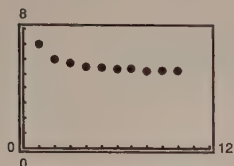
Matches (d)

7. $a_n = \frac{5n+2}{n}$

9. $\lim_{n \rightarrow \infty} \frac{n+1}{n^2} = 0$

Converges

11. $\lim_{n \rightarrow \infty} \frac{n^3}{n^2+1} = \infty$



The sequence seems to converge to 5.

$$\begin{aligned}\lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{5n+2}{n} \\ &= \lim_{n \rightarrow \infty} \left(5 + \frac{2}{n}\right) = 5\end{aligned}$$

$$13. \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0 \quad \text{Converges}$$

15. $\lim_{n \rightarrow \infty} \frac{\sin \sqrt{n}}{\sqrt{n}} = 0$

Converges

$$17. A_n = 5000 \left(1 + \frac{0.05}{4}\right)^n = 5000(1.0125)^n$$

$$n = 1, 2, 3$$

(a) $A_1 \approx 5062.50 \quad A_5 \approx 5320.41$

$A_2 \approx 5125.78 \quad A_6 \approx 5386.92$

$A_3 \approx 5189.85 \quad A_7 \approx 5454.25$

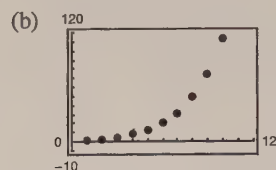
$A_4 \approx 5254.73 \quad A_8 \approx 5522.43$

(b) $A_{40} \approx 8218.10$

19. (a)

k	5	10	15	20	25
S_k	13.2	113.3	873.8	6448.5	50,500.3

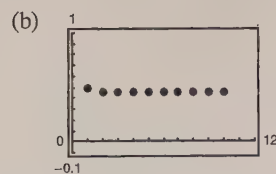
The series diverges (geometric $r = \frac{3}{2} > 1$)



21. (a)

k	5	10	15	20	25
S_k	0.4597	0.4597	0.4597	0.4597	0.4597

The series converges by the Alternating Series Test.



23. Converges. Geometric series, $r = 0.82$, $|r| < 1$.

25. Diverges. n th Term Test. $\lim_{n \rightarrow \infty} a_n \neq 0$.

$$27. \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$$

Geometric series with $a = 1$ and $r = \frac{2}{3}$.

$$S = \frac{a}{1-r} = \frac{1}{1-(2/3)} = \frac{1}{1/3} = 3$$

$$31. 0.\overline{09} = 0.09 + 0.0009 + 0.000009 + \cdots = 0.09(1 + 0.01 + 0.0001 + \cdots) = \sum_{n=0}^{\infty} (0.09)(0.01)^n = \frac{0.09}{1-0.01} = \frac{1}{11}$$

$$33. D_1 = 8$$

$$D_2 = 0.7(8) + 0.7(8) = 16(0.7)$$

\vdots

$$D = 8 + 16(0.7) + 16(0.7)^2 + \cdots + 16(0.7)^n + \cdots$$

$$= -8 + \sum_{n=0}^{\infty} 16(0.7)^n = -8 + \frac{16}{1-0.7} = 45\frac{1}{3} \text{ meters}$$

$$37. \int_1^{\infty} x^{-4} \ln(x) dx = \lim_{b \rightarrow \infty} \left[-\frac{\ln x}{3x^3} - \frac{1}{9x^3} \right]_1^b$$

$$= 0 + \frac{1}{9} = \frac{1}{9}$$

By the Integral Test, the series converges.

$$41. \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2n}}$$

$$\lim_{n \rightarrow \infty} \frac{1/\sqrt{n^3+2n}}{1/(n^{3/2})} = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{\sqrt{n^3+2n}} = 1$$

By a limit comparison test with the convergent p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}, \text{ the series converges.}$$

45. Converges by the Alternating Series Test (Conditional convergence)

$$49. \sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{e^{n^2}(n+1)}{e^{n^2+2n+1}n} \right|$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{e^{2n+1}} \right) \left(\frac{n+1}{n} \right)$$

$$= (0)(1) = 0 < 1$$

By the Ratio Test, the series converges.

$$29. \sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n} \right) = \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^n$$

$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/3)} = 2 - \frac{3}{2} = \frac{1}{2}$$

35. See Exercise 86 in Section 8.2.

$$A = \frac{P(e^r - 1)}{e^{r/12} - 1}$$

$$= \frac{200(e^{(0.06)(2)} - 1)}{e^{0.06/12} - 1}$$

$$\approx \$5087.14$$

$$39. \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n}$$

Since the second series is a divergent p -series while the first series is a convergent p -series, the difference diverges.

$$43. \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

$$= \left(\frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n-1}{2n-2} \right) \frac{1}{2n} > \frac{1}{2n}$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), so does the original series.

47. Diverges by the n th Term Test

$$51. \sum_{n=1}^{\infty} \frac{2^n}{n^3}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{2n^3}{(n+1)^3} = 2$$

Therefore, by the Ratio Test, the series diverges.

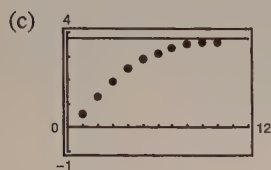
53. (a) Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)(3/5)^{n+1}}{n(3/5)^n}$

$$= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right) \left(\frac{3}{5} \right) = \frac{3}{5} < 1$$

Converges

(b)

x	5	10	15	20	25
S_n	2.8752	3.6366	3.7377	3.7488	3.7499



(d) The sum is approximately 3.75.

55. (a) $\int_N^\infty \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_N^\infty = \frac{1}{N}$

N	5	10	20	30	40
$\sum_{n=1}^N \frac{1}{n^2}$	1.4636	1.5498	1.5962	1.6122	1.6202
$\int_N^\infty \frac{1}{x^2} dx$	0.2000	0.1000	0.0500	0.0333	0.0250

(b) $\int_N^\infty \frac{1}{x^5} dx = \left[-\frac{1}{4x^4} \right]_N^\infty = \frac{1}{4N^4}$

N	5	10	20	30	40
$\sum_{n=1}^N \frac{1}{n^5}$	1.0367	1.0369	1.0369	1.0369	1.0369
$\int_N^\infty \frac{1}{x^5} dx$	0.0004	0.0000	0.0000	0.0000	0.0000

The series in part (b) converges more rapidly. The integral values represent the remainders of the partial sums.

57. $f(x) = e^{-x/2} \quad f(0) = 1$

$$f'(x) = -\frac{1}{2}e^{-x/2} \quad f'(0) = -\frac{1}{2}$$

$$f''(x) = \frac{1}{4}e^{-x/2} \quad f''(0) = \frac{1}{4}$$

$$f'''(x) = -\frac{1}{8}e^{-x/2} \quad f'''(0) = -\frac{1}{8}$$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$$

$$= 1 - \frac{1}{2}x + \frac{1}{4} \frac{x^2}{2!} - \frac{1}{8} \frac{x^3}{3!}$$

$$= 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3$$

59. Since $\frac{(95\pi)^9}{180^9 \cdot 9!} < 0.001$, use four terms $\sin(95^\circ) = \sin\left(\frac{95\pi}{180}\right) \approx \frac{95\pi}{180} - \frac{(95\pi)^3}{180^3 3!} + \frac{(95\pi)^5}{180^5 5!} - \frac{(95\pi)^7}{180^7 7!} \approx 0.99594$

61. $\ln(1.75) \approx (0.75) - \frac{(0.75)^2}{2} + \frac{(0.75)^3}{3} - \frac{(0.75)^4}{4} + \frac{(0.75)^5}{5} - \frac{(0.75)^6}{6} + \dots - \frac{(0.75)^{14}}{14} \approx 0.559062$

63. $f(x) = \cos x, c = 0$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$

$$|f^{(n+1)}(z)| \leq 1 \Rightarrow R_n(x) \leq \frac{x^{n+1}}{(n+1)!}$$

(a) $R_n(x) \leq \frac{(0.5)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for $n = 4$.

(c) $R_n(x) \leq \frac{(0.5)^{n+1}}{(n+1)!} < 0.0001$

This inequality is true for $n = 5$.

(b) $R_n(x) \leq \frac{(1)^{n+1}}{(n+1)!} < 0.001$

This inequality is true for $n = 6$.

(d) $R_n(x) \leq \frac{2^{n+1}}{(n+1)!} < 0.0001$

This inequality is true for $n = 10$.

65. $\sum_{n=0}^{\infty} \left(\frac{x}{10}\right)^n$

Geometric series which converges only if $|x/10| < 1$ or $-10 < x < 10$.

67. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{(n+1)^2}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (x-2)^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{(-1)^n (x-2)^n} \right|$$

$$= |x-2|$$

$R = 1$

Center: 2

Since the series converges when $x = 1$ and when $x = 3$, the interval of convergence is $1 \leq x \leq 3$.

69. $\sum_{n=0}^{\infty} n! (x-2)^n$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! (x-2)^{n+1}}{n! (x-2)^n} \right| = \infty$$

which implies that the series converges only at the center $x = 2$.

71.

$$y = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n (n!)^2}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{4^n (n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2) x^{2n+1}}{4^{n+1} [(n+1)!]^2}$$

$$y'' = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2)(2n+1) x^{2n}}{4^{n+1} [(n+1)!]^2}$$

$$\begin{aligned} x^2 y'' + xy' + x^2 y &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2)(2n+1) x^{2n+2}}{4^{n+1} [(n+1)!]^2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n+2) x^{2n+2}}{4^{n+1} [(n+1)!]^2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{4^n (n!)^2} \\ &= \sum_{n=0}^{\infty} \left[(-1)^{n+1} \frac{(2n+2)(2n+1)}{4^{n+1} [(n+1)!]^2} + \frac{(-1)^{n+1} (2n+2)}{4^{n+1} [(n+1)!]^2} + \frac{(-1)^n}{4^n (n!)^2} \right] x^{2n+2} \\ &= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1} (2n+2)(2n+1+1)}{4^{n+1} [(n+1)!]^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2} \\ &= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1} 4(n+1)^2}{4^{n+1} [(n+1)!]^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2} \\ &= \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1} 1}{4^n (n!)^2} + (-1)^n \frac{1}{4^n (n!)^2} \right] x^{2n+2} = 0 \end{aligned}$$

73. $\frac{2}{3-x} = \frac{2/3}{1-(x/3)} = \frac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{2}{3} \left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2x^n}{3^{n+1}}$$

75. Derivative: $\sum_{n=1}^{\infty} \frac{2nx^{n-1}}{3^{n+1}}$

$$77. 1 + \frac{2}{3}x + \frac{4}{9}x^2 + \frac{8}{27}x^3 + \cdots = \sum_{n=0}^{\infty} \left(\frac{2x}{3}\right)^n = \frac{1}{1 - (2x/3)} = \frac{3}{3 - 2x}, \quad -\frac{3}{2} < x < \frac{3}{2}$$

$$79. f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

$$f'''(x) = -\cos(x), \dots$$

$$\begin{aligned} \sin(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)[x - (3\pi/4)]^n}{n!} \\ &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{3\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!}\left(x - \frac{3\pi}{4}\right)^2 + \cdots = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}[x - (3\pi/4)]^n}{n!} \end{aligned}$$

$$81. 3^x = (e^{\ln(3)})^x = e^{x \ln(3)} \text{ and since } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ we have}$$

$$\begin{aligned} 3^x &= \sum_{n=0}^{\infty} \frac{(x \ln 3)^n}{n!} \\ &= 1 + x \ln 3 + \frac{x^2 \ln^2 3}{2!} + \frac{x^3 \ln^3 3}{3!} + \frac{x^4 \ln^4 3}{4!} + \cdots \end{aligned}$$

$$83. f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

$$f'''(x) = -\frac{6}{x^4}, \dots$$

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)(x+1)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{-n!(x+1)^n}{n!} = -\sum_{n=0}^{\infty} (x+1)^n, \quad -2 < x < 0$$

$$85. (1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$$

$$(1+x)^{1/5} = 1 + \frac{x}{5} + \frac{(1/5)(-4/5)x^2}{2!} + \frac{1/5(-4/5)(-9/5)x^3}{3!} + \cdots$$

$$= 1 + \frac{1}{5}x - \frac{1 \cdot 4x^2}{5^2 2!} + \frac{1 \cdot 4 \cdot 9x^3}{5^3 3!} - \cdots$$

$$= 1 + \frac{x}{5} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 4 \cdot 9 \cdot 14 \cdots (5n-6)x^n}{5^n n!}$$

$$= 1 + \frac{x}{5} - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \cdots$$

$$87. \ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \quad 0 < x \leq 2$$

$$\ln\left(\frac{5}{4}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{(5/4)-1}{n}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{4^n n} \approx 0.2231$$

$$89. e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

$$e^{1/2} = \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \approx 1.6487$$

$$91. \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$$

$$\cos\left(\frac{2}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{3^{2n}(2n)!} \approx 0.7859$$

$$95. (a) f(x) = e^{2x} \quad f(0) = 1$$

$$f'(x) = 2e^{2x} \quad f'(0) = 2$$

$$f''(x) = 4e^{2x} \quad f''(0) = 4$$

$$f'''(x) = 8e^{2x} \quad f'''(0) = 8$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots$$

$$(c) e^{2x} = e^x \cdot e^x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right) \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)$$

$$= 1 + (x + x) + \left(x^2 + \frac{x^2}{2} + \frac{x^2}{2}\right) + \left(\frac{x^3}{6} + \frac{x^3}{6} + \frac{x^3}{2} + \frac{x^3}{2}\right) + \dots = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots$$

$$97. \sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

$$\frac{\sin t}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$$

$$\int_0^x \frac{\sin t}{t} dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!} \right]_0^x$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

$$101. \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$$

$$\frac{\arctan x}{\sqrt{x}} = \sqrt{x} - \frac{x^{5/2}}{3} + \frac{x^{9/2}}{5} - \frac{x^{13/2}}{7} + \frac{x^{17/2}}{9} - \dots$$

$$\lim_{x \rightarrow 0^+} \frac{\arctan x}{\sqrt{x}} = 0$$

$$\text{By L'Hôpital's Rule, } \lim_{x \rightarrow 0^+} \frac{\arctan x}{\sqrt{x}} = \lim_{x \rightarrow 0^+} \frac{\left(\frac{1}{1+x^2}\right)}{\left(\frac{1}{2\sqrt{x}}\right)} = \lim_{x \rightarrow 0^+} \frac{2\sqrt{x}}{1+x^2} = 0.$$

93. The series for Exercise 41 converges very slowly because the terms approach 0 at a slow rate.

$$(b) e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \dots$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots$$

$$99. \frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$$

$$\ln(1+t) = \int \frac{1}{1+t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$$

$$\frac{\ln(t+1)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n+1}$$

$$\int_0^x \frac{\ln(t+1)}{t} dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{(n+1)^2} \right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2}$$

Problem Solving for Chapter 8

$$1. (a) 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \dots = \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^n = \frac{1/3}{1 - (2/3)} = 1$$

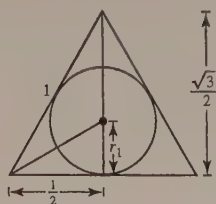
$$(b) 0, \frac{1}{3}, \frac{2}{3}, 1, \text{ etc.}$$

$$(c) \lim_{n \rightarrow \infty} C_n = 1 - \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^n = 1 - 1 = 0$$

3. If there are n rows, then $a_n = \frac{n(n+1)}{2}$.

For one circle,

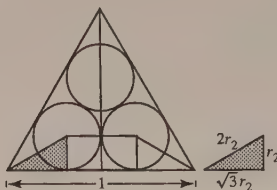
$$a_1 = 1 \text{ and } r_1 = \frac{1}{3} \left(\frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}$$



For three circles,

$$a_2 = 3 \text{ and } 1 = 2\sqrt{3}r_2 + 2r_2$$

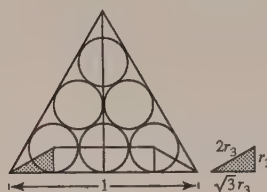
$$r_2 = \frac{1}{2 + 2\sqrt{3}}$$



For six circles,

$$a_3 = 6 \text{ and } 1 = 2\sqrt{3}r_3 + 4r_3$$

$$r_3 = \frac{1}{2\sqrt{3} + 4}$$



Continuing this pattern, $r_n = \frac{1}{2\sqrt{3} + 2(n-1)}$.

$$\text{Total Area} = (\pi r_n^2) a_n = \pi \left(\frac{1}{2\sqrt{3} + 2(n-1)} \right)^2 \frac{n(n+1)}{2}$$

$$A_n = \frac{\pi}{2} \frac{n(n+1)}{[2\sqrt{3} + 2(n+1)]^2}$$

$$\lim_{n \rightarrow \infty} A_n = \frac{\pi}{2} \cdot \frac{1}{4} = \frac{\pi}{8}$$

5. (a) $\sum a_n x^n = 1 + 2x + 3x^2 + x^3 + 2x^4 + 3x^5 + \dots$

$$= (1 + x^3 + x^6 + \dots) + 2(x + x^4 + x^7 + \dots) + 3(x^2 + x^5 + x^8 + \dots)$$

$$= (1 + x^3 + x^6 + \dots)[1 + 2x + 3x^2]$$

$$= (1 + 2x + 3x^2) \frac{1}{1 - x^3}$$

$R = 1$ because each series in the second line has $R = 1$.

$$(b) \sum a_n x^n = (a_0 + a_1 x + \dots + a_{p-1} x^{p-1}) + (a_0 x^p + a_1 x^{p+1} + \dots) + \dots$$

$$= a_0(1 + x^p + \dots) + a_1 x(1 + x^p + \dots) + \dots + a_{p-1} x^{p-1}(1 + x^p + \dots)$$

$$= (a_0 + a_1 x + \dots + a_{p-1} x^{p-1})(1 + x^p + \dots)$$

$$= (a_0 + a_1 x + \dots + a_{p-1} x^{p-1}) \frac{1}{1 - x^p}$$

$$R = 1$$

(Assume all $a_n > 0$.)

$$7. \quad e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$xe^x = x + x^2 + \frac{x^3}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

$$\int xe^x dx = xe^x - e^x + C = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}$$

Letting $x = 0$, $C = 1$. Letting $x = 1$,

$$1 = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+2)n!}$$

$$\text{Thus, } \sum_{n=1}^{\infty} \frac{1}{(n+2)n!} = \frac{1}{2}.$$

$$9. \text{ Let } a_1 = \int_0^{\pi} \frac{\sin x}{x} dx, a_2 = -\int_{\pi}^{2\pi} \frac{\sin x}{x} dx, a_3 = \int_{2\pi}^{3\pi} \frac{\sin x}{x} dx, \text{ etc.}$$

Then,

$$\int_0^{\infty} \frac{\sin x}{x} dx = a_1 - a_2 + a_3 - a_4 + \cdots$$

Since $\lim_{n \rightarrow \infty} a_n = 0$ and $a_{n+1} < a_n$, this series converges.

$$11. (a) \quad a_1 = 3.0$$

$$a_2 \approx 1.73205$$

$$a_3 \approx 2.17533$$

$$a_4 \approx 2.27493$$

$$a_5 \approx 2.29672$$

$$a_6 \approx 2.30146$$

$$\lim_{n \rightarrow \infty} a_n = \frac{1 + \sqrt{13}}{2} \text{ [See part (b) for proof.]}$$

$$(b) \text{ Use mathematical induction to show the sequence is increasing. Clearly, } a_2 = \sqrt{a + a_1} = \sqrt{a\sqrt{a}} > \sqrt{a} = a_1.$$

Now assume $a_n > a_{n-1}$. Then

$$a_n + a > a_{n-1} + a$$

$$\sqrt{a_n + a} > \sqrt{a_{n-1} + a}$$

$$a_{n+1} > a_n.$$

Use mathematical induction to show that the sequence is bounded above by a . Clearly, $a_1 = \sqrt{a} < a$.

Now assume $a_n < a$. Then $a > a_n$ and $a - 1 > 1$ implies

$$a(a - 1) > a_n(1)$$

$$a^2 - a > a_n$$

$$a^2 > a_n + a$$

$$a > \sqrt{a_n + a} = a_{n+1}.$$

Hence, the sequence converges to some number L . To find L , assume $a_{n+1} \approx a_n \approx L$:

$$L = \sqrt{a + L} \Rightarrow L^2 = a + L \Rightarrow L^2 - L - a = 0$$

$$L = \frac{1 \pm \sqrt{1 + 4a}}{2}.$$

$$\text{Hence, } L = \frac{1 + \sqrt{1 + 4a}}{2}.$$

$$13. (a) \sum_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}} = \frac{1}{2^{1-1}} + \frac{1}{2^{2+1}} + \frac{1}{2^{3-1}} + \frac{1}{2^{4+1}} + \frac{1}{2^{5-1}} + \dots$$

$$S_1 = \frac{1}{2^0} = 1$$

$$S_2 = 1 + \frac{1}{8} = \frac{9}{8}$$

$$S_3 = \frac{9}{8} + \frac{1}{4} = \frac{11}{8}$$

$$S_4 = \frac{11}{8} + \frac{1}{32} = \frac{45}{32}$$

$$S_5 = \frac{45}{32} + \frac{1}{16} = \frac{47}{32}$$

$$(b) \frac{a_{n+1}}{a_n} = \frac{2^{n+(-1)^n}}{2^{(n+1)+(-1)^{n+1}}} = \frac{2^{(-1)^n}}{2^{1+(-1)^{n+1}}}$$

This sequence is $\frac{1}{8}, 2, \frac{1}{8}, 2, \dots$ which diverges.

$$(c) \sqrt[n]{\frac{1}{2^{n+(-1)^n}}} = \left(\frac{1}{2^n \cdot 2^{(-1)^n}} \right)^{1/n}$$

$$= \frac{1}{2 \cdot \sqrt[n]{2^{(-1)^n}}} \rightarrow \frac{1}{2} < 1 \text{ converges because } \{2^{(-1)^n}\} = \frac{1}{2}, 2, \frac{1}{2}, 2, \dots \text{ and } \sqrt[n]{1/2} \rightarrow 1 \text{ and } \sqrt[n]{2} \rightarrow 1.$$

$$15. S_6 = 130 + 70 + 40 = 240$$

$$S_7 = 240 + 130 + 70 = 440$$

$$S_8 = 440 + 240 + 130 = 810$$

$$S_9 = 810 + 440 + 240 = 1490$$

$$S_{10} = 1490 + 810 + 440 = 2740$$

C H A P T E R 9

Conics, Parametric Equations, and Polar Coordinates

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CHAPTER 9

Conics, Parametric Equations, and Polar Coordinates

Section 9.1 Conics and Calculus

Solutions to Odd-Numbered Exercises

1. $y^2 = 4x$

Vertex: $(0, 0)$

$p = 1 > 0$

Opens to the right

Matches graph (h).

5. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Center: $(0, 0)$

Ellipse

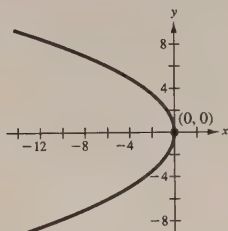
Matches (f)

9. $y^2 = -6x = 4\left(-\frac{3}{2}\right)x$

Vertex: $(0, 0)$

Focus: $\left(-\frac{3}{2}, 0\right)$

Directrix: $x = \frac{3}{2}$



13. $y^2 - 4y - 4x = 0$

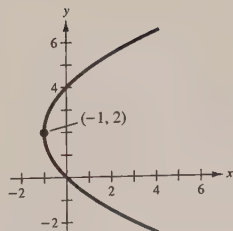
$y^2 - 4y + 4 = 4x + 4$

$(y - 2)^2 = 4(1)(x + 1)$

Vertex: $(-1, 2)$

Focus: $(0, 2)$

Directrix: $x = -2$



3. $(x + 3)^2 = -2(y - 2)$

Vertex: $(-3, 2)$

$p = -\frac{1}{2} < 0$

Opens downward

Matches graph (e).

7. $\frac{y^2}{16} - \frac{x^2}{1} = 1$

Hyperbola

Center: $(0, 0)$

Vertical transverse axis.

Matches (c)

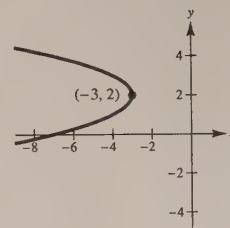
11. $(x + 3) + (y - 2)^2 = 0$

$(y - 2)^2 = 4\left(-\frac{1}{4}\right)(x + 3)$

Vertex: $(-3, 2)$

Focus: $(-3.25, 2)$

Directrix: $x = -2.75$



15. $x^2 + 4x + 4y - 4 = 0$

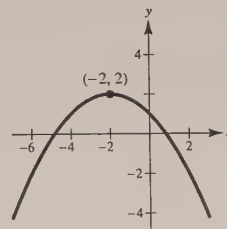
$x^2 + 4x + 4 = -4y + 4 + 4$

$(x + 2)^2 = 4(-1)(y - 2)$

Vertex: $(-2, 2)$

Focus: $(-2, 1)$

Directrix: $y = 3$



17. $y^2 + x + y = 0$

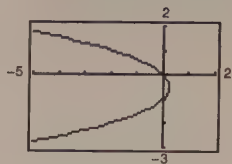
$$y^2 + y + \frac{1}{4} = -x + \frac{1}{4}$$

$$\left(y + \frac{1}{2}\right)^2 = 4\left(-\frac{1}{4}\right)\left(x - \frac{1}{4}\right)$$

Vertex: $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Focus: $\left(0, -\frac{1}{2}\right)$

Directrix: $x = \frac{1}{2}$



19. $y^2 - 4x - 4 = 0$

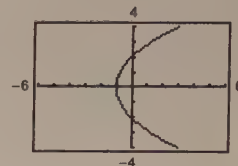
$$y^2 = 4x + 4$$

$$= 4(1)(x + 1)$$

Vertex: $(-1, 0)$

Focus: $(0, 0)$

Directrix: $x = -2$



21. $(y - 2)^2 = 4(-2)(x - 3)$

$$y^2 - 4y + 8x - 20 = 0$$

23. $(x - h)^2 = 4p(y - k)$

$$x^2 = 4(6)(y - 4)$$

$$x^2 - 24y + 96 = 0$$

25. $y = 4 - x^2$

$$x^2 + y - 4 = 0$$

27. Since the axis of the parabola is vertical, the form of the equation is $y = ax^2 + bx + c$. Now, substituting the values of the given coordinates into this equation, we obtain

$$3 = c, 4 = 9a + 3b + c, 11 = 16a + 4b + c.$$

Solving this system, we have $a = \frac{5}{3}$, $b = -\frac{14}{3}$, $c = 3$. Therefore,

$$y = \frac{5}{3}x^2 - \frac{14}{3}x + 3 \text{ or } 5x^2 - 14x - 3y + 9 = 0.$$

29. $x^2 + 4y^2 = 4$

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

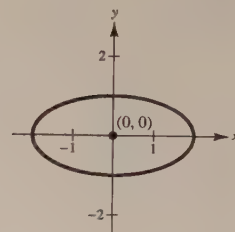
$$a^2 = 4, b^2 = 1, c^2 = 3$$

Center: $(0, 0)$

Foci: $(\pm\sqrt{3}, 0)$

Vertices: $(\pm 2, 0)$

$$e = \frac{\sqrt{3}}{2}$$



31. $\frac{(x - 1)^2}{9} + \frac{(y - 5)^2}{25} = 1$

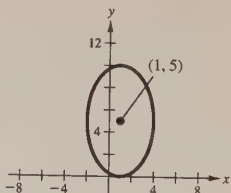
$$a^2 = 25, b^2 = 9, c^2 = 16$$

Center: $(1, 5)$

Foci: $(1, 9), (1, 1)$

Vertices: $(1, 10), (1, 0)$

$$e = \frac{4}{5}$$



33. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

$$9(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -36 + 36 + 36$$

$$= 36$$

$$\frac{(x + 2)^2}{4} + \frac{(y - 3)^2}{9} = 1$$

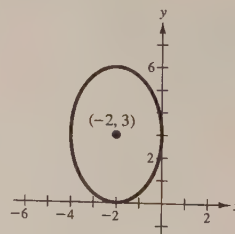
$$a^2 = 9, b^2 = 4, c^2 = 5$$

Center: $(-2, 3)$

Foci: $(-2, 3 \pm \sqrt{5})$

Vertices: $(-2, 6), (-2, 0)$

$$e = \frac{\sqrt{5}}{3}$$



35. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

$$12\left(x^2 - x + \frac{1}{4}\right) + 20(y^2 + 2y + 1) = 37 + 3 + 20$$

$$= 60$$

$$\frac{[x - (1/2)]^2}{5} + \frac{(y + 1)^2}{3} = 1$$

$$a^2 = 5, b^2 = 3, c^2 = 2$$

$$\text{Center: } \left(\frac{1}{2}, -1\right)$$

$$\text{Foci: } \left(\frac{1}{2} \pm \sqrt{2}, -1\right)$$

$$\text{Vertices: } \left(\frac{1}{2} \pm \sqrt{5}, -1\right)$$

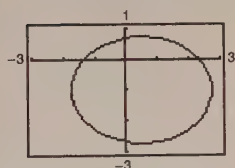
Solve for y:

$$20(y^2 + 2y + 1) = -12x^2 + 12x + 37 + 20$$

$$(y + 1)^2 = \frac{57 + 12x - 12x^2}{20}$$

$$y = -1 \pm \sqrt{\frac{57 + 12x - 12x^2}{20}}$$

(Graph each of these separately.)



39. Center: (0, 0)

Focus: (2, 0)

Vertex: (3, 0)

Horizontal major axis

$$a = 3, c = 2 \Rightarrow b = \sqrt{5}$$

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

43. Center: (0, 0)

Horizontal major axis

Points on ellipse: (3, 1), (4, 0)

Since the major axis is horizontal,

$$\left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) = 1.$$

Substituting the values of the coordinates of the given points into this equation, we have

$$\left(\frac{9}{a^2}\right) + \left(\frac{1}{b^2}\right) = 1, \text{ and } \frac{16}{a^2} = 1.$$

The solution to this system is $a^2 = 16, b^2 = 16/7$.

Therefore,

$$\frac{x^2}{16} + \frac{y^2}{16/7} = 1, \frac{x^2}{16} + \frac{7y^2}{16} = 1.$$

37. $x^2 + 2y^2 - 3x + 4y + 0.25 = 0$

$$\left(x^2 - 3x + \frac{9}{4}\right) + 2(y^2 + 2y + 1) = -\frac{1}{4} + \frac{9}{4} + 2 = 4$$

$$\frac{[x - (3/2)]^2}{4} + \frac{(y + 1)^2}{2} = 1$$

$$a^2 = 4, b^2 = 2, c^2 = 2$$

$$\text{Center: } \left(\frac{3}{2}, -1\right)$$

$$\text{Foci: } \left(\frac{3}{2} \pm \sqrt{2}, -1\right)$$

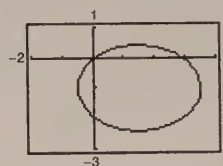
$$\text{Vertices: } \left(-\frac{1}{2}, -1\right), \left(\frac{7}{2}, -1\right)$$

$$\text{Solve for y: } 2(y^2 + 2y + 1) = -x^2 + 3x - \frac{1}{4} + 2$$

$$(y + 1)^2 = \frac{1}{2}\left(\frac{7}{4} + 3x - x^2\right)$$

$$y = -1 \pm \sqrt{\frac{7 + 12x - 4x^2}{8}}$$

(Graph each of these separately.)



41. Vertices: (3, 1), (3, 9)

Minor axis length: 6

Vertical major axis

Center: (3, 5)

$$a = 4, b = 3$$

$$\frac{(x - 3)^2}{9} + \frac{(y - 5)^2}{16} = 1$$

$$45. \frac{y^2}{1} - \frac{x^2}{4} = 1$$

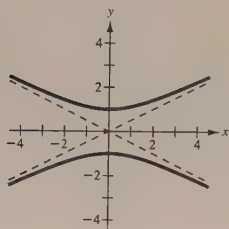
$$a = 1, b = 2, c = \sqrt{5}$$

Center: (0, 0)

Vertices: (0, ± 1)

Foci: (0, $\pm \sqrt{5}$)

Asymptotes: $y = \pm \frac{1}{2}x$



$$47. \frac{(x-1)^2}{4} - \frac{(y+2)^2}{1} = 1$$

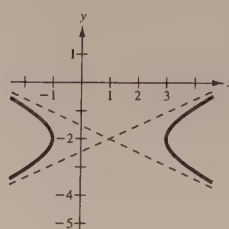
$$a = 2, b = 1, c = \sqrt{5}$$

Center: (1, -2)

Vertices: (-1, -2), (3, -2)

Foci: (1 $\pm \sqrt{5}$, -2)

Asymptotes: $y = -2 \pm \frac{1}{2}(x - 1)$



$$49. 9x^2 - y^2 - 36x - 6y + 18 = 0$$

$$9(x^2 - 4x + 4) - (y^2 + 6y + 9) = -18 + 36 - 9$$

$$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$$

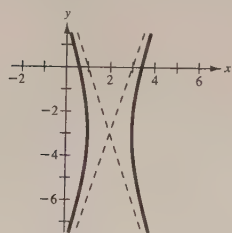
$$a = 1, b = 3, c = \sqrt{10}$$

Center: (2, -3)

Vertices: (1, -3), (3, -3)

Foci: (2 $\pm \sqrt{10}$, -3)

Asymptotes: $y = -3 \pm 3(x - 2)$



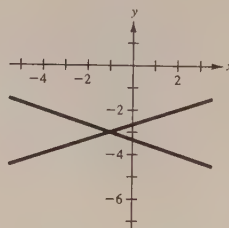
$$51. x^2 - 9y^2 + 2x - 54y - 80 = 0$$

$$(x^2 + 2x + 1) - 9(y^2 + 6y + 9) = 80 + 1 - 81 = 0$$

$$(x+1)^2 - 9(y+3)^2 = 0$$

$$y + 3 = \pm \frac{1}{3}(x + 1)$$

Degenerate hyperbola is two lines intersecting at (-1, -3).



$$53. 9y^2 - x^2 + 2x + 54y + 62 = 0$$

$$9(y^2 + 6y + 9) - (x^2 - 2x + 1) = -62 - 1 + 81 = 18$$

$$\frac{(y+3)^2}{2} - \frac{(x-1)^2}{18} = 1$$

$$a = \sqrt{2}, b = 3\sqrt{2}, c = 2\sqrt{5}$$

Center: (1, -3)

Vertices: (1, -3 $\pm \sqrt{2}$)

Foci: (1, -3 $\pm 2\sqrt{5}$)

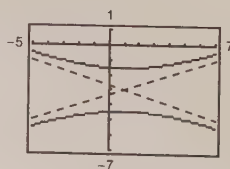
Solve for y:

$$9(y^2 + 6y + 9) = x^2 - 2x - 62 + 81$$

$$(y+3)^2 = \frac{x^2 - 2x + 19}{9}$$

$$y = -3 \pm \frac{1}{3}\sqrt{x^2 - 2x + 19}$$

(Graph each curve separately.)



$$55. 3x^2 - 2y^2 - 6x - 12y - 27 = 0$$

$$3(x^2 - 2x + 1) - 2(y^2 + 6y + 9) = 27 + 3 - 18 = 12$$

$$\frac{(x-1)^2}{4} - \frac{(y+3)^2}{6} = 1$$

$$a = 2, b = \sqrt{6}, c = \sqrt{10}$$

Center: (1, -3)

Vertices: (-1, -3), (3, -3)

Foci: (1 $\pm \sqrt{10}$, -3)

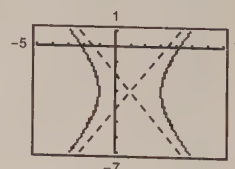
Solve for y:

$$2(y^2 + 6y + 9) = 3x^2 - 6x - 27 + 18$$

$$(y+3)^2 = \frac{3x^2 - 6x - 9}{2}$$

$$y = -3 \pm \sqrt{\frac{3(x^2 - 2x - 3)}{2}}$$

(Graph each curve separately.)



57. Vertices: $(\pm 1, 0)$ Asymptotes: $y = \pm 3x$

Horizontal transverse axis

Center: $(0, 0)$

$$a = 1, \pm \frac{b}{a} = \pm \frac{b}{1} = \pm 3 \Rightarrow b = 3$$

$$\text{Therefore, } \frac{x^2}{1} - \frac{y^2}{9} = 1.$$

61. Center: $(0, 0)$ Vertex: $(0, 2)$ Focus: $(0, 4)$

Vertical transverse axis

$$a = 2, c = 4, b^2 = c^2 - a^2 = 12$$

$$\text{Therefore, } \frac{y^2}{4} - \frac{x^2}{12} = 1.$$

$$65. (a) \frac{x^2}{9} - y^2 = 1, \frac{2x}{9} - 2yy' = 0, \frac{x}{9y} = y'$$

$$\text{At } x = 6: y = \pm\sqrt{3}, y' = \frac{\pm 6}{9\sqrt{3}} = \frac{\pm 2\sqrt{3}}{9}$$

$$\text{At } (6, \sqrt{3}): y - \sqrt{3} = \frac{2\sqrt{3}}{9}(x - 6)$$

$$\text{or } 2x - 3\sqrt{3}y - 3 = 0$$

$$\text{At } (6, -\sqrt{3}): y + \sqrt{3} = \frac{-2\sqrt{3}}{9}(x - 6)$$

$$\text{or } 2x + 3\sqrt{3}y - 3 = 0$$

67.

$$x^2 + 4y^2 - 6x + 16y + 21 = 0$$

$$x^2 - 6x + 4(y^2 + 4y) = -21$$

$$x^2 - 6x + 9 + 4(y^2 + 4y + 4) = -21 + 9 + 16$$

$$(x - 3)^2 + 4(y + 2)^2 = 4$$

$$\frac{(x - 3)^2}{4} + \frac{(y + 2)^2}{1} = 1$$

Ellipse

59. Vertices: $(2, \pm 3)$ Point on graph: $(0, 5)$

Vertical transverse axis

Center: $(2, 0)$

$$a = 3$$

Therefore, the equation is of the form

$$\frac{y^2}{9} - \frac{(x - 2)^2}{b^2} = 1.$$

Substituting the coordinates of the point $(0, 5)$, we have

$$\frac{25}{9} - \frac{4}{b^2} = 1 \quad \text{or} \quad b^2 = \frac{9}{4}.$$

$$\text{Therefore, the equation is } \frac{y^2}{9} - \frac{(x - 2)^2}{9/4} = 1.$$

63. Vertices: $(0, 2), (6, 2)$

$$\text{Asymptotes: } y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$$

Horizontal transverse axis

Center: $(3, 2)$

$$a = 3$$

$$\text{Slopes of asymptotes: } \pm \frac{b}{a} = \pm \frac{2}{3}$$

Thus, $b = 2$. Therefore,

$$\frac{(x - 3)^2}{9} - \frac{(y - 2)^2}{4} = 1.$$

(b) From part (a) we know that the slopes of the normal lines must be $\mp 9/(2\sqrt{3})$.

$$\text{At } (6, \sqrt{3}): y - \sqrt{3} = -\frac{9}{2\sqrt{3}}(x - 6)$$

$$\text{or } 9x + 2\sqrt{3}y - 60 = 0$$

$$\text{At } (6, -\sqrt{3}): y + \sqrt{3} = \frac{9}{2\sqrt{3}}(x - 6)$$

$$\text{or } 9x - 2\sqrt{3}y - 60 = 0$$

69.

$$y^2 - 4y - 4x = 0$$

$$y^2 - 4y + 4 - 4x = 4$$

$$(y - 2)^2 - 4x = 4$$

$$(y - 2)^2 = 4x + 4$$

$$(y - 2)^2 = 4(x + 1)$$

Parabola

71.

$$4x^2 + 4y^2 - 16y + 15 = 0$$

$$4x^2 + 4(y^2 - 4y) = -15$$

$$x^2 + (y^2 - 4y) = -\frac{15}{4}$$

$$x^2 + y^2 - 4y + 4 = -\frac{15}{4} + 4$$

$$x^2 + (y - 2)^2 = \frac{1}{4}$$

Circle

$$\begin{aligned}
 73. \quad & 9x^2 + 9y^2 - 36x + 6y + 34 = 0 \\
 & 9x^2 - 36x + 9y^2 + 6y = -34 \\
 & 9(x^2 - 4x + 4) + 9\left(y^2 + \frac{2}{3}y + \frac{1}{9}\right) = -34 + 36 + 1 \\
 & 9(x-2)^2 = 9\left(y + \frac{1}{3}\right)^2 = 3 \\
 & (x-2)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{1}{3}
 \end{aligned}$$

Circle

77. (a) A parabola is the set of all points (x, y) that are equidistant from a fixed line (directrix) and a fixed point (focus) not on the line.

(b) $(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$

(c) See Theorem 9.2.

81. Assume that the vertex is at the origin.

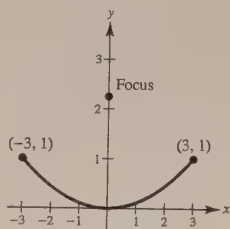
$$x^2 = 4py$$

$$(3)^2 = 4p(1)$$

$$\frac{9}{4} = p$$

The pipe is located

$\frac{9}{4}$ meters from the vertex.



$$\begin{aligned}
 75. \quad & 3(x-1)^2 = 6 + 2(y+1)^2 \\
 & 3(x-1)^2 - (y+1)^2 = 6 \\
 & \frac{(x-1)^2}{2} - \frac{(y+1)^2}{3} = 1
 \end{aligned}$$

Hyperbola

79. (a) A hyperbola is the set of all points (x, y) for which the absolute value of the difference between the distances from two distance fixed points (foci) is constant.

(b) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

(c) $y = k \pm \frac{b}{a}(x-h)$ or $y = k \pm \frac{a}{b}(x-h)$

83. $y = ax^2$

$$y' = 2ax$$

The equation of the tangent line is

$$y - ax_0^2 = 2ax_0(x - x_0)$$

$$\text{or } y = 2ax_0x - ax_0^2.$$

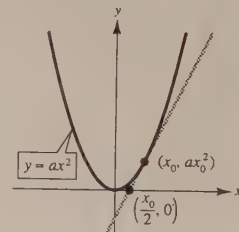
Let $y = 0$. Then:

$$-ax_0^2 = 2ax_0x - 2ax_0^2$$

$$ax_0^2 = 2ax_0x$$

$$x = \frac{x_0}{2}$$

Therefore, $\left(\frac{x_0}{2}, 0\right)$ is the x -intercept.



85. (a) Consider the parabola $x^2 = 4py$. Let m_0 be the slope of the one tangent line at (x_1, y_1) and therefore, $-1/m_0$ is the slope of the second at (x_2, y_2) . Differentiating, $2x = 4py'$ or $y' = \frac{x}{2p}$, and we have:

$$m_0 = \frac{1}{2p}x_1 \text{ or } x_1 = 2pm_0$$

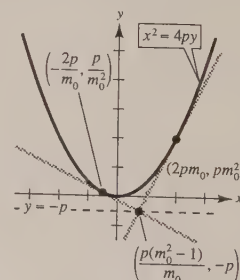
$$\frac{-1}{m_0} = \frac{1}{2p}x_2 \text{ or } x_2 = \frac{-2p}{m_0}$$

Substituting these values of x into the equation $x^2 = 4py$, we have the coordinates of the points of tangency $(2pm_0, pm_0^2)$ and $(-2p/m_0, p/m_0^2)$ and the equations of the tangent lines are

$$(y - pm_0^2) = m_0(x - 2pm_0) \quad \text{and} \quad \left(y - \frac{p}{m_0^2}\right) = \frac{-1}{m_0}\left(x + \frac{2p}{m_0}\right).$$

The point of intersection of these lines is

$$\left(\frac{p(m_0^2 - 1)}{m_0}, -p\right) \text{ and is on the directrix, } y = -p.$$



85. —CONTINUED—

$$(b) x^2 - 4x - 4y + 8 = 0$$

$$(x - 2)^2 = 4(y - 1). \text{ Vertex } (2, 1)$$

$$2x - 4 - 4 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{1}{2}x - 1$$

$$\text{At } (-2, 5), dy/dx = -2. \text{ At } (3, \frac{5}{4}), dy/dx = \frac{1}{2}.$$

$$\text{Tangent line at } (-2, 5): y - 5 = -2(x + 2) \Rightarrow 2x + y - 1 = 0.$$

$$\text{Tangent line at } (3, \frac{5}{4}): y - \frac{5}{4} = \frac{1}{2}(x - 3) \Rightarrow 2x - 4y - 1 = 0.$$

$$\text{Since } m_1 m_2 = (-2)(\frac{1}{2}) = -1, \text{ the lines are perpendicular.}$$

$$\text{Point of intersection: } -2x + 1 = \frac{1}{2}x - \frac{1}{4}$$

$$-\frac{5}{2}x = -\frac{5}{4}$$

$$x = \frac{1}{2}$$

$$y = 0$$

Directrix: $y = 0$ and the point of intersection $(\frac{1}{2}, 0)$ lies on this line.

$$87. y = x - x^2$$

$$\frac{dy}{dx} = 1 - 2x$$

$$\text{At } (x_1, y_1) \text{ on the mountain, } m = 1 - 2x_1. \text{ Also, } m = \frac{y_1 - 1}{x_1 + 1}.$$

$$\frac{y_1 - 1}{x_1 + 1} = 1 - 2x_1$$

$$(x_1 - x_1^2) - 1 = (1 - 2x_1)(x_1 + 1)$$

$$-x_1^2 + x_1 - 1 = -2x_1^2 - x_1 + 1$$

$$x_1^2 + 2x_1 - 2 = 0$$

$$x_1 = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$$

Choosing the positive value for x_1 , we have $x_1 = -1 + \sqrt{3}$.

$$m = 1 - 2(-1 + \sqrt{3}) = 3 - 2\sqrt{3}$$

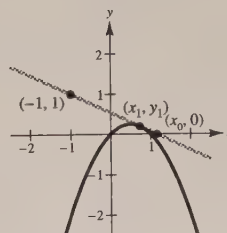
$$m = \frac{0 - 1}{x_0 + 1} = -\frac{1}{x_0 + 1}$$

$$\text{Thus, } -\frac{1}{x_0 + 1} = 3 - 2\sqrt{3}$$

$$\frac{-1}{3 - 2\sqrt{3}} = x_0 + 1$$

$$\frac{3 + 2\sqrt{3}}{3} - 1 = x_0$$

$$\frac{2\sqrt{3}}{3} = x_0.$$



The closest the receiver can be to the hill is $(2\sqrt{3}/3) - 1 \approx 0.155$.

89. Parabola

Vertex: (0, 4)

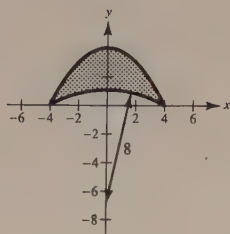
$$x^2 = 4p(y - 4)$$

$$4^2 = 4p(0 - 4)$$

$$p = -1$$

$$x^2 = -4(y - 4)$$

$$y = 4 - \frac{x^2}{4}$$



Circle

Center: (0, k)

Radius: 8

$$x^2 + (y - k)^2 = 64$$

$$4^2 + (0 - k)^2 = 64$$

$$k^2 = 48$$

$$k = -4\sqrt{3} \quad (\text{Center is on the negative } y\text{-axis.})$$

$$x^2 + (y + 4\sqrt{3})^2 = 64$$

$$y = -4\sqrt{3} \pm \sqrt{64 - x^2}$$

 Since the y-value is positive when $x = 0$, we have $y = -4\sqrt{3} + \sqrt{64 - x^2}$.

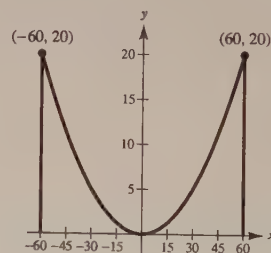
$$\begin{aligned} A &= 2 \int_0^4 \left[\left(4 - \frac{x^2}{4} \right) - (-4\sqrt{3} + \sqrt{64 - x^2}) \right] dx \\ &= 2 \left[4x - \frac{x^3}{12} + 4\sqrt{3}x - \frac{1}{2} \left(x\sqrt{64 - x^2} + 64 \arcsin \frac{x}{8} \right) \right]_0^4 \\ &= 2 \left[16 - \frac{64}{12} + 16\sqrt{3} - 2\sqrt{48} - 32 \arcsin \frac{1}{2} \right] \\ &= \frac{16(4 + 3\sqrt{3} - 2\pi)}{3} \approx 15.536 \text{ square feet} \end{aligned}$$

 91. (a) Assume that $y = ax^2$.

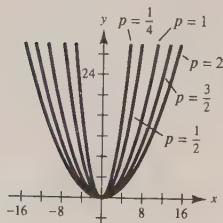
$$20 = a(60)^2 \Rightarrow a = \frac{2}{360} = \frac{1}{180} \Rightarrow y = \frac{1}{180}x^2$$

$$(b) f(x) = \frac{1}{180}x^2, f'(x) = \frac{1}{90}x$$

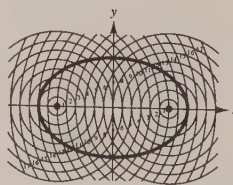
$$\begin{aligned} S &= 2 \int_0^{60} \sqrt{1 + \left(\frac{1}{90}x \right)^2} dx = \frac{2}{90} \int_0^{60} \sqrt{90^2 + x^2} dx \\ &= \frac{2}{90} \frac{1}{2} \left[x\sqrt{90^2 + x^2} + 90^2 \ln |x + \sqrt{90^2 + x^2}| \right]_0^{60} \quad (\text{formula 26}) \\ &= \frac{1}{90} [60\sqrt{11,700} + 90^2 \ln(60 + \sqrt{11,700}) - 90^2 \ln 90] \\ &= \frac{1}{90} [1800\sqrt{13} + 90^2 \ln(60 + 30\sqrt{13}) - 90^2 \ln 90] \\ &= 20\sqrt{13} + 90 \ln \left(\frac{60 + 30\sqrt{13}}{90} \right) \\ &= 10 \left[2\sqrt{13} + 9 \ln \left(\frac{2 + \sqrt{13}}{3} \right) \right] \approx 128.4 \text{ m} \end{aligned}$$



93. $x^2 = 4py, p = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}, 2$

 As p increases, the graph becomes wider.


95.



$$97. a = \frac{5}{2}, b = 2, c = \sqrt{\left(\frac{5}{2}\right)^2 - (2)^2} = \frac{3}{2}$$

The tacks should be placed 1.5 feet from the center. The string should be $2a = 5$ feet long.

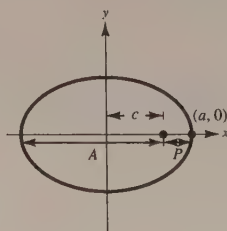
$$99. e = \frac{c}{a}$$

$$A + P = 2a$$

$$a = \frac{A + P}{2}$$

$$c = a - P = \frac{A + P}{2} - P = \frac{A - P}{2}$$

$$e = \frac{c}{a} = \frac{(A - P)/2}{(A + P)/2} = \frac{A - P}{A + P}$$



$$101. e = \frac{A - P}{A + P} = \frac{35.34au - 0.59au}{35.34au + 0.59au} \approx 0.9672$$

$$103. \frac{x^2}{10^2} + \frac{y^2}{5^2} = 1$$

$$\frac{2x}{10^2} + \frac{2yy'}{5^2} = 0$$

$$y' = \frac{-5^2x}{10^2y} = \frac{-x}{4y}$$

$$\text{At } (-8, 3): y' = \frac{8}{12} = \frac{2}{3}$$

The equation of the tangent line is $y - 3 = \frac{2}{3}(x + 8)$. It will cross the y -axis when $x = 0$ and $y = \frac{2}{3}(8) + 3 = \frac{25}{3}$.

$$105. 16x^2 + 9y^2 + 96x + 36y + 36 = 0$$

$$32x + 18yy' + 96 + 36y' = 0$$

$$y'(18y + 36) = -(32x + 96)$$

$$y' = \frac{-(32x + 96)}{18y + 36}$$

$y' = 0$ when $x = -3$. y' is undefined when $y = -2$.

At $x = -3$, $y = 2$ or -6 .

Endpoints of major axis: $(-3, 2)$, $(-3, -6)$

At $y = -2$, $x = 0$ or -6 .

Endpoints of minor axis: $(0, -2)$, $(-6, -2)$

Note: Equation of ellipse is $\frac{(x + 3)^2}{9} + \frac{(y + 2)^2}{16} = 1$

$$107. (a) A = 4 \int_0^2 \frac{1}{2} \sqrt{4 - x^2} dx = \left[x\sqrt{4 - x^2} + 4 \arcsin\left(\frac{x}{2}\right) \right]_0^2 = 2\pi \quad [\text{or, } A = \pi ab = \pi(2)(1) = 2\pi]$$

$$(b) \text{ Disk: } V = 2\pi \int_0^2 \frac{1}{4}(4 - x^2) dx = \frac{1}{2}\pi \left[4x - \frac{1}{3}x^3 \right]_0^2 = \frac{8\pi}{3}$$

$$y = \frac{1}{2}\sqrt{4 - x^2}$$

$$y' = \frac{-x}{2\sqrt{4 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{16 - 4x^2}} = \sqrt{\frac{16 - 3x^2}{4y}}$$

$$S = 2(2\pi) \int_0^2 y \left(\frac{\sqrt{16 - 3x^2}}{4y} \right) dx = \frac{\pi}{2\sqrt{3}} \left[\sqrt{3}x\sqrt{16 - 3x^2} + 16 \arcsin\left(\frac{\sqrt{3}x}{4}\right) \right]_0^2 = \frac{2\pi}{9}(9 + 4\sqrt{3}\pi) \approx 21.48$$

—CONTINUED—

107. —CONTINUED—

(c) Shell: $V = 2\pi \int_0^2 x \sqrt{4 - x^2} dx = -\pi \int_0^2 2x(4 - x^2)^{1/2} dx = -\frac{2\pi}{3} \left[(4 - x^2)^{3/2} \right]_0^2 = \frac{16\pi}{3}$

$$x = 2\sqrt{1 - y^2}$$

$$x' = \frac{-2y}{\sqrt{1 - y^2}}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + \frac{4y^2}{1 - y^2}} = \frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}}$$

$$S = 2(2\pi) \int_0^1 2\sqrt{1 - y^2} \frac{\sqrt{1 + 3y^2}}{\sqrt{1 - y^2}} dy = 8\pi \int_0^1 \sqrt{1 + 3y^2} dy$$

$$= \frac{8\pi}{2\sqrt{3}} \left[\sqrt{3}y \sqrt{1 + 3y^2} + \ln |\sqrt{3}y + \sqrt{1 + 3y^2}| \right]_0^1 = \frac{4\pi}{3} \left[6 + \sqrt{3} \ln(2 + \sqrt{3}) \right] \approx 34.69$$

109. From Example 5,

$$C = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta$$

For $\frac{x^2}{25} + \frac{y^2}{49} = 1$, we have

$$a = 7, b = 5, c = \sqrt{49 - 25} = 2\sqrt{6}, e = \frac{c}{a} = \frac{2\sqrt{6}}{7}$$

$$C = 4(7) \int_0^{\pi/2} \sqrt{1 - \frac{24}{49} \sin^2 \theta} d\theta$$

$$\approx 28(1.3558) \approx 37.9614$$

111. Area circle = $\pi r^2 = 100\pi$ Area ellipse = $\pi ab = \pi a(10)$

$$2(100\pi) = 10\pi a \Rightarrow a = 20$$

Hence, the length of the major axis is $2a = 40$.113. The transverse axis is horizontal since $(2, 2)$ and $(10, 2)$ are the foci (see definition of hyperbola).Center: $(6, 2)$

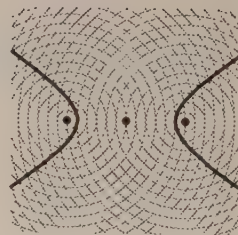
$$c = 4, 2a = 6, b^2 = c^2 - a^2 = 7$$

Therefore, the equation is

$$\frac{(x - 6)^2}{9} - \frac{(y - 2)^2}{7} = 1.$$

115. $2a = 10 \Rightarrow a = 5$

$$c = 6 \Rightarrow b = \sqrt{11}$$

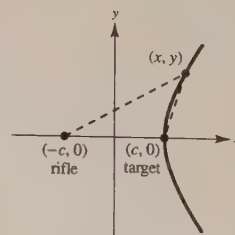
117. Time for sound of bullet hitting target to reach (x, y) : $\frac{2c}{v_m} + \frac{\sqrt{(x - c)^2 + y^2}}{v_s}$ Time for sound of rifle to reach (x, y) : $\frac{\sqrt{(x + c)^2 + y^2}}{v_s}$ Since the times are the same, we have: $\frac{2c}{v_m} + \frac{\sqrt{(x - c)^2 + y^2}}{v_s} = \frac{\sqrt{(x + c)^2 + y^2}}{v_s}$

$$\frac{4c^2}{v_m^2} + \frac{4c}{v_m v_s} \sqrt{(x - c)^2 + y^2} + \frac{(x - c)^2 + y^2}{v_s^2} = \frac{(x + c)^2 + y^2}{v_s^2}$$

$$\sqrt{(x - c)^2 + y^2} = \frac{v_m^2 x - v_s^2 c}{v_s v_m}$$

$$\left(1 - \frac{v_m^2}{v_s^2}\right)x^2 + y^2 = \left(\frac{v_s^2}{v_m^2} - 1\right)c^2$$

$$\frac{x^2}{c^2 v_s^2 / v_m^2} - \frac{y^2}{c^2 (v_m^2 - v_s^2) / v_m^2} = 1$$



119. The point (x, y) lies on the line between $(0, 10)$ and $(10, 0)$. Thus, $y = 10 - x$. The point also lies on the hyperbola $(x^2/36) - (y^2/64) = 1$. Using substitution, we have:

$$\frac{x^2}{36} - \frac{(10-x)^2}{64} = 1$$

$$16x^2 - 9(10-x)^2 = 576$$

$$7x^2 + 180x - 1476 = 0$$

$$x = \frac{-180 \pm \sqrt{180^2 - 4(7)(-1476)}}{2(7)} = \frac{-180 \pm 192\sqrt{2}}{14} = \frac{-90 \pm 96\sqrt{2}}{7}$$

Choosing the positive value for x we have:

$$x = \frac{-90 + 96\sqrt{2}}{7} \approx 6.538 \text{ and } y = \frac{160 - 96\sqrt{2}}{7} \approx 3.462$$

121.

$$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2y^2}{b^2} = 1 - \frac{x^2}{a^2}, \quad c^2 = a^2 - b^2$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2y^2}{b^2} = \frac{x^2}{a^2 - b^2} - 1$$

$$1 - \frac{x^2}{a^2} = \frac{x^2}{a^2 - b^2} - 1 \Rightarrow 2 = x^2 \left(\frac{1}{a^2} + \frac{1}{a^2 - b^2} \right)$$

$$x^2 = \frac{2a^2(a^2 - b^2)}{2a^2 - b^2} \Rightarrow x = \pm \frac{\sqrt{2}a\sqrt{a^2 - b^2}}{\sqrt{2a^2 - b^2}} = \pm \frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}$$

$$\frac{2y^2}{b^2} = 1 - \frac{1}{a^2} \left(\frac{2a^2c^2}{2a^2 - b^2} \right) \Rightarrow \frac{2y^2}{b^2} = \frac{b^2}{2a^2 - b^2}$$

$$y^2 = \frac{b^4}{2(2a^2 - b^2)} \Rightarrow y = \pm \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}$$

There are four points of intersection: $\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right), \left(-\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \pm \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)$

$$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{4yy'}{b^2} = 0 \Rightarrow y'_e = -\frac{b^2x}{2a^2y}$$

$$\frac{x^2}{a^2 - b^2} - \frac{2y^2}{b^2} = 1 \Rightarrow \frac{2x}{c^2} - \frac{4yy'}{b^2} = 0 \Rightarrow y'_h = \frac{b^2x}{2c^2y}$$

At $\left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)$, the slopes of the tangent lines are:

$$y'_e = \frac{-b^2 \left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}} \right)}{2a^2 \left(\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)} = -\frac{c}{a} \quad \text{and} \quad y'_h = \frac{b^2 \left(\frac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}} \right)}{2c^2 \left(\frac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}} \right)} = \frac{a}{c}$$

Since the slopes are negative reciprocals, the tangent lines are perpendicular. Similarly, the curves are perpendicular at the other three points of intersection.

123. False. See the definition of a parabola.

125. True

127. False. $y^2 - x^2 + 2x + 2y = 0$ yields two intersecting lines.

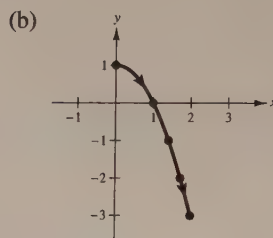
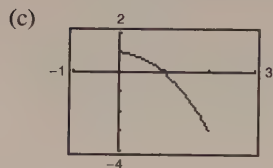
129. True

Section 9.2 Plane Curves and Parametric Equations

1. $x = \sqrt{t}$, $y = 1 - t$

(a)

t	0	1	2	3	4
x	0	1	$\sqrt{2}$	$\sqrt{3}$	2
y	1	0	-1	-2	-3



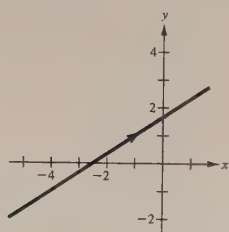
(d) $x^2 = t$
 $y = 1 - x^2, x \geq 0$

3. $x = 3t - 1$

$y = 2t + 1$

$y = 2\left(\frac{x+1}{3}\right) + 1$

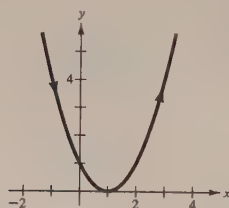
$2x - 3y + 5 = 0$



5. $x = t + 1$

$y = t^2$

$y = (x - 1)^2$

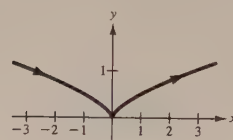


7. $x = t^3$

$y = \frac{1}{2}t^2$

$x = t^3$ implies $t = x^{1/3}$

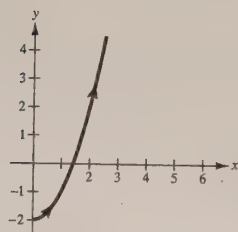
$y = \frac{1}{2}x^{2/3}$



9. $x = \sqrt{t}, t \geq 0$

$y = t - 2$

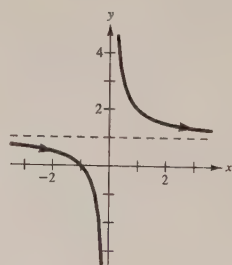
$y = x^2 - 2, x \geq 0$



11. $x = t - 1$

$y = \frac{t}{t-1}$

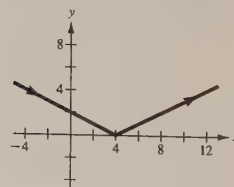
$y = \frac{x+1}{x}$



13. $x = 2t$

$y = |t - 2|$

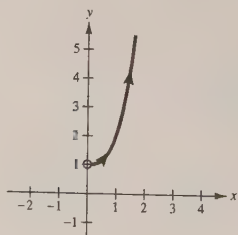
$y = \left|\frac{x}{2} - 2\right| = \frac{|x - 4|}{2}$



15. $x = e^t, x > 0$

$y = e^{3t} + 1$

$y = x^3 + 1, x > 0$



17. $x = \sec \theta$

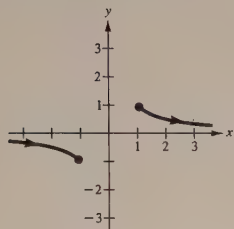
$y = \cos \theta$

$0 \leq \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \leq \pi$

$xy = 1$

$y = \frac{1}{x}$

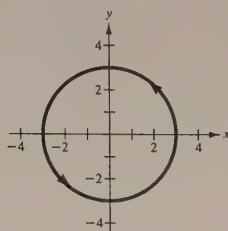
$|x| \geq 1, |y| \leq 1$



19. $x = 3 \cos \theta, y = 3 \sin \theta$

Squaring both equations and adding, we have

$x^2 + y^2 = 9.$



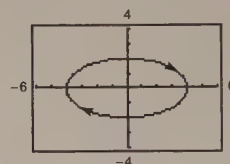
21. $x = 4 \sin 2\theta$

$y = 2 \cos 2\theta$

$\frac{x^2}{16} = \sin^2 2\theta$

$\frac{y^2}{4} = \cos^2 2\theta$

$\frac{x^2}{16} + \frac{y^2}{4} = 1$



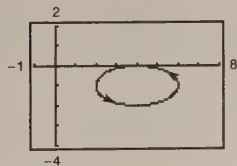
23. $x = 4 + 2 \cos \theta$

$y = -1 + \sin \theta$

$\frac{(x-4)^2}{4} = \cos^2 \theta$

$\frac{(y+1)^2}{1} = \sin^2 \theta$

$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{1} = 1$



25.

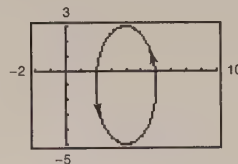
$x = 4 + 2 \cos \theta$

$y = -1 + 4 \sin \theta$

$\frac{(x-4)^2}{4} = \cos^2 \theta$

$\frac{(y+1)^2}{16} = \sin^2 \theta$

$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{16} = 1$



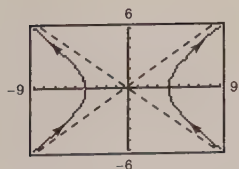
27. $x = 4 \sec \theta$

$y = 3 \tan \theta$

$\frac{x^2}{16} = \sec^2 \theta$

$\frac{y^2}{9} = \tan^2 \theta$

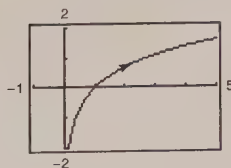
$\frac{x^2}{16} - \frac{y^2}{9} = 1$



29. $x = t^3$

$y = 3 \ln t$

$y = 3 \ln \sqrt[3]{x} = \ln x$



31. $x = e^{-t}$

$y = e^{3t}$

$e^t = \frac{1}{x}$

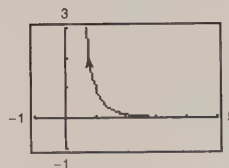
$e^t = \sqrt[3]{y}$

$\sqrt[3]{y} = \frac{1}{x}$

$y = \frac{1}{x^3}$

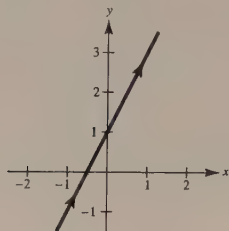
$x > 0$

$y > 0$



33. By eliminating the parameters in (a) – (d), we get $y = 2x + 1$. They differ from each other in orientation and in restricted domains. These curves are all smooth except for (b).

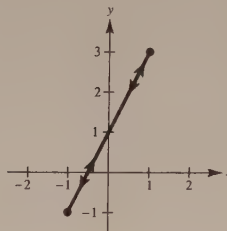
(a) $x = t, y = 2t + 1$



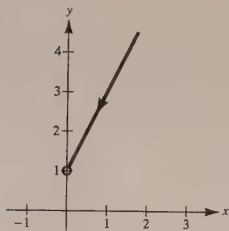
(b) $x = \cos \theta, y = 2 \cos \theta + 1$

$-1 \leq x \leq 1, -1 \leq y \leq 3$

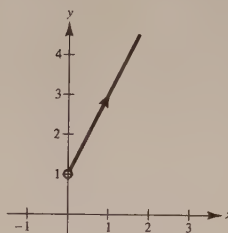
$\frac{dx}{d\theta} = \frac{dy}{d\theta} = 0$ when $\theta = 0, \pm\pi, \pm2\pi, \dots$



(c) $x = e^{-t}, y = 2e^{-t} + 1$
 $x > 0, y > 1$

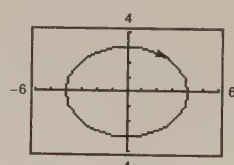
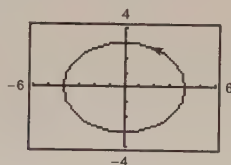


(d) $x = e^t, y = 2e^t + 1$
 $x > 0, y > 1$



35. The curves are identical on $0 < \theta < \pi$. They are both smooth. Represent $y = 2(1 - x^2)$

37. (a)



(b) The orientation of the second curve is reversed.

(c) The orientation will be reversed.

(d) Many answers possible. For example, $x = 1 + t, y = 1 + 2t$, and $x = 1 - t, y = 1 - 2t$.

39. $x = x_1 + t(x_2 - x_1)$
 $y = y_1 + t(y_2 - y_1)$

$\frac{x - x_1}{x_2 - x_1} = t$

$y = y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right)(y_2 - y_1)$

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$y - y_1 = m(x - x_1)$

41. $x = h + a \cos \theta$
 $y = k + b \sin \theta$

$\frac{x - h}{a} = \cos \theta$

$\frac{y - k}{b} = \sin \theta$

$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$

43. From Exercise 39 we have

$x = 5t$

$y = -2t$

Solution not unique

45. From Exercise 40 we have

$x = 2 + 4 \cos \theta$

$y = 1 + 4 \sin \theta$

Solution not unique

47. From Exercise 41 we have

$a = 5, c = 4 \Rightarrow b = 3$

$x = 5 \cos \theta$

$y = 3 \sin \theta$

Center: $(0, 0)$

Solution not unique

49. From Exercise 42 we have

$$a = 4, c = 5 \Rightarrow b = 3$$

$$x = 4 \sec \theta$$

$$y = 3 \tan \theta.$$

Center: $(0, 0)$

Solution not unique

51. $y = 3x - 2$

Example

$$x = t, \quad y = 3t - 2$$

$$x = t - 3, \quad y = 3t - 11$$

53. $y = x^3$

Example

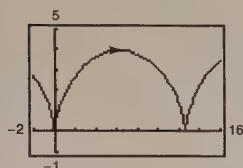
$$x = t, \quad y = t^3$$

$$x = \sqrt[3]{t}, \quad y = t$$

$$x = \tan t, \quad y = \tan^3 t$$

55. $x = 2(\theta - \sin \theta)$

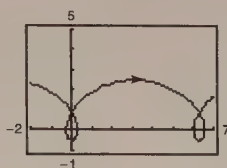
$$y = 2(1 - \cos \theta)$$



Not smooth at $\theta = 2n\pi$

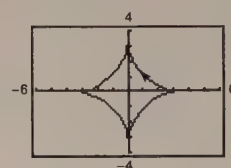
57. $x = \theta - \frac{3}{2} \sin \theta$

$$y = 1 - \frac{3}{2} \cos \theta$$



59. $x = 3 \cos^3 \theta$

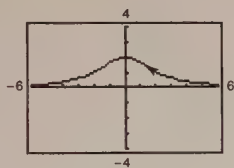
$$y = 3 \sin^3 \theta$$



Not smooth at $(x, y) = (\pm 3, 0)$ and $(0, \pm 3)$, or $\theta = \frac{1}{2}n\pi$.

61. $x = 2 \cot \theta$

$$y = 2 \sin^2 \theta$$



Smooth everywhere

63. See definition on page 665.

65. A plane curve C , represented by $x = f(t)$, $y = g(t)$, is smooth if f' and g' are continuous and not simultaneously 0. See page 670.

67. $x = 4 \cos \theta$

$$y = 2 \sin 2\theta$$

Matches (d)

69. $x = \cos \theta + \theta \sin \theta$

$$y = \sin \theta - \theta \cos \theta$$

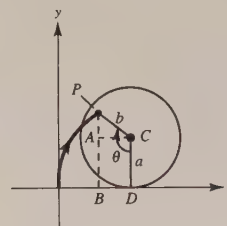
Matches (b)

71. When the circle has rolled θ radians, we know that the center is at $(a\theta, a)$.

$$\sin \theta = \sin(180^\circ - \theta) = \frac{|AC|}{b} = \frac{|BD|}{b} \quad \text{or} \quad |BD| = b \sin \theta$$

$$\cos \theta = -\cos(180^\circ - \theta) = \frac{|AP|}{-b} \quad \text{or} \quad |AP| = -b \cos \theta$$

Therefore, $x = a\theta - b \sin \theta$ and $y = a - b \cos \theta$.



73. False

$$x = t^2 \Rightarrow x \geq 0$$

$$x = t^2 \Rightarrow y \geq 0$$

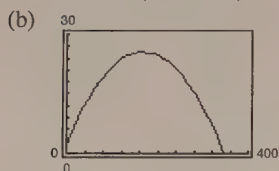
The graph of the parametric equations is only a portion of the line $y = x$.

$$75. (a) 100 \text{ mi/hr} = \frac{(100)(5280)}{3600} = \frac{440}{3} \text{ ft/sec}$$

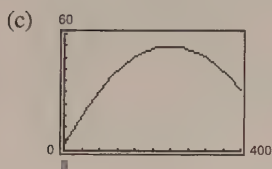
$$x = (v_0 \cos \theta)t = \left(\frac{440}{3} \cos \theta\right)t$$

$$y = h + (v_0 \sin \theta)t - 16t^2$$

$$= 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2$$



It is not a home run—when $x = 400$, $y < 10$.



Yes, it's a home run when $x = 400$, $y > 10$.

(d) We need to find the angle θ (and time t) such that

$$x = \left(\frac{440}{3} \cos \theta\right)t = 400$$

$$y = 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2 = 10.$$

From the first equation $t = 1200/440 \cos \theta$. Substituting into the second equation,

$$10 = 3 + \left(\frac{440}{3} \sin \theta\right)\left(\frac{1200}{440 \cos \theta}\right) - 16\left(\frac{1200}{440 \cos \theta}\right)^2$$

$$7 = 400 \tan \theta - 16\left(\frac{120}{44}\right)^2 \sec^2 \theta$$

$$= 400 \tan \theta - 16\left(\frac{120}{44}\right)^2 (\tan^2 \theta + 1).$$

We now solve the quadratic for $\tan \theta$:

$$16\left(\frac{120}{44}\right)^2 \tan^2 \theta - 400 \tan \theta + 7 + 16\left(\frac{120}{44}\right)^2 = 0$$

$$\tan \theta \approx 0.35185 \Rightarrow \theta \approx 19.4^\circ$$

Section 9.3 Parametric Equations and Calculus

$$1. \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4}{2t} = \frac{-2}{t}$$

$$5. x = 2t, y = 3t - 1$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2}$$

$$\frac{d^2y}{dx^2} = 0 \text{ Line}$$

$$9. x = 2 \cos \theta, y = 2 \sin \theta$$

$$\frac{dy}{dx} = \frac{2 \cos \theta}{-2 \sin \theta} = -\cot \theta = -1 \text{ when } \theta = \frac{\pi}{4}.$$

$$\frac{d^2y}{dx^2} = \frac{\csc^2 \theta}{-2 \sin \theta} = \frac{-\csc^3 \theta}{2} = -\sqrt{2} \text{ when } \theta = \frac{\pi}{4}.$$

concave downward

$$3. \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2 \cos \theta \sin \theta}{2 \sin \theta \cos \theta} = -1$$

$$\left[\text{Note: } x + y = 1 \Rightarrow y = 1 - x \text{ and } \frac{dy}{dx} = -1 \right]$$

$$7. x = t + 1, y = t^2 + 3t$$

$$\frac{dy}{dx} = \frac{2t + 3}{1} = 1 \text{ when } t = -1.$$

$$\frac{d^2y}{dx^2} = 2 \text{ concave upwards}$$

$$11. x = 2 + \sec \theta, y = 1 + 2 \tan \theta$$

$$\frac{dy}{dx} = \frac{2 \sec^2 \theta}{\sec \theta \tan \theta}$$

$$= \frac{2 \sec \theta}{\tan \theta} = 2 \csc \theta = 4 \text{ when } \theta = \frac{\pi}{6}.$$

$$\frac{d^2y}{dx^2} = \frac{-2 \csc \theta \cot \theta}{\sec \theta \tan \theta}$$

$$= -2 \cot^3 \theta = -6\sqrt{3} \text{ when } \theta = \frac{\pi}{6}.$$

concave downward

13. $x = \cos^3 \theta, y = \sin^3 \theta$

$$\frac{dy}{dx} = \frac{3 \sin^2 \theta \cos \theta}{-3 \cos^2 \theta \sin \theta}$$

$$= -\tan \theta = -1 \text{ when } \theta = \frac{\pi}{4}.$$

$$\frac{d^2y}{dx^2} = \frac{-\sec^2 \theta}{-3 \cos^2 \theta \sin \theta} = \frac{1}{3 \cos^4 \theta \sin \theta}$$

$$= \frac{\sec^4 \theta \csc \theta}{3} = \frac{4\sqrt{2}}{3} \text{ when } \theta = \frac{\pi}{4}.$$

concave upward

15. $x = 2 \cot \theta, y = 2 \sin^2 \theta$

$$\frac{dy}{dx} = \frac{4 \sin \theta \cos \theta}{-2 \csc^2 \theta} = -2 \sin^3 \theta \cos \theta$$

$$\text{At } \left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right), \theta = \frac{2\pi}{3}, \text{ and } \frac{dy}{dx} = \frac{3\sqrt{3}}{8}.$$

$$\text{Tangent line: } y - \frac{3}{2} = \frac{3\sqrt{3}}{8} \left(x + \frac{2}{\sqrt{3}}\right)$$

$$3\sqrt{3}x - 8y + 18 = 0$$

$$\text{At } (0, 2), \theta = \frac{\pi}{2}, \text{ and } \frac{dy}{dx} = 0.$$

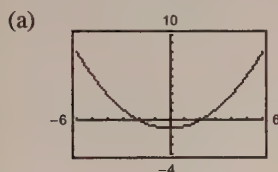
$$\text{Tangent line: } y - 2 = 0$$

$$\text{At } \left(2\sqrt{3}, \frac{1}{2}\right), \theta = \frac{\pi}{6}, \text{ and } \frac{dy}{dx} = -\frac{\sqrt{3}}{8}.$$

$$\text{Tangent line: } y - \frac{1}{2} = -\frac{\sqrt{3}}{8}(x - 2\sqrt{3})$$

$$\sqrt{3}x + 8y - 10 = 0$$

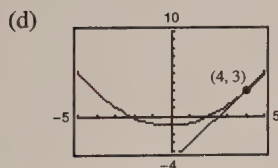
17. $x = 2t, y = t^2 - 1, t = 2$



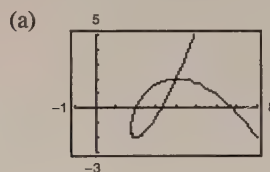
(b) At $t = 2, (x, y) = (4, 3)$, and

$$\frac{dx}{dt} = 2, \frac{dy}{dt} = 4, \frac{dy}{dx} = 2$$

(c) $\frac{dy}{dx} = 2$. At $(4, 3), y - 3 = 2(x - 4)$
 $y = 2x - 5$



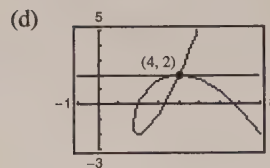
19. $x = t^2 - t + 2, y = t^3 - 3t, t = -1$



(b) At $t = -1, (x, y) = (4, 2)$, and

$$\frac{dx}{dt} = -3, \frac{dy}{dt} = 0, \frac{dy}{dx} = 0$$

(c) $\frac{dy}{dx} = 0$. At $(4, 2), y - 2 = 0(x - 4)$
 $y = 2$



21. $x = 2 \sin 2t, y = 3 \sin t$ crosses itself at the origin, $(x, y) = (0, 0)$.

At this point, $t = 0$ or $t = \pi$.

$$\frac{dy}{dx} = \frac{3 \cos t}{4 \cos 2t}$$

At $t = 0: \frac{dy}{dx} = \frac{3}{4}$ and $y = \frac{3}{4}x$. Tangent Line

At $t = \pi, \frac{dy}{dx} = -\frac{3}{4}$ and $y = -\frac{3}{4}x$. Tangent Line

23. $x = \cos \theta + \theta \sin \theta$, $y = \sin \theta - \theta \cos \theta$

Horizontal tangents: $\frac{dy}{d\theta} = \theta \sin \theta = 0$ when $\theta = \pm\pi, \pm 2\pi, \pm 3\pi, \dots$

Points: $(-1, [2n-1]\pi)$, $(1, 2n\pi)$ where n is an integer.

Points shown: $(1, 0)$, $(-1, \pi)$, $(1, -2\pi)$

Vertical tangents: $\frac{dx}{d\theta} = \theta \cos \theta = 0$ when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$

Note: $\theta = 0$ corresponds to the cusp at $(x, y) = (1, 0)$.

$$\frac{dy}{dx} = \frac{\theta \sin \theta}{\theta \cos \theta} = \tan \theta = 0 \text{ at } \theta = 0.$$

$$\text{Points: } \left(\frac{(-1)^{n+1}(2n-1)\pi}{2}, (-1)^{n+1} \right)$$

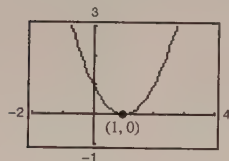
$$\text{Points shown: } \left(\frac{\pi}{2}, 1 \right), \left(-\frac{3\pi}{2}, -1 \right), \left(\frac{5\pi}{2}, 1 \right)$$

25. $x = 1 - t$, $y = t^2$

Horizontal tangents: $\frac{dy}{dt} = 2t = 0$ when $t = 0$.

Point: $(1, 0)$

Vertical tangents: $\frac{dx}{dt} = -1 \neq 0$; none

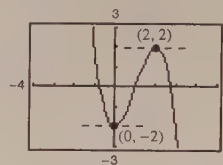


27. $x = 1 - t$, $y = t^3 - 3t$

Horizontal tangents: $\frac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$.

Points: $(0, -2)$, $(2, 2)$

Vertical tangents: $\frac{dx}{dt} = -1 \neq 0$; none



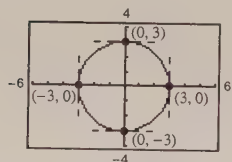
29. $x = 3 \cos \theta$, $y = 3 \sin \theta$

Horizontal tangents: $\frac{dy}{d\theta} = 3 \cos \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points: $(0, 3)$, $(0, -3)$

Vertical tangents: $\frac{dx}{d\theta} = -3 \sin \theta = 0$ when $\theta = 0, \pi$.

Points: $(3, 0)$, $(-3, 0)$



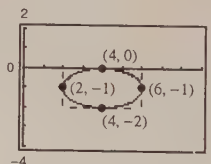
31. $x = 4 + 2 \cos \theta$, $y = -1 + \sin \theta$

Horizontal tangents: $\frac{dy}{d\theta} = \cos \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points: $(4, 0)$, $(4, -2)$

Vertical tangents: $\frac{dx}{d\theta} = -2 \sin \theta = 0$ when $\theta = 0, \pi$.

Points: $(6, -1)$, $(2, -1)$

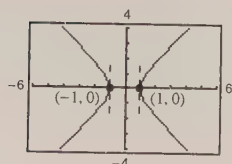


33. $x = \sec \theta$, $y = \tan \theta$

Horizontal tangents: $\frac{dy}{d\theta} = \sec^2 \theta \neq 0$; none

Vertical tangents: $\frac{dx}{d\theta} = \sec \theta \tan \theta = 0$ when $x = 0, \pi$.

Points: $(1, 0)$, $(-1, 0)$



35. $x = t^2$, $y = 2t$, $0 \leq t \leq 2$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 2, \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 = 4t^2 + 4 = 4(t^2 + 1)$$

$$\begin{aligned} s &= 2 \int_0^2 \sqrt{t^2 + 1} \, dt \\ &= \left[t\sqrt{t^2 + 1} + \ln|t + \sqrt{t^2 + 1}| \right]_0^2 \\ &= 2\sqrt{5} + \ln(2 + \sqrt{5}) \approx 5.916 \end{aligned}$$

37. $x = e^{-t} \cos t, y = e^{-t} \sin t, 0 \leq t \leq \frac{\pi}{2}$

$$\frac{dx}{dt} = -e^{-t}(\sin t + \cos t), \frac{dy}{dt} = e^{-t}(\cos t - \sin t)$$

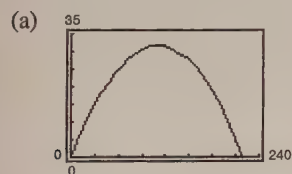
$$\begin{aligned} s &= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{\pi/2} \sqrt{2e^{-2t}} dt = -\sqrt{2} \int_0^{\pi/2} e^{-t}(-1) dt \\ &= \left[-\sqrt{2}e^{-t}\right]_0^{\pi/2} = \sqrt{2}(1 - e^{-\pi/2}) \approx 1.12 \end{aligned}$$

41. $x = a \cos^3 \theta, y = a \sin^3 \theta, \frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta,$

$$\frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$$

$$\begin{aligned} s &= 4 \int_0^{\pi/2} \sqrt{9a^2 \cos^4 \theta \sin^2 + 9a^2 \sin^4 \theta \cos^2} d\theta \\ &= 12a \int_0^{\pi/2} \sin \theta \cos \theta \sqrt{\cos^2 \theta + \sin^2 \theta} d\theta \\ &= 6a \int_0^{\pi/2} \sin 2\theta d\theta = \left[-3a \cos 2\theta\right]_0^{\pi/2} = 6a \end{aligned}$$

45. $x = (90 \cos 30^\circ)t, y = (90 \sin 30^\circ)t - 16t^2$



(b) Range: $\left(t = \frac{45}{16}\right)$

(c) $\frac{dx}{dt} = 90 \cos 30^\circ, \frac{dy}{dt} = 90 \sin 30^\circ - 32t.$

$$y = 0 \text{ for } t = \frac{45}{16}.$$

$$\begin{aligned} s &= \int_0^{45/16} \sqrt{(90 \cos 30^\circ)^2 + (90 \sin 30^\circ - 32t)^2} dt \\ &\approx 230.8 \text{ ft} \end{aligned}$$

39. $x = \sqrt{t}, y = 3t - 1, \frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \frac{dy}{dt} = 3$

$$\begin{aligned} s &= \int_0^1 \sqrt{\frac{1}{4t} + 9} dt = \frac{1}{2} \int_0^1 \frac{\sqrt{1 + 36t}}{\sqrt{t}} dt \\ &= \frac{1}{6} \int_0^6 \sqrt{1 + u^2} du \\ &= \frac{1}{12} \left[\ln(\sqrt{1 + u^2} + u) + u\sqrt{1 + u^2} \right]_0^6 \\ &= \frac{1}{12} \left[\ln(\sqrt{37} + 6) + 6\sqrt{37} \right] \approx 3.249 \end{aligned}$$

$$u = 6\sqrt{t}, du = \frac{3}{\sqrt{t}} dt$$

43. $x = a(\theta - \sin \theta), y = a(1 - \cos \theta),$

$$\frac{dx}{d\theta} = a(1 - \cos \theta), \frac{dy}{d\theta} = a \sin \theta$$

$$\begin{aligned} s &= 2 \int_0^\pi \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta \\ &= 2\sqrt{2}a \int_0^\pi \sqrt{1 - \cos \theta} d\theta \\ &= 2\sqrt{2}a \int_0^\pi \frac{\sin \theta}{\sqrt{1 + \cos \theta}} d\theta \\ &= \left[-4\sqrt{2}a\sqrt{1 + \cos \theta}\right]_0^\pi = 8a \end{aligned}$$

(d) $y = 0 \Rightarrow (90 \sin \theta)t = 16t^2 \Rightarrow t = 0, \frac{90}{16} \sin \theta$

$$x = (90 \cos \theta)t = (90 \cos \theta) \frac{90}{16} \sin \theta$$

$$= \frac{90^2}{16} \sin \theta \cos \theta = \frac{90^2}{32} \sin 2\theta$$

$$x'(\theta) = \frac{90^2}{32} 2 \cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}$$

By the First Derivative Test, $\theta = \frac{\pi}{4} (45^\circ)$ maximizes the range ($x = 253.125$ feet).

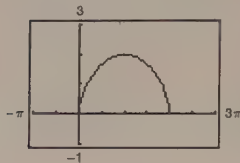
To maximize the arc length, we have

$$\frac{dx}{dt} = 90 \cos \theta, \frac{dy}{dt} = 90 \sin \theta - 32t$$

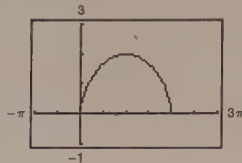
$$\begin{aligned} s &= \int_0^{(90/16)\sin \theta} \sqrt{(90 \cos \theta)^2 + (90 \sin \theta - 32t)^2} dt \\ &= \frac{2025}{8} \sin \theta + \frac{2025}{16} \cos^2 \theta \ln \left[\frac{1 + \sin \theta}{1 - \sin \theta} \right] \end{aligned}$$

Using a graphing utility, we see that s is a maximum of approximately 303.67 feet at $\theta \approx 0.9855 (56.5^\circ)$.

47. (a) $x = t - \sin t$
 $y = 1 - \cos t$
 $0 \leq t \leq 2\pi$



$x = 2t - \sin(2t)$
 $y = 1 - \cos(2t)$
 $0 \leq t \leq \pi$



(b) The average speed of the particle on the second path is twice the average speed of a particle on the first path.

(c) $x = \frac{1}{2}t - \sin(\frac{1}{2}t)$
 $y = 1 - \cos(\frac{1}{2}t)$

The time required for the particle to traverse the same path is $t = 4\pi$.

49. $x = t$, $y = 2t$, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2$

(a) $S = 2\pi \int_0^4 2t\sqrt{1+4} dt = 4\sqrt{5}\pi \int_0^4 t dt$
 $= \left[2\sqrt{5}\pi t^2 \right]_0^4 = 32\pi\sqrt{5}$

(b) $S = 2\pi \int_0^4 t\sqrt{1+4} dt = 2\sqrt{5}\pi \int_0^4 t dt$
 $= \left[\sqrt{5}\pi t^2 \right]_0^4 = 16\pi\sqrt{5}$

51. $x = 4 \cos \theta$, $y = 4 \sin \theta$, $\frac{dx}{d\theta} = -4 \sin \theta$, $\frac{dy}{d\theta} = 4 \cos \theta$

$S = 2\pi \int_0^{\pi/2} 4 \cos \theta \sqrt{(-4 \sin \theta)^2 + (4 \cos \theta)^2} d\theta$
 $= 32\pi \int_0^{\pi/2} \cos \theta d\theta = \left[32\pi \sin \theta \right]_0^{\pi/2} = 32\pi$

53. $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, $\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$, $\frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$

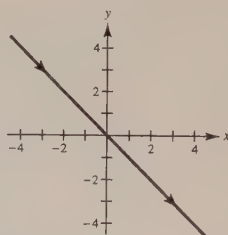
$S = 4\pi \int_0^{\pi/2} a \sin^3 \theta \sqrt{9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta} d\theta = 12a^2\pi \int_0^{\pi/2} \sin^4 \theta \cos \theta d\theta = \frac{12\pi a^2}{5} \left[\sin^5 \theta \right]_0^{\pi/2} = \frac{12}{5}\pi a^2$

55. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

See Theorem 9.7, page 675.

57. One possible answer is the graph given by

$x = t$, $y = -t$.

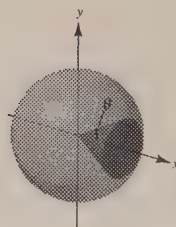


59. $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

See Theorem 9.8, page 678.

61. $x = r \cos \phi$, $y = r \sin \phi$

$S = 2\pi \int_0^\theta r \sin \phi \sqrt{r^2 \sin^2 \phi + r^2 \cos^2 \phi} d\phi$
 $= 2\pi r^2 \int_0^\theta \sin \phi d\phi$
 $= \left[-2\pi r^2 \cos \phi \right]_0^\theta$
 $= 2\pi r^2(1 - \cos \theta)$



63. $x = \sqrt{t}$, $y = 4 - t$, $0 \leq t \leq 4$

$$A = \int_0^4 (4 - t) \frac{1}{2\sqrt{t}} dt = \frac{1}{2} \int_0^4 (4t^{-1/2} - t^{1/2}) dt = \left[\frac{1}{2} \left(8\sqrt{t} - \frac{2}{3}t\sqrt{t} \right) \right]_0^4 = \frac{16}{3}$$

$$\bar{x} = \frac{3}{16} \int_0^4 (4 - t) \sqrt{t} \left(\frac{1}{2\sqrt{t}} \right) dt = \frac{3}{32} \int_0^4 (4 - t) dt = \left[\frac{3}{32} \left(4t - \frac{t^2}{2} \right) \right]_0^4 = \frac{3}{4}$$

$$\bar{y} = \frac{3}{32} \int_0^4 (4 - t)^2 \frac{1}{2\sqrt{t}} dt = \frac{3}{64} \int_0^4 [16t^{-1/2} - 8t^{1/2} + t^{3/2}] dt = \frac{3}{64} \left[32\sqrt{t} - \frac{16}{3}t\sqrt{t} + \frac{2}{5}t^2\sqrt{t} \right]_0^4 = \frac{8}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{8}{5} \right)$$

65. $x = 3 \cos \theta$, $y = 3 \sin \theta$, $\frac{dx}{d\theta} = -3 \sin \theta$

$$V = 2\pi \int_{\pi/2}^0 (3 \sin \theta)^2 (-3 \sin \theta) d\theta$$

$$= -54\pi \int_{\pi/2}^0 \sin^3 \theta d\theta$$

$$= -54\pi \int_{\pi/2}^0 (1 - \cos^2 \theta) \sin \theta d\theta$$

$$= -54\pi \left[-\cos \theta + \frac{\cos^3 \theta}{3} \right]_{\pi/2}^0 = 36\pi$$

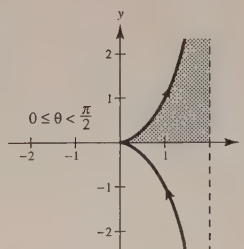
67. $x = 2 \sin^2 \theta$

$$y = 2 \sin^2 \theta \tan \theta$$

$$\frac{dx}{d\theta} = 4 \sin \theta \cos \theta$$

$$A = \int_0^{\pi/2} 2 \sin^2 \theta \tan \theta (4 \sin \theta \cos \theta) d\theta = 8 \int_0^{\pi/2} \sin^4 \theta d\theta$$

$$= 8 \left[\frac{-\sin^3 \theta \cos \theta}{4} - \frac{3}{8} \sin \theta \cos \theta + \frac{3}{8} \theta \right]_0^{\pi/2} = \frac{3\pi}{2}$$



69. πab is area of ellipse (d).

71. $6\pi a^2$ is area of cardioid (f).

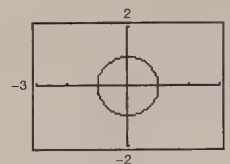
73. $\frac{8}{3}ab$ is area of hourglass (a).

75. (a) $x = \frac{1 - t^2}{1 + t^2}$, $y = \frac{2t}{1 + t^2}$, $-20 \leq t \leq 20$

The graph (for $-\infty < t < \infty$) is the circle $x^2 + y^2 = 1$, except the point $(-1, 0)$.

$$\text{Verify: } x^2 + y^2 = \left(\frac{1 - t^2}{1 + t^2} \right)^2 + \left(\frac{2t}{1 + t^2} \right)^2 = \frac{1 - 2t^2 + t^4 + 4t^2}{(1 + t^2)^2} = \frac{(1 + t^2)^2}{(1 + t^2)^2} = 1$$

(b) As t increases from -20 to 0 , the speed increases, and as t increases from 0 to 20 , the speed decreases.



77. False

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left[\frac{g'(t)}{f'(t)} \right]}{f'(t)} = \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$$

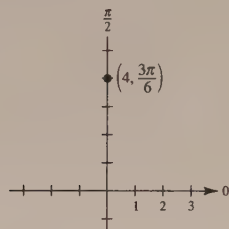
Section 9.4 Polar Coordinates and Polar Graphs

1. $\left(4, \frac{\pi}{2}\right)$

$$x = 4 \cos\left(\frac{\pi}{2}\right) = 0$$

$$y = 4 \sin\left(\frac{\pi}{2}\right) = 4$$

$$(x, y) = (0, 4)$$

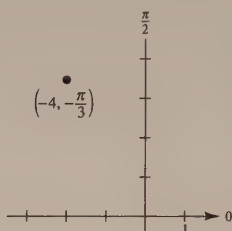


3. $\left(-4, -\frac{\pi}{3}\right)$

$$x = -4 \cos\left(-\frac{\pi}{3}\right) = -2$$

$$y = -4 \sin\left(-\frac{\pi}{3}\right) = 2\sqrt{3}$$

$$(x, y) = (-2, 2\sqrt{3})$$

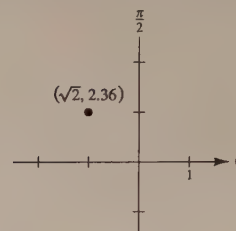


5. $(\sqrt{2}, 2.36)$

$$x = \sqrt{2} \cos(2.36) \approx -1.004$$

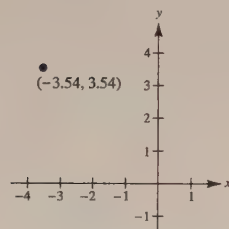
$$y = \sqrt{2} \sin(2.36) \approx 0.996$$

$$(x, y) = (-1.004, 0.996)$$



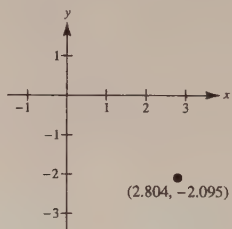
7. $(r, \theta) = \left(5, \frac{3\pi}{4}\right)$

$$(x, y) = (-3.5355, 3.5355)$$



9. $(r, \theta) = (-3.5, 2.5)$

$$(x, y) = (2.804, -2.095)$$

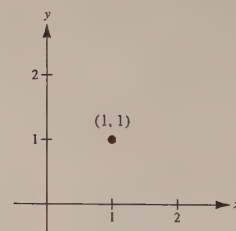


11. $(x, y) = (1, 1)$

$$r = \pm\sqrt{2}$$

$$\tan \theta = 1$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \left(\sqrt{2}, \frac{\pi}{4}\right), \left(-\sqrt{2}, \frac{5\pi}{4}\right)$$

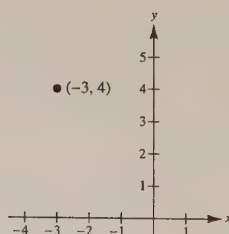


13. $(x, y) = (-3, 4)$

$$r = \pm\sqrt{9 + 16} = \pm 5$$

$$\tan \theta = -\frac{4}{3}$$

$$\theta \approx 2.214, 5.356, (5, 2.214), (-5, 5.356)$$



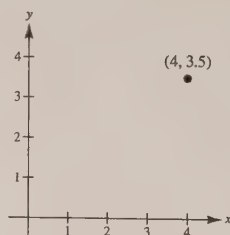
15. $(x, y) = (3, -2)$

$$(r, \theta) = (3.606, -0.588)$$

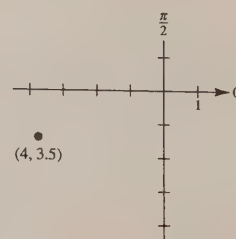
17. $(x, y) = \left(\frac{5}{2}, \frac{4}{3}\right)$

$$(r, \theta) = (2.833, 0.490)$$

19. (a) $(x, y) = (4, 3.5)$

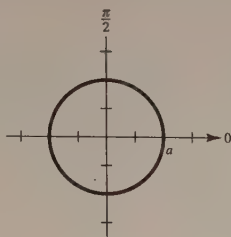


(b) $(r, \theta) = (4, 3.5)$



21. $x^2 + y^2 = a^2$

$$r = a$$



23. $y = 4$

$$r \sin \theta = 4$$

$$r = 4 \csc \theta$$

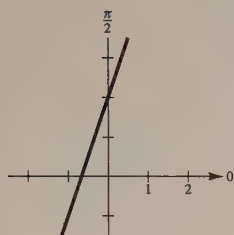


25. $3x - y + 2 = 0$

$$3r \cos \theta - r \sin \theta + 2 = 0$$

$$r(3 \cos \theta - \sin \theta) = -2$$

$$r = \frac{-2}{3 \cos \theta - \sin \theta}$$

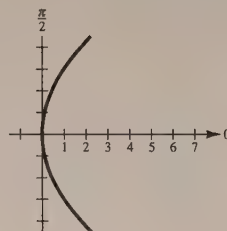


27. $y^2 = 9x$

$$r^2 \sin^2 \theta = 9r \cos \theta$$

$$r = \frac{9 \cos \theta}{\sin^2 \theta}$$

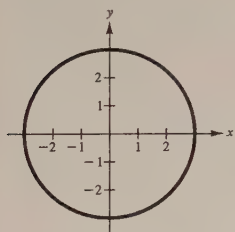
$$r = 9 \csc^2 \theta \cos \theta$$



29. $r = 3$

$$r^2 = 9$$

$$x^2 + y^2 = 9$$



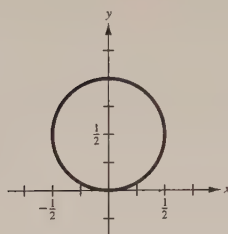
31. $r = \sin \theta$

$$r^2 = r \sin \theta$$

$$x^2 + y^2 = y$$

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$x^2 + y^2 - y = 0$$

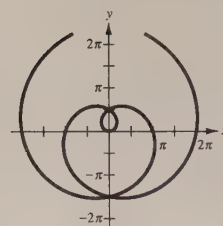


33. $r = \theta$

$$\tan r = \tan \theta$$

$$\tan \sqrt{x^2 + y^2} = \frac{y}{x}$$

$$\sqrt{x^2 + y^2} = \arctan \frac{y}{x}$$

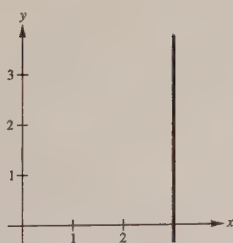


35. $r = 3 \sec \theta$

$$r \cos \theta = 3$$

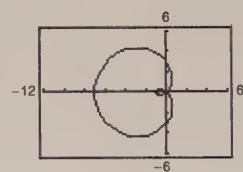
$$x = 3$$

$$x - 3 = 0$$



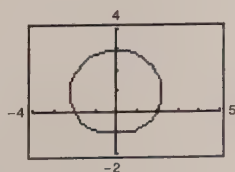
37. $r = 3 - 4 \cos \theta$

$$0 \leq \theta < 2\pi$$



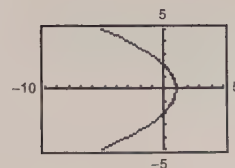
39. $r = 2 + \sin \theta$

$$0 \leq \theta < 2\pi$$



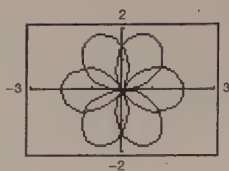
41. $r = \frac{2}{1 + \cos \theta}$

Traced out once on
 $-\pi < \theta < \pi$



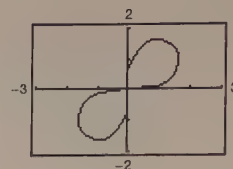
43. $r = 2 \cos\left(\frac{3\theta}{2}\right)$

$0 \leq \theta < 4\pi$



45. $r^2 = 4 \sin 2\theta$

$0 \leq \theta < \frac{\pi}{2}$



47.

$$r = 2(h \cos \theta + k \sin \theta)$$

$$r^2 = 2r(h \cos \theta + k \sin \theta)$$

$$r^2 = 2[h(r \cos \theta) + k(r \sin \theta)]$$

$$x^2 + y^2 = 2(hx + ky)$$

$$x^2 + y^2 - 2hx - 2ky = 0$$

$$(x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) = 0 + h^2 + k^2$$

$$(x - h)^2 + (y - k)^2 = h^2 + k^2$$

Radius: $\sqrt{h^2 + k^2}$

Center: (h, k)

49. $\left(4, \frac{2\pi}{3}\right), \left(2, \frac{\pi}{6}\right)$

$$d = \sqrt{4^2 + 2^2 - 2(4)(2) \cos\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)}$$

$$= \sqrt{20 - 16 \cos \frac{\pi}{2}} = 2\sqrt{5} \approx 4.5$$

51. $(2, 0.5), (7, 1.2)$

$$d = \sqrt{2^2 + 7^2 - 2(2)(7) \cos(0.5 - 1.2)}$$

$$= \sqrt{53 - 28 \cos(-0.7)} \approx 5.6$$

53. $r = 2 + 3 \sin \theta$

$$\frac{dy}{dx} = \frac{3 \cos \theta \sin \theta + \cos \theta(2 + 3 \sin \theta)}{3 \cos \theta \cos \theta - \sin \theta(2 + 3 \sin \theta)}$$

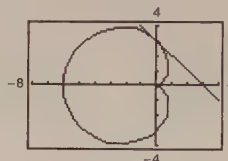
$$= \frac{2 \cos \theta(3 \sin \theta + 1)}{3 \cos 2\theta - 2 \sin \theta} = \frac{2 \cos \theta(3 \sin \theta + 1)}{6 \cos^2 \theta - 2 \sin \theta - 3}$$

At $\left(5, \frac{\pi}{2}\right), \frac{dy}{dx} = 0$.

At $(2, \pi), \frac{dy}{dx} = -\frac{2}{3}$.

At $\left(-1, \frac{3\pi}{2}\right), \frac{dy}{dx} = 0$.

55. (a), (b) $r = 3(1 - \cos \theta)$



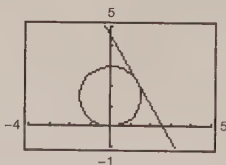
$$(r, \theta) = \left(3, \frac{\pi}{2}\right) \Rightarrow (x, y) = (0, 3)$$

Tangent line: $y - 3 = -1(x - 0)$

$$y = -x + 3$$

(c) At $\theta = \frac{\pi}{2}, \frac{dy}{dx} = -1.0$.

57. (a), (b) $r = 3 \sin \theta$



$$(r, \theta) = \left(\frac{3\sqrt{3}}{2}, \frac{\pi}{3}\right) \Rightarrow (x, y) = \left(\frac{3\sqrt{3}}{4}, \frac{9}{4}\right)$$

Tangent line: $y - \frac{9}{4} = -\sqrt{3}\left(x - \frac{3\sqrt{3}}{4}\right)$

$$y = -\sqrt{3}x + \frac{9}{2}$$

(c) At $\theta = \frac{\pi}{3}, \frac{dy}{dx} = -\sqrt{3} \approx -1.732$.

59. $r = 1 - \sin \theta$

$$\frac{dy}{d\theta} = (1 - \sin \theta) \cos \theta - \cos \theta \sin \theta$$

$$= \cos \theta (1 - 2 \sin \theta) = 0$$

$$\cos \theta = 0, \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Horizontal tangents: $\left(2, \frac{3\pi}{2}\right), \left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$

$$\frac{dx}{d\theta} = (-1 + \sin \theta) \sin \theta - \cos \theta \cos \theta$$

$$= -\sin \theta + \sin^2 \theta + \sin^2 \theta - 1$$

$$= 2 \sin^2 \theta - \sin \theta - 1$$

$$= (2 \sin \theta + 1)(\sin \theta - 1) = 0$$

$$\sin \theta = 1, \sin \theta = -\frac{1}{2} \Rightarrow \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Vertical tangents: $\left(\frac{3}{2}, \frac{7\pi}{6}\right), \left(\frac{3}{2}, \frac{11\pi}{6}\right)$

61. $r = 2 \csc \theta + 3$

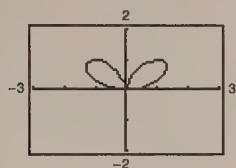
$$\frac{dy}{d\theta} = (2 \csc \theta + 3) \cos \theta + (-2 \csc \theta \cot \theta) \sin \theta$$

$$= 3 \cos \theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

Horizontal: $\left(5, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right)$

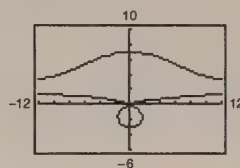
63. $r = 4 \sin \theta \cos^2 \theta$



Horizontal tangents:

$$(0, 0), (1.4142, 0.7854), (1.4142, 2.3562)$$

65. $r = 2 \csc \theta + 5$



Horizontal tangents: $\left(7, \frac{\pi}{2}\right), \left(3, \frac{3\pi}{2}\right)$

67. $r = 3 \sin \theta$

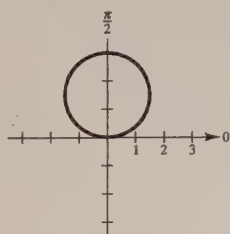
$$r^2 = 3r \sin \theta$$

$$x^2 + y^2 = 3y$$

$$x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$$

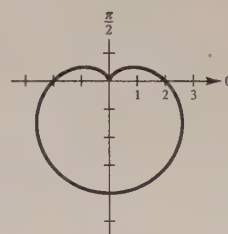
Circle $r = \frac{3}{2}$

Center: $\left(0, \frac{3}{2}\right)$

 Tangent at the pole: $\theta = 0$


69. $r = 2(1 - \sin \theta)$

Cardioid

 Symmetric to y-axis, $\theta = \frac{\pi}{2}$


71. $r = 2 \cos(3\theta)$

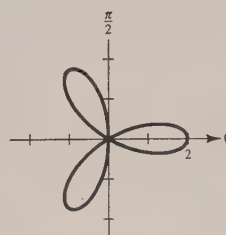
Rose curve with three petals

Symmetric to the polar axis

Relative extrema: $(2, 0), \left(-2, \frac{\pi}{3}\right), \left(2, \frac{2\pi}{3}\right)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	2	0	$-\sqrt{2}$	-2	0	2	0	-2

Tangents at the pole: $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$



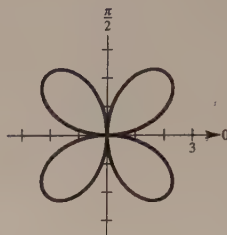
73. $r = 3 \sin 2\theta$

Rose curve with four petals

 Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and pole

 Relative extrema: $(\pm 3, \frac{\pi}{4}), (\pm 3, \frac{5\pi}{4})$

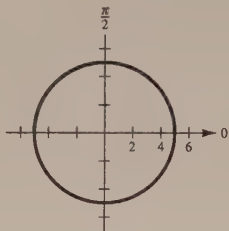
 Tangents at the pole: $\theta = 0, \frac{\pi}{2}$

 ($\theta = \pi, 3\pi/2$ give the same tangents.)


75. $r = 5$

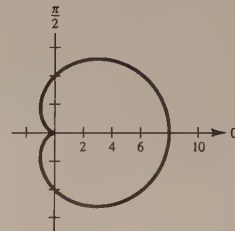
Circle radius: 5

$x^2 + y^2 = 25$



77. $r = 4(1 + \cos \theta)$

Cardioid

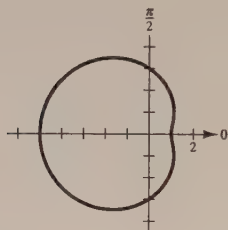


79. $r = 3 - 2 \cos \theta$

Limaçon

Symmetric to polar axis

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	2	3	4	5



81. $r = 3 \csc \theta$

$r \sin \theta = 3$

$y = 3$

Horizontal line

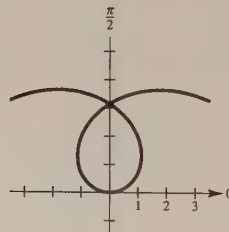


83. $r = 2\theta$

Spiral of Archimedes

 Symmetric to $\theta = \frac{\pi}{2}$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π


 Tangent at the pole: $\theta = 0$

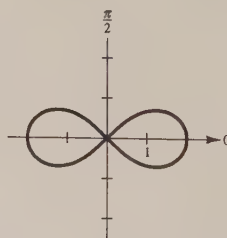
85. $r^2 = 4 \cos(2\theta)$

Lemniscate

 Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and pole

 Relative extrema: $(\pm 2, 0)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	± 2	$\pm \sqrt{2}$	0

 Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$


87. Since

$$r = 2 - \sec \theta = 2 - \frac{1}{\cos \theta},$$

the graph has polar axis symmetry and the lengths at the pole are

$$\theta = \frac{\pi}{3}, \frac{\pi}{3}.$$

Furthermore,

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{\pi}{2}^-$$

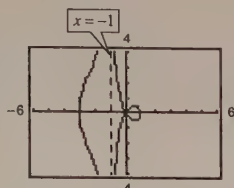
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \frac{\pi}{2}^+.$$

$$\text{Also, } r = 2 - \frac{1}{\cos \theta} = 2 - \frac{r}{r \cos \theta} = 2 - \frac{r}{x}$$

$$rx = 2x - r$$

$$r = \frac{2x}{1+x}.$$

Thus, $r \Rightarrow \pm\infty$ as $x \Rightarrow -1$.



$$89. r = \frac{2}{\theta}$$

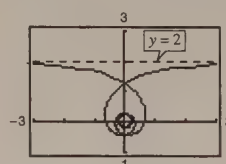
Hyperbolic spiral

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow 0$$

$$r = \frac{2}{\theta} \Rightarrow \theta = \frac{2}{r} = \frac{2 \sin \theta}{r \sin \theta} = \frac{2 \sin \theta}{y}$$

$$y = \frac{2 \sin \theta}{\theta}$$

$$\lim_{\theta \rightarrow 0} \frac{2 \sin \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{2 \cos \theta}{1} = 2$$



91. The rectangular coordinate system consists of all points of the form (x, y) where x is the directed distance from the y -axis to the point, and y is the directed distance from the x -axis to the point. Every point has a unique representation.

The polar coordinate system uses (r, θ) to designate the location of a point.

r is the directed distance to the origin and θ is the angle the point makes with the positive x -axis, measured clockwise.

Point do not have a unique polar representation.

93. $r = a$ circle

$\theta = b$ line

95. $r = 2 \sin \theta$ circle

Matches (c)

97. $r = 3(1 + \cos \theta)$

Cardioid

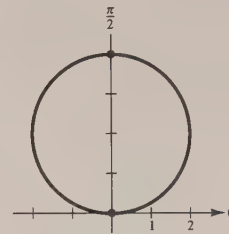
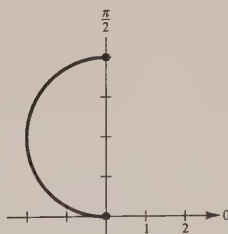
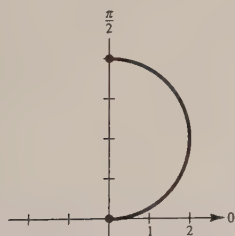
Matches (a)

99. $r = 4 \sin \theta$

$$(a) 0 \leq \theta \leq \frac{\pi}{2}$$

$$(b) \frac{\pi}{2} \leq \theta \leq \pi$$

$$(c) -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

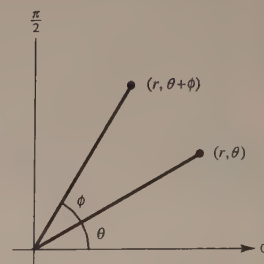


101. Let the curve $r = f(\theta)$ be rotated by ϕ to form the curve $r = g(\theta)$. If (r_1, θ_1) is a point on $r = f(\theta)$, then $(r_1, \theta_1 + \phi)$ is on $r = g(\theta)$. That is,

$$g(\theta_1 + \phi) = r_1 = f(\theta_1).$$

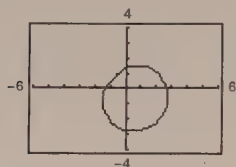
Letting $\theta = \theta_1 + \phi$, or $\theta_1 = \theta - \phi$, we see that

$$g(\theta) = g(\theta_1 + \phi) = f(\theta_1) = f(\theta - \phi).$$

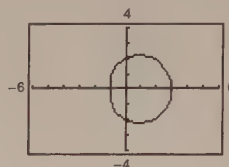


103. $r = 2 - \sin \theta$

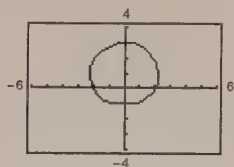
(a) $r = 2 - \sin\left(\theta - \frac{\pi}{4}\right) = 2 - \frac{\sqrt{2}}{2}(\sin \theta - \cos \theta)$



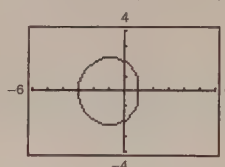
(b) $r = 2 - \sin\left(\theta - \frac{\pi}{2}\right) = 2 - (-\cos \theta) = 2 + \cos \theta$



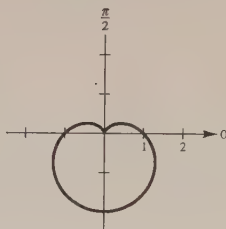
(c) $r = 2 - \sin(\theta - \pi) = 2 - (-\sin \theta) = 2 + \sin \theta$



(d) $r = 2 - \sin\left(\theta - \frac{3\pi}{2}\right) = 2 - \cos \theta$



105. (a) $r = 1 - \sin \theta$

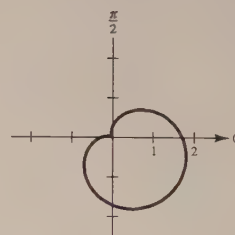


(b) $r = 1 - \sin\left(\theta - \frac{\pi}{4}\right)$

Rotate the graph of

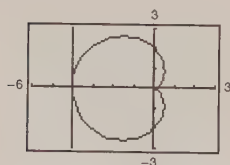
$$r = 1 - \sin \theta$$

through the angle $\pi/4$.



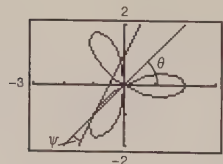
107. $\tan \psi = \frac{r}{dr/d\theta} = \frac{2(1 - \cos \theta)}{2 \sin \theta}$

At $\theta = \pi$, $\tan \psi$ is undefined $\Rightarrow \psi = \frac{\pi}{2}$.



109. $\tan \psi = \frac{r}{dr/d\theta} = \frac{2 \cos 3\theta}{-6 \sin 3\theta}$

At $\theta = \frac{\pi}{4}$, $\tan \psi = \frac{1}{3} \Rightarrow \psi \approx 18.4^\circ$

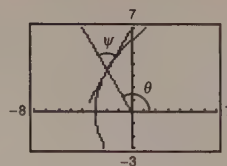


$$111. \quad r = \frac{6}{1 - \cos \theta} = 6(1 - \cos \theta)^{-1} \Rightarrow \frac{dr}{d\theta} = \frac{6 \sin \theta}{(1 - \cos \theta)^2}$$

$$\tan \psi = \frac{r}{\frac{dr}{d\theta}} = \frac{\frac{6}{1 - \cos \theta}}{\frac{6 \sin \theta}{(1 - \cos \theta)^2}} = \frac{1 - \cos \theta}{\sin \theta}$$

$$\text{At } \theta = \frac{2\pi}{3}, \tan \psi = \frac{1 - \left(-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} = \sqrt{3}.$$

$$\psi = \frac{\pi}{3}, (60^\circ)$$

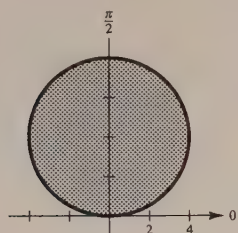


113. True

115. True

Section 9.5 Area and Arc Length in Polar Coordinates

$$1. (a) \quad r = 8 \sin \theta$$



$$A = \pi(4)^2 = 16\pi$$

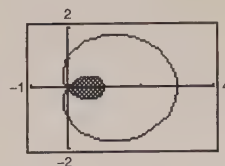
$$3. \quad A = 2 \left[\frac{1}{2} \int_0^{\pi/6} (2 \cos 3\theta)^2 d\theta \right] = 2 \left[\theta + \frac{1}{6} \sin 6\theta \right]_0^{\pi/6} = \frac{\pi}{3}$$

$$7. \quad A = 2 \left[\frac{1}{2} \int_{-\pi/2}^{\pi/2} (1 - \sin \theta)^2 d\theta \right] \\ = \left[\frac{3}{2} \theta + 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$$

$$(b) \quad A = 2 \left(\frac{1}{2} \right) \int_0^{\pi/2} [8 \sin \theta]^2 d\theta \\ = 64 \int_0^{\pi/2} \sin^2 \theta d\theta \\ = 32 \int_0^{\pi/2} (1 - \cos 2\theta) d\theta \\ = 32 \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = 16\pi$$

$$5. \quad A = 2 \left[\frac{1}{2} \int_0^{\pi/4} (\cos 2\theta)^2 d\theta \right] \\ = \frac{1}{2} \left[\theta + \frac{1}{4} \sin 4\theta \right]_0^{\pi/4} = \frac{\pi}{8}$$

$$9. \quad A = 2 \left[\frac{1}{2} \int_{2\pi/3}^{\pi} (1 + 2 \cos \theta)^2 d\theta \right] \\ = \left[3\theta + 4 \sin \theta + \sin 2\theta \right]_{2\pi/3}^{\pi} = \frac{2\pi - 3\sqrt{3}}{2}$$

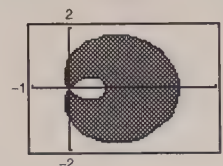


11. The area inside the outer loop is

$$2 \left[\frac{1}{2} \int_0^{2\pi/3} (1 + 2 \cos \theta)^2 d\theta \right] = \left[3\theta + 4 \sin \theta + \sin 2\theta \right]_0^{2\pi/3} = \frac{4\pi + 3\sqrt{3}}{2}.$$

From the result of Exercise 9, the area between the loops is

$$A = \left(\frac{4\pi + 3\sqrt{3}}{2} \right) - \left(\frac{2\pi - 3\sqrt{3}}{2} \right) = \pi + 3\sqrt{3}.$$



13. $r = 1 + \cos \theta$

$r = 1 - \cos \theta$

Solving simultaneously,

$1 + \cos \theta = 1 - \cos \theta$

$2 \cos \theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos \theta = 1 - \cos \theta$, $\cos \theta = 1$, $\theta = 0$. Both curves pass through the pole, $(0, \pi)$, and $(0, 0)$, respectively.

Points of intersection: $\left(1, \frac{\pi}{2}\right)$, $\left(1, \frac{3\pi}{2}\right)$, $(0, 0)$

15. $r = 1 + \cos \theta$

$r = 1 - \sin \theta$

Solving simultaneously,

$1 + \cos \theta = 1 - \sin \theta$

$\cos \theta = -\sin \theta$

$\tan \theta = -1$

$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos \theta = 1 - \sin \theta$, $\sin \theta + \cos \theta = 2$, which has no solution. Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\frac{2 - \sqrt{2}}{2}, \frac{3\pi}{4}\right)$, $\left(\frac{2 + \sqrt{2}}{2}, \frac{7\pi}{4}\right)$, $(0, 0)$

17. $r = 4 - 5 \sin \theta$

$r = 3 \sin \theta$

Solving simultaneously,

$4 - 5 \sin \theta = 3 \sin \theta$

$\sin \theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Both curves pass through the pole, $(0, \arcsin 4/5)$, and $(0, 0)$, respectively.

Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{6}\right)$, $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$, $(0, 0)$

19. $r = \frac{\theta}{2}$

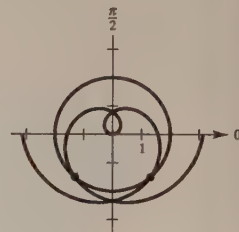
$r = 2$

Solving simultaneously, we have

$\theta/2 = 2, \theta = 4$

Points of intersection:

$(2, 4), (-2, -4)$



21. $r = 4 \sin 2\theta$

$r = 2$

$r = 4 \sin 2\theta$ is the equation of a rose curve with four petals and is symmetric to the polar axis, $\theta = \pi/2$, and the pole. Also, $r = 2$ is the equation of a circle of radius 2 centered at the pole.

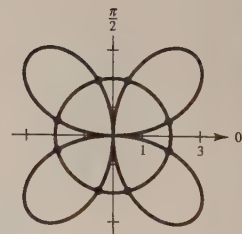
Solving simultaneously,

$4 \sin 2\theta = 2$

$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

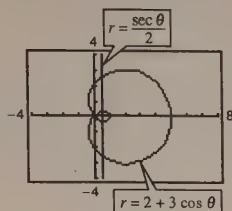
$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$

Therefore, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. By symmetry, the other points of intersection are $(2, 7\pi/12)$, $(2, 11\pi/12)$, $(2, 13\pi/12)$, $(2, 17\pi/12)$, $(2, 19\pi/12)$, and $(2, 23\pi/12)$.



23. $r = 2 + 3 \cos \theta$

$$r = \frac{\sec \theta}{2}$$



The graph of $r = 2 + 3 \cos \theta$ is a limaçon with an inner loop ($b > a$) and is symmetric to the polar axis. The graph of $r = (\sec \theta)/2$ is the vertical line $x = 1/2$. Therefore, there are four points of intersection. Solving simultaneously,

$$2 + 3 \cos \theta = \frac{\sec \theta}{2}$$

$$6 \cos^2 \theta + 4 \cos \theta - 1 = 0$$

$$\cos \theta = \frac{-2 \pm \sqrt{10}}{6}$$

$$\theta = \arccos\left(\frac{-2 + \sqrt{10}}{6}\right) \approx 1.376$$

$$\theta = \arccos\left(\frac{-2 - \sqrt{10}}{6}\right) \approx 2.6068.$$

Points of intersection: $(-0.581, \pm 2.607)$, $(2.581, \pm 1.376)$

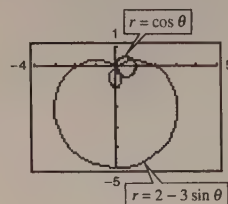
25. $r = \cos \theta$

$$r = 2 - 3 \sin \theta$$

Points of intersection:

$$(0, 0), (0.935, 0.363), (0.535, -1.006)$$

The graphs reach the pole at different times (θ values).



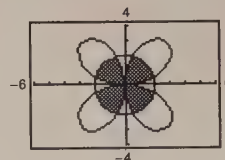
27. From Exercise 21, the points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. The area within one petal is

$$A = \frac{1}{2} \int_0^{\pi/12} (4 \sin 2\theta)^2 d\theta + \frac{1}{2} \int_{\pi/12}^{5\pi/12} (2)^2 d\theta + \frac{1}{2} \int_{5\pi/12}^{\pi/2} (4 \sin 2\theta)^2 d\theta$$

$$= 16 \int_0^{\pi/12} \sin^2(2\theta) d\theta + 2 \int_{\pi/12}^{5\pi/12} d\theta \quad (\text{by symmetry of the petal})$$

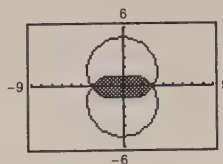
$$= 8 \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/12} + \left[2\theta \right]_{\pi/12}^{5\pi/12} = \frac{4\pi}{3} - \sqrt{3}.$$

$$\text{Total area} = 4 \left(\frac{4\pi}{3} - \sqrt{3} \right) = \frac{16\pi}{3} - 4\sqrt{3} = \frac{4}{3} (4\pi - 3\sqrt{3})$$



29. $A = 4 \left[\frac{1}{2} \int_0^{\pi/2} (3 - 2 \sin \theta)^2 d\theta \right]$

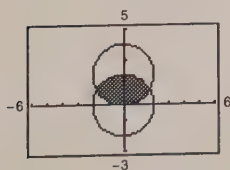
$$= 2 \left[11\theta + 12 \cos \theta - \sin(2\theta) \right]_0^{\pi/2} = 11\pi - 24$$



31. $A = 2 \left[\frac{1}{2} \int_0^{\pi/6} (4 \sin \theta)^2 d\theta + \frac{1}{2} \int_{\pi/6}^{\pi/2} (2)^2 d\theta \right]$

$$= 16 \left[\frac{1}{2} \theta - \frac{1}{4} \sin(2\theta) \right]_0^{\pi/6} + \left[4\theta \right]_{\pi/6}^{\pi/2}$$

$$= \frac{8\pi}{3} - 2\sqrt{3} = \frac{2}{3} (4\pi - 3\sqrt{3})$$

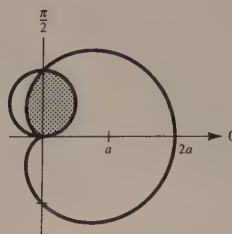


33. $A = 2 \left[\frac{1}{2} \int_0^{\pi} [a(1 + \cos \theta)]^2 d\theta \right] - \frac{a^2 \pi}{4}$

$$= a^2 \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{\sin 2\theta}{4} \right]_0^{\pi} - \frac{a^2 \pi}{4}$$

$$= \frac{3a^2 \pi}{2} - \frac{a^2 \pi}{4} = \frac{5a^2 \pi}{4}$$

$$\begin{aligned}
 35. A &= \frac{\pi a^2}{8} + \frac{1}{2} \int_{\pi/2}^{\pi} [a(1 + \cos \theta)]^2 d\theta \\
 &= \frac{\pi a^2}{8} + \frac{a^2}{2} \int_{\pi/2}^{\pi} \left(\frac{3}{2} + 2 \cos \theta + \frac{\cos 2\theta}{2} \right) d\theta \\
 &= \frac{\pi a^2}{8} + \frac{a^2}{2} \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{\sin 2\theta}{4} \right]_{\pi/2}^{\pi} \\
 &= \frac{\pi a^2}{8} + \frac{a^2}{2} \left[\frac{3\pi}{2} - \frac{3\pi}{4} - 2 \right] = \frac{a^2}{2} [\pi - 2]
 \end{aligned}$$

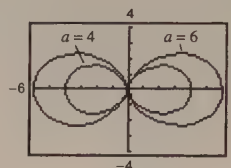


$$37. (a) \quad r = a \cos^2 \theta$$

$$r^3 = ar^2 \cos^2 \theta$$

$$(x^2 + y^2)^{3/2} = ax^2$$

(b)



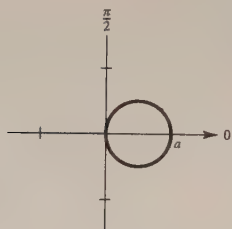
$$\begin{aligned}
 (c) A &= 4 \left(\frac{1}{2} \right) \int_0^{\pi/2} [(6 \cos^2 \theta)^2 - (4 \cos^2 \theta)^2] d\theta = 40 \int_0^{\pi/2} \cos^4 \theta d\theta = 10 \int_0^{\pi/2} (1 + \cos 2\theta)^2 d\theta \\
 &= 10 \int_0^{\pi/2} \left(1 + 2 \cos 2\theta + \frac{1 - \cos 4\theta}{2} \right) d\theta = 10 \left[\frac{3}{2} \theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right]_0^{\pi/2} = \frac{15\pi}{2}
 \end{aligned}$$

$$39. \quad r = a \cos(n\theta)$$

For $n = 1$:

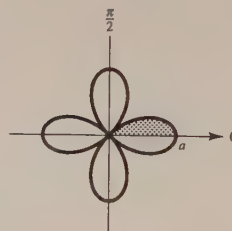
$$r = a \cos \theta$$

$$A = \pi \left(\frac{a}{2} \right)^2 = \frac{\pi a^2}{4}$$

For $n = 2$:

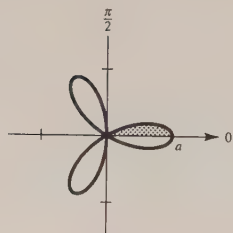
$$r = a \cos 2\theta$$

$$A = 8 \left(\frac{1}{2} \right) \int_0^{\pi/4} (a \cos 2\theta)^2 d\theta = \frac{\pi a^2}{2}$$

For $n = 3$:

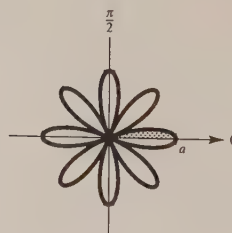
$$r = a \cos 3\theta$$

$$A = 6 \left(\frac{1}{2} \right) \int_0^{\pi/6} (a \cos 3\theta)^2 d\theta = \frac{\pi a^2}{4}$$

For $n = 4$:

$$r = a \cos 4\theta$$

$$A = 16 \left(\frac{1}{2} \right) \int_0^{\pi/8} (a \cos 4\theta)^2 d\theta = \frac{\pi a^2}{2}$$



In general, the area of the region enclosed by $r = a \cos(n\theta)$ for $n = 1, 2, 3, \dots$ is $(\pi a^2)/4$ if n is odd and is $(\pi a^2)/2$ if n is even.

41. $r = a$

$r' = 0$

$$s = \int_0^{2\pi} \sqrt{a^2 + 0^2} d\theta = \left[a\theta \right]_0^{2\pi} = 2\pi a$$

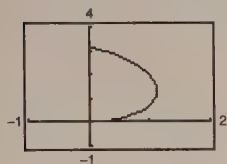
 (circumference of circle of radius a)

43. $r = 1 + \sin \theta$

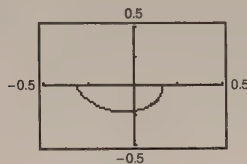
$r' = \cos \theta$

$$\begin{aligned} s &= 2 \int_{\pi/2}^{3\pi/2} \sqrt{(1 + \sin \theta)^2 + (\cos \theta)^2} d\theta \\ &= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \sqrt{1 + \sin \theta} d\theta \\ &= 2\sqrt{2} \int_{\pi/2}^{3\pi/2} \frac{-\cos \theta}{\sqrt{1 - \sin \theta}} d\theta \\ &= \left[4\sqrt{2} \sqrt{1 - \sin \theta} \right]_{\pi/2}^{3\pi/2} \\ &= 4\sqrt{2} (\sqrt{2} - 0) = 8 \end{aligned}$$

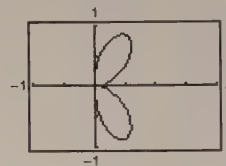
45. $r = 2\theta, 0 \leq \theta \leq \frac{\pi}{2}$


 Length ≈ 4.16

47. $r = \frac{1}{\theta}, \pi \leq \theta \leq 2\pi$


 Length ≈ 0.71

49. $r = \sin(3 \cos \theta), 0 \leq \theta \leq \pi$


 Length ≈ 4.39

51. $r = 6 \cos \theta$

$r' = -6 \sin \theta$

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} 6 \cos \theta \sin \theta \sqrt{36 \cos^2 \theta + 36 \sin^2 \theta} d\theta \\ &= 72\pi \int_0^{\pi/2} \sin \theta \cos \theta d\theta \\ &= \left[36\pi \sin^2 \theta \right]_0^{\pi/2} \\ &= 36\pi \end{aligned}$$

53. $r = e^{a\theta}$

$r' = ae^{a\theta}$

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} e^{a\theta} \cos \theta \sqrt{(e^{a\theta})^2 + (ae^{a\theta})^2} d\theta \\ &= 2\pi \sqrt{1 + a^2} \int_0^{\pi/2} e^{2a\theta} \cos \theta d\theta \\ &= 2\pi \sqrt{1 + a^2} \left[\frac{e^{2a\theta}}{4a^2 + 1} (2a \cos \theta + \sin \theta) \right]_0^{\pi/2} \\ &= \frac{2\pi \sqrt{1 + a^2}}{4a^2 + 1} (e^{\pi a} - 2a) \end{aligned}$$

55. $r = 4 \cos 2\theta$

$r' = -8 \sin 2\theta$

$$\begin{aligned} S &= 2\pi \int_0^{\pi/4} 4 \cos 2\theta \sin \theta \sqrt{16 \cos^2 2\theta + 64 \sin^2 2\theta} d\theta \\ &= 32\pi \int_0^{\pi/4} \cos 2\theta \sin \theta \sqrt{\cos^2 2\theta + 4 \sin^2 2\theta} d\theta \approx 21.87 \end{aligned}$$

57. Area $= \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$

Arc length $= \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta$

59. (a) is correct: $s \approx 33.124$.

61. Revolve
- $r = a$
- about the line
- $r = b \sec \theta$
- where
- $b > a > 0$
- .

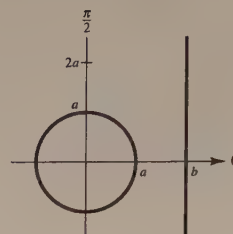
$$f(\theta) = a$$

$$f'(\theta) = 0$$

$$S = 2\pi \int_0^{2\pi} [b - a \cos \theta] \sqrt{a^2 + 0^2} d\theta$$

$$= 2\pi a \left[b\theta - a \sin \theta \right]_0^{2\pi}$$

$$= 2\pi a(2\pi b) = 4\pi^2 ab$$



63. False.
- $f(\theta) = 1$
- and
- $g(\theta) = -1$
- have the same graphs.

65. In parametric form,

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Using θ instead of t , we have $x = r \cos \theta = f(\theta) \cos \theta$ and $y = r \sin \theta = f(\theta) \sin \theta$. Thus,

$$\frac{dx}{d\theta} = f'(\theta) \cos \theta - f(\theta) \sin \theta \text{ and } \frac{dy}{d\theta} = f'(\theta) \sin \theta + f(\theta) \cos \theta.$$

It follows that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2.$$

$$\text{Therefore, } s = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$$

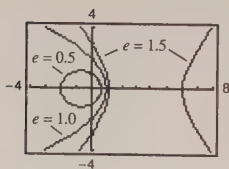
Section 9.6 Polar Equations of Conics and Kepler's Laws

$$1. r = \frac{2e}{1 + e \cos \theta}$$

$$(a) e = 1, r = \frac{2}{1 + \cos \theta}, \text{ parabola}$$

$$(b) e = 0.5, r = \frac{1}{1 + 0.5 \cos \theta} = \frac{2}{2 + \cos \theta}, \text{ ellipse}$$

$$(c) e = 1.5, r = \frac{3}{1 + 1.5 \cos \theta} = \frac{6}{2 + 3 \cos \theta}, \text{ hyperbola}$$

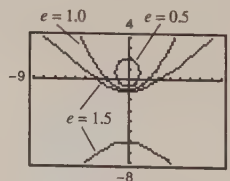


$$3. r = \frac{2e}{1 - e \sin \theta}$$

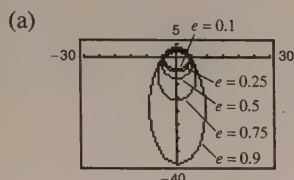
$$(a) e = 1, r = \frac{2}{1 - \sin \theta}, \text{ parabola}$$

$$(b) e = 0.5, r = \frac{1}{1 - 0.5 \sin \theta} = \frac{2}{2 - \sin \theta}, \text{ ellipse}$$

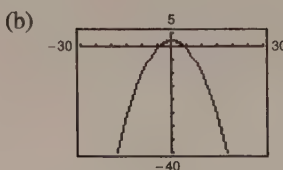
$$(c) e = 1.5, r = \frac{3}{1 - 1.5 \sin \theta} = \frac{6}{2 - 3 \sin \theta}, \text{ hyperbola}$$



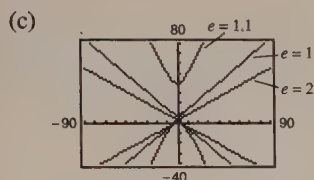
$$5. r = \frac{4}{1 + e \sin \theta}$$



The conic is an ellipse. As $e \rightarrow 1^-$, the ellipse becomes more elliptical, and as $e \rightarrow 0^+$, it becomes more circular.



The conic is a parabola.



The conic is a hyperbola. As $e \rightarrow 1^+$, the hyperbolas opens more slowly, and as $e \rightarrow \infty$, they open more rapidly.

7. Parabola; Matches (c)

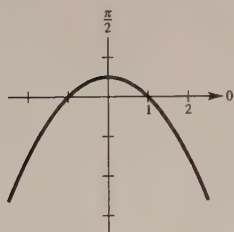
9. Hyperbola; Matches (a)

11. Ellipse; Matches (b)

$$13. r = \frac{-1}{1 - \sin \theta}$$

Parabola since $e = 1$

Vertex: $\left(-\frac{1}{2}, \frac{3\pi}{2}\right)$

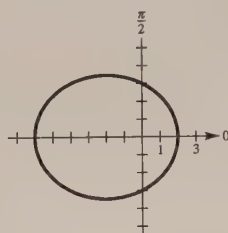


$$15. r = \frac{6}{2 + \cos \theta}$$

$$= \frac{3}{1 + (1/2) \cos \theta}$$

Ellipse since $e = \frac{1}{2} < 1$

Vertices: $(2, 0), (6, \pi)$



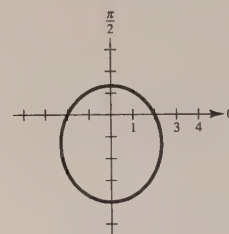
$$17. r(2 + \sin \theta) = 4$$

$$r = \frac{4}{2 + \sin \theta}$$

$$= \frac{2}{1 + (1/2) \sin \theta}$$

Ellipse since $e = \frac{1}{2} < 1$

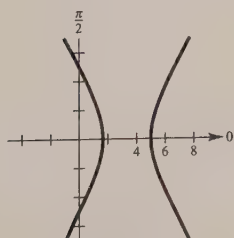
Vertices: $\left(\frac{4}{3}, \frac{\pi}{2}\right), \left(4, \frac{3\pi}{2}\right)$



$$19. r = \frac{5}{-1 + 2 \cos \theta} = \frac{-5}{1 - 2 \cos \theta}$$

Hyperbola since $e = 2 > 1$

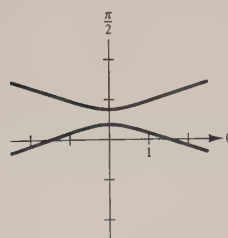
Vertices: $(5, 0), \left(-\frac{5}{3}, \pi\right)$



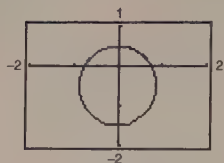
$$21. r = \frac{3}{2 + 6 \sin \theta} = \frac{3/2}{1 + 3 \sin \theta}$$

Hyperbola since $e = 3 > 1$

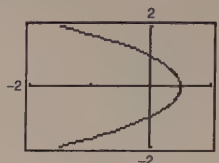
Vertices: $\left(\frac{3}{8}, \frac{\pi}{2}\right), \left(-\frac{3}{4}, \frac{3\pi}{2}\right)$



23. Ellipse



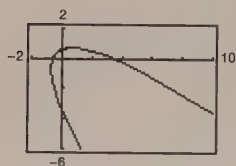
25. Parabola



$$27. r = \frac{-1}{1 - \sin\left(\theta - \frac{\pi}{4}\right)}$$

Rotate the graph of

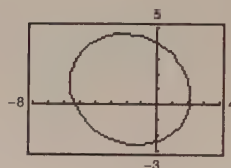
$$r = \frac{-1}{1 - \sin \theta}$$

 counterclockwise through the angle $\frac{\pi}{4}$.


$$29. r = \frac{6}{2 + \cos\left(\theta + \frac{\pi}{6}\right)}$$

Rotate the graph of

$$r = \frac{6}{2 + \cos \theta}$$

 clockwise through the angle $\frac{\pi}{6}$.


$$31. \text{ Change } \theta \text{ to } \theta + \frac{\pi}{4}; r = \frac{5}{5 + 3 \cos\left(\theta + \frac{\pi}{4}\right)}$$

33. Parabola

$$e = 1, x = -1, d = 1$$

$$r = \frac{ed}{1 - e \cos \theta} = \frac{1}{1 - \cos \theta}$$

35. Ellipse

$$e = \frac{1}{2}, y = 1, d = 1$$

$$r = \frac{ed}{1 + e \sin \theta}$$

$$= \frac{1/2}{1 + (1/2) \sin \theta}$$

$$= \frac{1}{2 + \sin \theta}$$

37. Hyperbola

$$e = 2, x = 1, d = 1$$

$$r = \frac{ed}{1 + e \cos \theta} = \frac{2}{1 + 2 \cos \theta}$$

39. Parabola

$$\text{Vertex: } \left(1, -\frac{\pi}{2}\right)$$

$$e = 1, d = 2, r = \frac{2}{1 - \sin \theta}$$

41. Ellipse

 Vertices: $(2, 0), (8, \pi)$

$$e = \frac{3}{5}, d = \frac{16}{3}$$

$$r = \frac{ed}{1 + e \cos \theta}$$

$$= \frac{16/5}{1 + (3/5) \cos \theta}$$

$$= \frac{16}{5 + 3 \cos \theta}$$

43. Hyperbola

$$\text{Vertices: } \left(1, \frac{3\pi}{2}\right), \left(9, \frac{3\pi}{2}\right)$$

$$e = \frac{5}{4}, d = \frac{9}{5}$$

$$r = \frac{ed}{1 - e \sin \theta}$$

$$= \frac{9/4}{1 - (5/4) \sin \theta}$$

$$= \frac{9}{4 - 5 \sin \theta}$$

 45. Ellipse if $0 < e < 1$, parabola if $e = 1$, hyperbola if $e > 1$.

47. (a) Hyperbola ($e = 2 > 1$)

 (b) Ellipse ($e = \frac{1}{2} < 1$)

 (c) Parabola ($e = 1$)

 (d) Rotated hyperbola ($e = 3$)

49. $a = 5, c = 4, e = \frac{4}{5}, b = 3$

$$r^2 = \frac{9}{1 - (16/25) \cos^2 \theta}$$

51. $a = 3, b = 4, c = 5, e = \frac{5}{3}$

$$r^2 = \frac{-16}{1 - (25/9) \cos^2 \theta}$$

53. $A = 2 \left[\frac{1}{2} \int_0^\pi \left(\frac{3}{2 - \cos \theta} \right)^2 d\theta \right]$

$$= 9 \int_0^\pi \frac{1}{(2 - \cos \theta)^2} d\theta \approx 10.88$$

 55. Vertices: (126,000, 0), (4119, π)

$$a = \frac{126,000 + 4119}{2} = 65,059.5, c = 65,059.5 - 4119 = 60,940.5, e = \frac{c}{a} = \frac{40,627}{43,373}, d = 4119 \left(\frac{84,000}{40,627} \right)$$

$$r = \frac{ed}{1 - e \cos \theta} = \frac{4119(84,000/43,373)}{1 - (40,627/43,373) \cos \theta} = \frac{345,996,000}{43,373 - 40,627 \cos \theta}$$

$$\text{When } \theta = 60^\circ, r = \frac{345,996,000}{23,059.5} \approx 15,004.49.$$

 Distance between the surface of the earth and the satellite is $r - 4000 = 11,004.49$ miles.

57. $a = 92.957 \times 10^6 \text{ mi}, e = 0.0167$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} = \frac{92,931,075.2223}{1 - 0.0167 \cos \theta}$$

 Perihelion distance: $a(1 - e) \approx 91,404,618 \text{ mi}$

 Aphelion distance: $a(1 + e) \approx 94,509,382 \text{ mi}$

59. $a = 5.900 \times 10^9 \text{ km}, e = 0.2481$

$$r = \frac{(1 - e^2)a}{1 - e \cos \theta} \approx \frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta}$$

 Perihelion distance: $a(1 - e) = 4.436 \times 10^9 \text{ km}$

 Aphelion distance: $a(1 + e) = 7.364 \times 10^9 \text{ km}$

61. $r = \frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta}$

(a) $A = \frac{1}{2} \int_0^{\pi/9} \left[\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right]^2 d\theta \approx 9.341 \times 10^{18} \text{ km}^2$

$$248 \left[\frac{1}{2} \int_0^{\pi/9} \left[\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right]^2 d\theta - \frac{1}{2} \int_0^{2\pi} \left[\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right]^2 d\theta \right] \approx 21.867 \text{ yr}$$

(b) $\frac{1}{2} \int_\pi^{\alpha - \pi} \left[\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right]^2 d\theta = 9.341 \times 10^{18}$

$$\alpha \approx \pi + 0.8995 \text{ rad}$$

In part (a) the ray swept through a smaller angle to generate the same area since the length of the ray is longer than in part (b).

(c) $r' = \frac{(-5.537 \times 10^9)(0.2481 \sin \theta)}{(1 - 0.2481 \cos \theta)^2}$

$$s = \int_0^{\pi/9} \sqrt{\left(\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right)^2 + \left[\frac{-1.3737297 \times 10^9 \sin \theta}{(1 - 0.2481 \cos \theta)^2} \right]^2} d\theta \approx 2.559 \times 10^9 \text{ km}$$

$$\frac{2.559 \times 10^9 \text{ km}}{21.867 \text{ yr}} \approx 1.17 \times 10^8 \text{ km/yr}$$

$$s = \int_\pi^{\pi + 0.899} \sqrt{\left(\frac{5.537 \times 10^9}{1 - 0.2481 \cos \theta} \right)^2 + \left[\frac{-1.3737297 \times 10^9 \sin \theta}{(1 - 0.2481 \cos \theta)^2} \right]^2} d\theta \approx 4.119 \times 10^9 \text{ km}$$

$$\frac{4.119 \times 10^9 \text{ km}}{21.867 \text{ yr}} \approx 1.88 \times 10^8 \text{ km/yr}$$

$$63. r_1 = \frac{ed}{1 + \sin \theta} \text{ and } r_2 = \frac{ed}{1 - \sin \theta}$$

Points of intersection: $(ed, 0)$, (ed, π)

$$r_1: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 + \sin \theta}\right)(\cos \theta) + \left(\frac{-ed \cos \theta}{(1 + \sin \theta)^2}\right)(\sin \theta)}{\left(\frac{-ed}{1 + \sin \theta}\right)(\sin \theta) + \left(\frac{-ed \cos \theta}{(1 + \sin \theta)^2}\right)(\cos \theta)}$$

At $(ed, 0)$, $\frac{dy}{dx} = -1$. At (ed, π) , $\frac{dy}{dx} = 1$.

$$r_2: \frac{dy}{dx} = \frac{\left(\frac{ed}{1 - \sin \theta}\right)(\cos \theta) + \left(\frac{ed \cos \theta}{(1 - \sin \theta)^2}\right)(\sin \theta)}{\left(\frac{-ed}{1 - \sin \theta}\right)(\sin \theta) + \left(\frac{ed \cos \theta}{(1 - \sin \theta)^2}\right)(\cos \theta)}$$

At $(ed, 0)$, $\frac{dy}{dx} = 1$. At (ed, π) , $\frac{dy}{dx} = -1$.

Therefore, at $(ed, 0)$ we have $m_1 m_2 = (-1)(1) = -1$, and at (ed, π) we have $m_1 m_2 = 1(-1) = -1$. The curves intersect at right angles.

Review Exercises for Chapter 9

1. Matches (d) - ellipse

$$5. 16x^2 + 16y^2 - 16x + 24y - 3 = 0$$

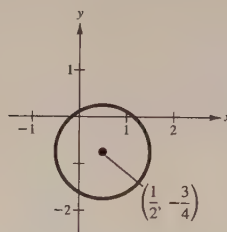
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) = \frac{3}{16} + \frac{1}{4} + \frac{9}{16}$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{4}\right)^2 = 1$$

Circle

Center: $\left(\frac{1}{2}, -\frac{3}{4}\right)$

Radius: 1



3. Matches (a) - parabola

$$7. 3x^2 - 2y^2 + 24x + 12y + 24 = 0$$

$$3(x^2 + 8x + 16) - 2(y^2 - 6y + 9) = -24 + 48 - 18$$

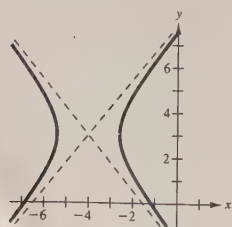
$$\frac{(x + 4)^2}{2} - \frac{(y - 3)^2}{3} = 1$$

Hyperbola

Center: $(-4, 3)$

Vertices: $(-4 \pm \sqrt{2}, 3)$

Asymptotes: $y = 3 \pm \sqrt{\frac{3}{2}}(x + 4)$



$$9. 3x^2 + 2y^2 - 12x + 12y + 29 = 0$$

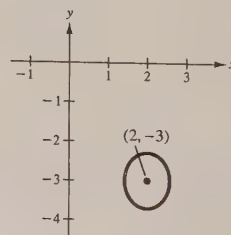
$$3(x^2 - 4x + 4) + 2(y^2 + 6y + 9) = -29 + 12 + 18$$

$$\frac{(x - 2)^2}{1/3} + \frac{(y + 3)^2}{1/2} = 1$$

Ellipse

Center: $(2, -3)$

Vertices: $\left(2, -3 \pm \frac{\sqrt{2}}{2}\right)$



11. Vertex: (0, 2)

Directrix: $x = -3$

Parabola opens to the right

 $p = 3$

$$(y - 2)^2 = 4(3)(x - 0)$$

$$y^2 - 4y - 12x + 4 = 0$$

13. Vertices: $(-3, 0), (7, 0)$ Foci: $(0, 0), (4, 0)$

Horizontal major axis

Center: $(2, 0)$

$$a = 5, c = 2, b = \sqrt{21}$$

$$\frac{(x - 2)^2}{25} + \frac{y^2}{21} = 1$$

15. Vertices: $(\pm 4, 0)$ Foci: $(\pm 6, 0)$ Center: $(0, 0)$

Horizontal transverse axis

$$a = 4, c = 6, b = \sqrt{36 - 16} = 2\sqrt{5}$$

$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$

17. $\frac{x^2}{9} + \frac{y^2}{4} = 1, a = 3, b = 2, c = \sqrt{5}, e = \frac{\sqrt{5}}{3}$

By Example 5 of Section 9.1,

$$C = 12 \int_0^{\pi/2} \sqrt{1 - \left(\frac{5}{9}\right) \sin^2 \theta} d\theta \approx 15.87.$$

19. $y = x - 2$ has a slope of 1. The perpendicular slope is -1 .

$$y = x^2 - 2x + 2$$

$$\frac{dy}{dx} = 2x - 2 = -1 \text{ when } x = \frac{1}{2} \text{ and } y = \frac{5}{4}.$$

$$\text{Perpendicular line: } y - \frac{5}{4} = -1\left(x - \frac{1}{2}\right)$$

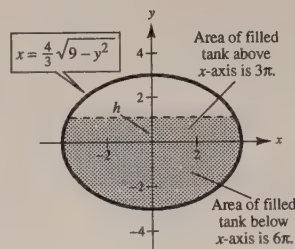
$$4x + 4y - 7 = 0$$

21. (a) $V = (\pi ab)(\text{Length}) = 12\pi(16) = 192\pi \text{ ft}^3$

$$\begin{aligned}
 \text{(b) } F &= 2(62.4) \int_{-3}^3 (3 - y) \frac{4}{3} \sqrt{9 - y^2} dy = \frac{8}{3}(62.4) \left[3 \int_{-3}^3 \sqrt{9 - y^2} dy - \int_{-3}^3 y \sqrt{9 - y^2} dy \right] \\
 &= \frac{8}{3}(62.4) \left[\frac{3}{2} \left(y \sqrt{9 - y^2} + 9 \arcsin \frac{y}{3} \right) + \frac{1}{3} (9 - y^2)^{3/2} \right]_{-3}^3 \\
 &= \frac{8}{3}(62.4) \left[\frac{3}{2} \left(\frac{9\pi}{2} \right) - \frac{3}{2} \left(-\frac{9\pi}{2} \right) \right] = \frac{8}{3}(62.4) \left(\frac{27\pi}{2} \right) \approx 7057.274
 \end{aligned}$$

(c) You want $\frac{3}{4}$ of the total area of 12π covered. Find h so that

$$\begin{aligned}
 2 \int_0^h \frac{4}{3} \sqrt{9 - y^2} dy &= 3\pi \\
 \int_0^h \sqrt{9 - y^2} dy &= \frac{9\pi}{8} \\
 \frac{1}{2} \left[y \sqrt{9 - y^2} + 9 \arcsin \left(\frac{y}{3} \right) \right]_0^h &= \frac{9\pi}{8} \\
 h \sqrt{9 - h^2} + 9 \arcsin \left(\frac{h}{3} \right) &= \frac{9\pi}{4}
 \end{aligned}$$

By Newton's Method, $h \approx 1.212$. Therefore, the total height of the water is $1.212 + 3 = 4.212$ ft.

(d) Area of ends $= 2(12\pi) = 24\pi$

Area of sides $= (\text{Perimeter})(\text{Length})$

$$\begin{aligned}
 &= 16 \int_0^{\pi/2} \left(\sqrt{1 - \left(\frac{7}{16}\right) \sin^2 \theta} \right) d\theta (16) \quad [\text{from Example 5 of Section 9.1}] \\
 &\approx 256 \left(\frac{\pi/2}{12} \right) \left[\sqrt{1 - \left(\frac{7}{16}\right) \sin^2(0)} + 4 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{8}\right)} + 2 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{4}\right)} \right. \\
 &\quad \left. + 4 \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{1 - \left(\frac{7}{16}\right) \sin^2\left(\frac{\pi}{2}\right)} \right] \approx 353.65
 \end{aligned}$$

Total area $= 24\pi + 353.65 \approx 429.05$

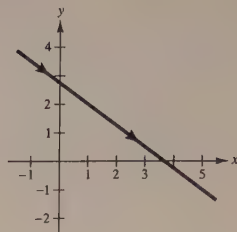
23. $x = 1 + 4t, y = 2 - 3t$

$$t = \frac{x-1}{4} \Rightarrow y = 2 - 3\left(\frac{x-1}{4}\right)$$

$$y = -\frac{3}{4}x + \frac{11}{4}$$

$$4y + 3x - 11 = 0$$

Line

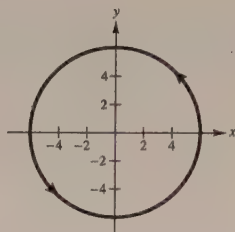


25. $x = 6 \cos \theta, y = 6 \sin \theta$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

$$x^2 + y^2 = 36$$

Circle

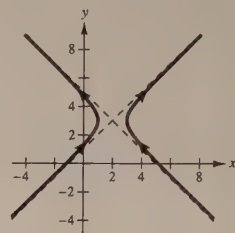


27. $x = 2 + \sec \theta, y = 3 + \tan \theta$

$$(x-2)^2 = \sec^2 \theta = 1 + \tan^2 \theta = 1 + (y-3)^2$$

$$(x-2)^2 - (y-3)^2 = 1$$

Hyperbola



29. $x = 3 + (3 - (-2))t = 3 + 5t$

$$y = 2 + (2 - 6)t = 2 - 4t$$

(other answers possible)

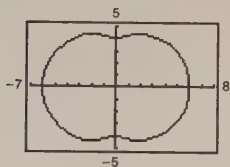
31. $\frac{(x+3)^2}{16} + \frac{(y-4)^2}{9} = 1$

$$\text{Let } \frac{(x+3)^2}{16} = \cos^2 \theta \text{ and } \frac{(y-4)^2}{9} = \sin^2 \theta.$$

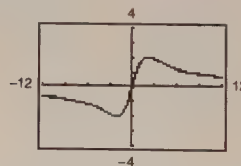
$$\text{Then } x = -3 + 4 \cos \theta \text{ and } y = 4 + 3 \sin \theta.$$

33. $x = \cos 3\theta + 5 \cos \theta$

$$y = \sin 3\theta + 5 \sin \theta$$



35. (a) $x = 2 \cot \theta, y = 4 \sin \theta \cos \theta, 0 < \theta < \pi$



(b) $(4 + x^2)y = (4 + 4 \cot^2 \theta)4 \sin \theta \cos \theta$

$$= 16 \csc^2 \theta \cdot \sin \theta \cdot \cos \theta$$

$$= 16 \frac{\cos \theta}{\sin \theta}$$

$$= 8(2 \cot \theta)$$

$$= 8x$$

37. $x = 1 + 4t$

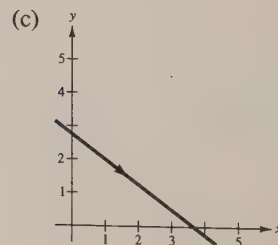
$$y = 2 - 3t$$

(a) $\frac{dy}{dx} = -\frac{3}{4}$

No horizontal tangents

(b) $t = \frac{x-1}{4}$

$$y = 2 - \frac{3}{4}(x-1) = \frac{-3x+11}{4}$$



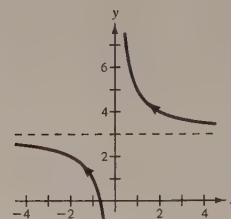
39. $x = \frac{1}{t}$

$y = 2t + 3$

(a) $\frac{dy}{dx} = \frac{2}{-1/t^2} = -2t^2$

(b) $t = \frac{1}{x}$

(c)



No horizontal tangents
($t \neq 0$)

$y = \frac{2}{x} + 3$

41. $x = \frac{1}{2t + 1}$

$y = \frac{1}{t^2 - 2t}$

(a) $\frac{dy}{dx} = \frac{\frac{-(2t-2)}{(t^2-2t)^2}}{\frac{-2}{(2t+1)^2}} = \frac{(t-1)(2t+1)^2}{t^2(t-2)^2} = 0$ when $t = 1$.

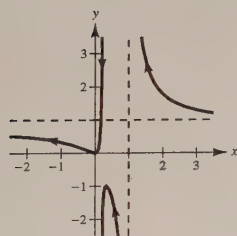
Point of horizontal tangency: $(\frac{1}{3}, -1)$

(b) $2t + 1 = \frac{1}{x} \Rightarrow t = \frac{1}{2}\left(\frac{1}{x} - 1\right)$

$$y = \frac{1}{\frac{1}{2}\left(\frac{1-x}{x}\right)\left[\frac{1}{2}\left(\frac{1-x}{x}\right) - 2\right]}$$

$$= \frac{4x^2}{(1-x)^2 - 4x(1-x)} = \frac{4x^2}{(5x-1)(x-1)}, x \neq 0$$

(c)



45. $x = \cos^3 \theta$

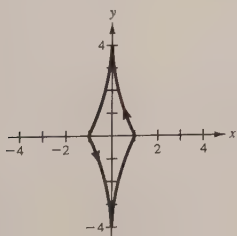
$y = 4 \sin^3 \theta$

(a) $\frac{dy}{dx} = \frac{12 \sin^2 \theta \cos \theta}{3 \cos^2 \theta (-\sin \theta)} = \frac{-4 \sin \theta}{\cos \theta} = -4 \tan \theta = 0$ when $\theta = 0, \pi$.

But, $\frac{dy}{dt} = \frac{dx}{dt} = 0$ at $\theta = 0, \pi$. Hence no points of horizontal tangency.

(b) $x^{2/3} + \left(\frac{y}{4}\right)^{2/3} = 1$

(c)



43. $x = 3 + 2 \cos \theta$

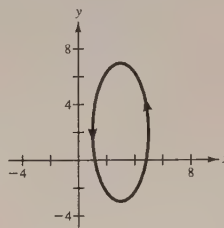
$y = 2 + 5 \sin \theta$

(a) $\frac{dy}{dx} = \frac{5 \cos \theta}{-2 \sin \theta} = -2.5 \cot \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of horizontal tangency: $(3, 7), (3, -3)$

(b) $\frac{(x-3)^2}{4} + \frac{(y-2)^2}{25} = 1$

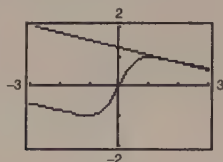
(c)



47. $x = \cot \theta$

$$y = \sin 2\theta = 2 \sin \theta \cos \theta$$

(a), (c)



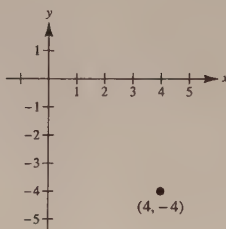
(b) At $\theta = \frac{\pi}{6}$, $\frac{dx}{d\theta} = -4$, $\frac{dy}{d\theta} = 1$, and $\frac{dy}{dx} = -\frac{1}{4}$

51. $(x, y) = (4, -4)$

$$r = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$$

$$\theta = 7\frac{\pi}{4}$$

$$(r, \theta) = \left(4\sqrt{2}, \frac{7\pi}{4}\right), \left(-4\sqrt{2}, \frac{3\pi}{4}\right)$$



53. $r = 3 \cos \theta$

$$r^2 = 3r \cos \theta$$

$$x^2 + y^2 = 3x$$

$$x^2 + y^2 - 3x = 0$$

57. $r^2 = \cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$$r^4 = r^2 \cos^2 \theta - r^2 \sin^2 \theta$$

$$(x^2 + y^2)^2 = x^2 - y^2$$

49. $x = r(\cos \theta + \theta \sin \theta)$

$$y = r(\sin \theta - \theta \cos \theta)$$

$$\frac{dx}{d\theta} = r\theta \cos \theta$$

$$\frac{dy}{d\theta} = r\theta \sin \theta$$

$$s = r \int_0^\pi \sqrt{\theta^2 \cos^2 \theta + \theta^2 \sin^2 \theta} d\theta$$

$$= r \int_0^\pi \theta d\theta = \frac{r}{2} [\theta^2]_0^\pi = \frac{1}{2} \pi^2 r$$

55. $r = -2(1 + \cos \theta)$

$$r^2 = -2r(1 + \cos \theta)$$

$$x^2 + y^2 = -2(\pm \sqrt{x^2 + y^2}) - 2x$$

$$(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$$

59. $r = 4 \cos 2\theta \sec \theta$

$$= 4(2 \cos^2 \theta - 1) \left(\frac{1}{\cos \theta} \right)$$

$$r \cos \theta = 8 \cos^2 \theta - 4$$

$$x = 8 \left(\frac{x^2}{x^2 + y^2} \right) - 4$$

$$x^3 + xy^2 = 4x^2 - 4y^2$$

$$y^2 = x^2 \left(\frac{4-x}{4+x} \right)$$

61. $(x^2 + y^2)^2 = ax^2y$

$$r^4 = a(r^2 \cos^2 \theta)(r \sin \theta)$$

$$r = a \cos^2 \theta \sin \theta$$

63. $x^2 + y^2 = a^2 \left(\arctan \frac{y}{x} \right)^2$

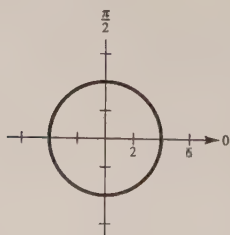
$$r^2 = a^2 \theta^2$$

65. $r = 4$

Circle of radius 4

Centered at the pole

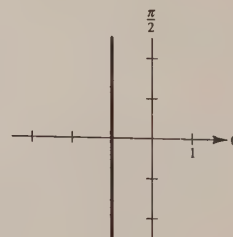
Symmetric to polar axis,

 $\theta = \pi/2$, and pole


67. $r = -\sec \theta = \frac{-1}{\cos \theta}$

$$r \cos \theta = -1, x = -1$$

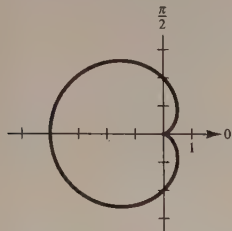
Vertical line



69. $r = -2(1 + \cos \theta)$

Cardioid

Symmetric to polar axis

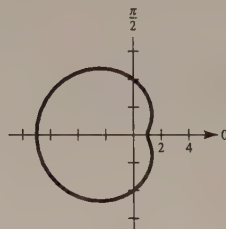


θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	-4	-3	-2	-1	0

71. $r = 4 - 3 \cos \theta$

Limaçon

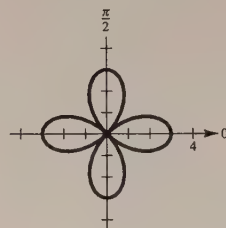
Symmetric to polar axis



θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	$\frac{5}{2}$	4	$\frac{11}{2}$	7

73. $r = -3 \cos(2\theta)$

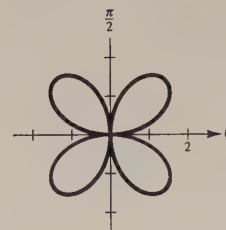
Rose curve with four petals

Symmetric to polar axis, $\theta = \frac{\pi}{2}$, and poleRelative extrema: $(-3, 0)$, $(3, \frac{\pi}{2})$, $(-3, \pi)$, $(3, \frac{3\pi}{2})$ Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ 

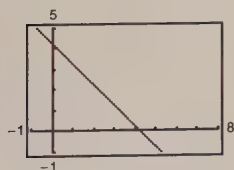
75. $r^2 = 4 \sin^2(2\theta)$

$r = \pm 2 \sin(2\theta)$

Rose curve with four petals

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and poleRelative extrema: $(\pm 2, \frac{\pi}{4})$, $(\pm 2, \frac{3\pi}{4})$ Tangents at the pole: $\theta = 0, \frac{\pi}{2}$ 

77. $r = \frac{3}{\cos[\theta - (\pi/4)]}$

Graph of $r = 3 \sec \theta$ rotated through an angle of $\pi/4$ 

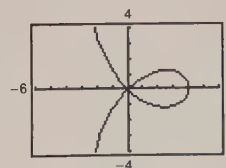
79. $r = 4 \cos 2\theta \sec \theta$

Strophoid

Symmetric to the polar axis

$r \Rightarrow -\infty$ as $\theta \Rightarrow \frac{\pi^-}{2}$

$r \Rightarrow -\infty$ as $\theta \Rightarrow \frac{-\pi^+}{2}$



81. $r = 1 - 2 \cos \theta$

(a) The graph has polar symmetry and the tangents at the pole are

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}.$$

$$(b) \frac{dy}{dx} = \frac{2 \sin^2 \theta + (1 - 2 \cos \theta) \cos \theta}{2 \sin \theta \cos \theta - (1 - 2 \cos \theta) \sin \theta}$$

$$\text{Horizontal tangents: } -4 \cos^2 \theta + \cos \theta + 2 = 0, \cos \theta = \frac{-1 \pm \sqrt{1 + 32}}{-8} = \frac{1 \pm \sqrt{33}}{8}$$

$$\text{When } \cos \theta = \frac{1 \pm \sqrt{33}}{8}, r = 1 - 2 \left(\frac{1 \pm \sqrt{33}}{8} \right) = \frac{3 \mp \sqrt{33}}{4},$$

$$\left[\frac{3 - \sqrt{33}}{4}, \arccos \left(\frac{1 + \sqrt{33}}{8} \right) \right] \approx (-0.686, 0.568)$$

$$\left[\frac{3 - \sqrt{33}}{4}, -\arccos \left(\frac{1 + \sqrt{33}}{8} \right) \right] \approx (-0.686, -0.568)$$

$$\left[\frac{3 + \sqrt{33}}{4}, \arccos \left(\frac{1 - \sqrt{33}}{8} \right) \right] \approx (2.186, 2.206)$$

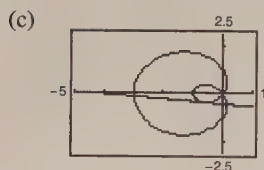
$$\left[\frac{3 + \sqrt{33}}{4}, -\arccos \left(\frac{1 - \sqrt{33}}{8} \right) \right] \approx (2.186, -2.206).$$

Vertical tangents:

$$\sin \theta (4 \cos \theta - 1) = 0, \sin \theta = 0, \cos \theta = \frac{1}{4},$$

$$\theta = 0, \pi, \theta = \pm \arccos \left(\frac{1}{4} \right), (-1, 0), (3, \pi)$$

$$\left(\frac{1}{2}, \pm \arccos \frac{1}{4} \right) \approx (0.5, \pm 1.318)$$



83. Circle: $r = 3 \sin \theta$

$$\frac{dy}{dx} = \frac{3 \cos \theta \sin \theta + 3 \sin \theta \cos \theta}{3 \cos \theta \cos \theta - 3 \sin \theta \sin \theta} = \frac{\sin 2\theta}{\cos^2 \theta - \sin^2 \theta} = \tan 2\theta \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \sqrt{3}$$

Limaçon: $r = 4 - 5 \sin \theta$

$$\frac{dy}{dx} = \frac{-5 \cos \theta \sin \theta + (4 - 5 \sin \theta) \cos \theta}{-5 \cos \theta \cos \theta - (4 - 5 \sin \theta) \sin \theta} \text{ at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \frac{\sqrt{3}}{9}$$

Let α be the angle between the curves:

$$\tan \alpha = \frac{\sqrt{3} - (\sqrt{3}/9)}{1 + (1/3)} = \frac{2\sqrt{3}}{3}.$$

$$\text{Therefore, } \alpha = \arctan \left(\frac{2\sqrt{3}}{3} \right) \approx 49.1^\circ.$$

85. $r = 1 + \cos \theta, r = 1 - \cos \theta$

The points $(1, \pi/2)$ and $(1, 3\pi/2)$ are the two points of intersection (other than the pole). The slope of the graph of $r = 1 + \cos \theta$ is

$$m_1 = \frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta} = \frac{-\sin^2 \theta + \cos \theta(1 + \cos \theta)}{-\sin \theta \cos \theta - \sin \theta(1 + \cos \theta)}.$$

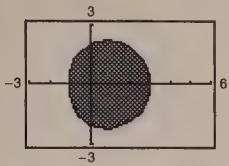
At $(1, \pi/2)$, $m_1 = -1/-1 = 1$ and at $(1, 3\pi/2)$, $m_1 = -1/1 = -1$. The slope of the graph of $r = 1 - \cos \theta$ is

$$m_2 = \frac{dy}{dx} = \frac{\sin^2 \theta + \cos \theta(1 - \cos \theta)}{\sin \theta \cos \theta - \sin \theta(1 - \cos \theta)}.$$

At $(1, \pi/2)$, $m_2 = 1/-1 = -1$ and at $(1, 3\pi/2)$, $m_2 = 1/1 = 1$. In both cases, $m_1 = -1/m_2$ and we conclude that the graphs are orthogonal at $(1, \pi/2)$ and $(1, 3\pi/2)$.

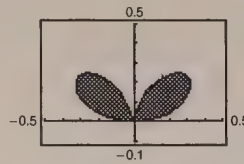
87. $r = 2 + \cos \theta$

$$A = 2 \left[\frac{1}{2} \int_0^\pi (2 + \cos \theta)^2 d\theta \right] \approx 14.14 \quad \left(\frac{9\pi}{2} \right)$$



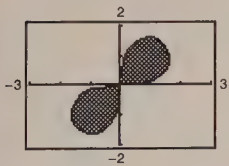
89. $r = \sin \theta \cdot \cos^2 \theta$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/2} (\sin \theta \cos^2 \theta)^2 d\theta \right] \approx 0.10 \quad \left(\frac{\pi}{32} \right)$$



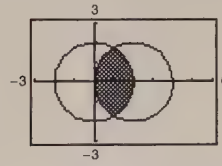
91. $r^2 = 4 \sin 2\theta$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/2} 4 \sin 2\theta d\theta \right] = 4$$



93. $r = 4 \cos \theta, r = 2$

$$A = 2 \left[\frac{1}{2} \int_0^{\pi/3} 4 d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} (4 \cos \theta)^2 d\theta \right] \approx 4.91$$

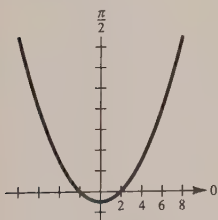


95. $s = 2 \int_0^\pi \sqrt{a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta} d\theta$

$$= 2\sqrt{2} a \int_0^\pi \sqrt{1 - \cos \theta} d\theta = 2\sqrt{2} a \int_0^\pi \frac{\sin \theta}{\sqrt{1 + \cos \theta}} d\theta = \left[-4\sqrt{2} a(1 + \cos \theta)^{1/2} \right]_0^\pi = 8a$$

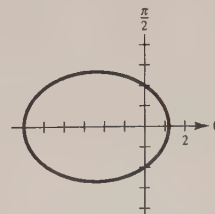
97. $r = \frac{2}{1 - \sin \theta}, e = 1$

Parabola



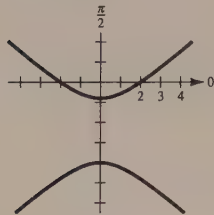
99. $r = \frac{6}{3 + 2 \cos \theta} = \frac{2}{1 + (2/3) \cos \theta}, e = \frac{2}{3}$

Ellipse



$$101. r = \frac{4}{2 - 3 \sin \theta} = \frac{2}{1 - (3/2) \sin \theta}, e = \frac{3}{2}$$

Hyperbola



105. Parabola

Vertex: $(2, \pi)$ Focus: $(0, 0)$

$$e = 1, d = 4$$

$$r = \frac{4}{1 - \cos \theta}$$

103. Circle

Center: $\left(5, \frac{\pi}{2}\right) = (0, 5)$ in rectangular coordinatesSolution point: $(0, 0)$

$$x^2 + (y - 5)^2 = 25$$

$$x^2 + y^2 - 10y = 0$$

$$r^2 - 10r \sin \theta = 0$$

$$r = 10 \sin \theta$$

107. Ellipse

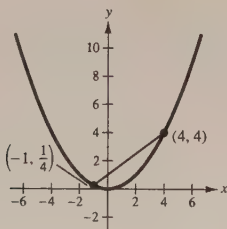
Vertices: $(5, 0), (1, \pi)$ Focus: $(0, 0)$

$$a = 3, c = 2, e = \frac{2}{3}, d = \frac{5}{2}$$

$$r = \frac{\left(\frac{2}{3}\right)\left(\frac{5}{2}\right)}{1 - \left(\frac{2}{3}\right) \cos \theta} = \frac{5}{3 - 2 \cos \theta}$$

Problem Solving for Chapter 9

1. (a)



$$(b) x^2 = 4y$$

$$2x = 4y'$$

$$y' = \frac{1}{2}x$$

$$y - 4 = 2(x - 4) \Rightarrow y = 2x - 4 \quad \text{Tangent line at } (4, 4)$$

$$y - \frac{1}{4} = -\frac{1}{2}(x + 1) \Rightarrow y = -\frac{1}{2}x - \frac{1}{4} \quad \text{Tangent line at } \left(-1, \frac{1}{4}\right)$$

Tangent lines have slopes of 2 and $-1/2 \Rightarrow$ perpendicular.

(c) Intersection:

$$2x - 4 = -\frac{1}{2}x - \frac{1}{4}$$

$$8x - 16 = -2x - 1$$

$$10x = 15$$

$$x = \frac{3}{2} \Rightarrow \left(\frac{3}{2}, -1\right)$$

Point of intersection, $(3/2, -1)$, is on directrix $y = -1$.

3. Consider $x^2 = 4py$ with focus $F = (0, p)$.

Let $P = (a, b)$ be point on parabola.

$$2x = 4py' \Rightarrow y' = \frac{x}{2p}$$

$$y - b = \frac{a}{2p}(x - a) \quad \text{Tangent line}$$

$$\text{For } x = 0, y = b + \frac{a}{2p}(-a) = b - \frac{a^2}{2p} = b - \frac{4pb}{2p} = -b.$$

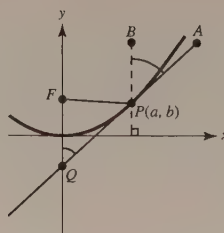
Thus, $Q = (0, -b)$.

$\triangle FQP$ is isosceles because

$$|FQ| = p + b$$

$$\begin{aligned} |FP| &= \sqrt{(a-0)^2 + (b-p)^2} = \sqrt{a^2 + b^2 - 2bp + p^2} \\ &= \sqrt{4pb + b^2 - 2bp + p^2} \\ &= \sqrt{(b+p)^2} \\ &= b + p. \end{aligned}$$

Thus, $\angle FQP = \angle BPA = \angle FPQ$.



5. (a) In $\triangle OCB$, $\cos \theta = \frac{2a}{OB} \Rightarrow OB = 2a \cdot \sec \theta$.

$$\text{In } \triangle OAC, \cos \theta = \frac{OA}{2a} \Rightarrow OA = 2a \cdot \cos \theta.$$

$$r = OP = AB = OB - OA = 2a(\sec \theta - \cos \theta)$$

$$= 2a \left(\frac{1}{\cos \theta} - \cos \theta \right)$$

$$= 2a \cdot \frac{\sin^2 \theta}{\cos \theta}$$

$$= 2a \cdot \tan \theta \sin \theta$$

$$(b) x = r \cos \theta = (2a \tan \theta \sin \theta) \cos \theta = 2a \sin^2 \theta$$

$$y = r \sin \theta = (2a \tan \theta \sin \theta) \sin \theta = 2a \tan \theta \cdot \sin^2 \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Let $t = \tan \theta, -\infty < t < \infty$.

$$\text{Then } \sin^2 \theta = \frac{t^2}{1+t^2} \text{ and } x = 2a \frac{t^2}{1+t^2}, y = 2a \frac{t^3}{1+t^2}.$$

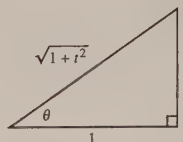
$$(c) \quad r = 2a \tan \theta \sin \theta$$

$$r \cos \theta = 2a \sin^2 \theta$$

$$r^3 \cos \theta = 2a r^2 \sin^2 \theta$$

$$(x^2 + y^2)x = 2ay^2$$

$$y^2 = \frac{x^3}{(2a-x)}$$



7. $y = a(1 - \cos \theta) \Rightarrow \cos \theta = \frac{a-y}{a}$

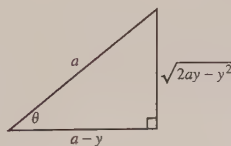
$$\theta = \arccos\left(\frac{a-y}{a}\right)$$

$$x = a(\theta - \sin \theta)$$

$$= a \left(\arccos\left(\frac{a-y}{a}\right) - \sin\left(\arccos\left(\frac{a-y}{a}\right)\right) \right)$$

$$= a \left(\arccos\left(\frac{a-y}{a}\right) - \frac{\sqrt{2ay-y^2}}{a} \right)$$

$$x = a \cdot \arccos\left(\frac{a-y}{a}\right) - \sqrt{2ay-y^2}, 0 \leq y \leq 2a$$



9. For $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$

$$y = \frac{2}{\pi}, \frac{-2}{3\pi}, \frac{2}{5\pi}, \frac{-2}{7\pi}, \dots$$

Hence, the curve has length greater than

$$\begin{aligned} S &= \frac{2}{\pi} + \frac{2}{3\pi} + \frac{2}{5\pi} + \frac{2}{7\pi} + \dots \\ &= \frac{2}{\pi} \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots \right) \\ &> \frac{2}{\pi} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots \right) \\ &= \infty. \end{aligned}$$

13. If a dog is located at (r, θ) in the first quadrant, then its neighbor is at $\left(r, \theta + \frac{\pi}{2}\right)$:

$$(x, y) = (r \cos \theta, r \sin \theta) \text{ and } (x, y) = (-r \sin \theta, r \cos \theta).$$

The slope joining these points is

$$\frac{r \cos \theta - r \sin \theta}{-r \sin \theta - r \cos \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta} = \text{slope of tangent line at } (r, \theta).$$

$$\frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}$$

$$\Rightarrow \frac{dr}{d\theta} = -r$$

$$\frac{dr}{r} = -d\theta$$

$$\ln r = -\theta + C_1$$

$$r = e^{-\theta + C_1}$$

$$r = Ce^{-\theta}$$

$$r\left(\frac{\pi}{4}\right) = \frac{d}{\sqrt{2}} \Rightarrow r = Ce^{-\pi/4} = \frac{d}{\sqrt{2}} \Rightarrow C = \frac{d}{\sqrt{2}} e^{\pi/4}$$

$$\text{Finally, } r = \frac{d}{\sqrt{2}} e^{((\pi/4) - \theta)}, \theta \geq \frac{\pi}{4}.$$

15. (a) The first plane makes an angle of 70° with the positive x -axis, and is 150 miles from P :

$$x_1 = \cos 70^\circ(150 - 375t)$$

$$y_1 = \sin 70^\circ(150 - 375t)$$

Similarly for the second plane,

$$x_2 = \cos 135^\circ(190 - 450t)$$

$$= \cos 45^\circ(-190 + 450t)$$

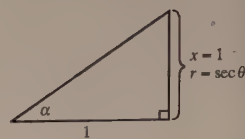
$$y_2 = \sin 135^\circ(190 - 450t)$$

$$= \sin 45^\circ(190 - 450t)$$

(b) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= [\cos 45^\circ(-190 + 450t) - \cos 70^\circ(150 - 375t)]^2 + [\sin 45^\circ(190 - 450t) - \sin 70^\circ(150 - 375t)]^2]^{1/2}$$

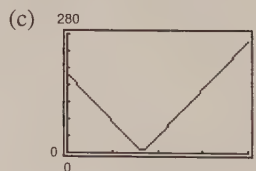
11. (a) $\text{Area} = \int_0^\alpha \frac{1}{2} r^2 d\theta$
 $= \frac{1}{2} \int_0^\alpha \sec^2 \theta d\theta$



(b) $\tan \alpha = \frac{h}{1} \Rightarrow \text{Area} = \frac{1}{2}(1)\tan \alpha$

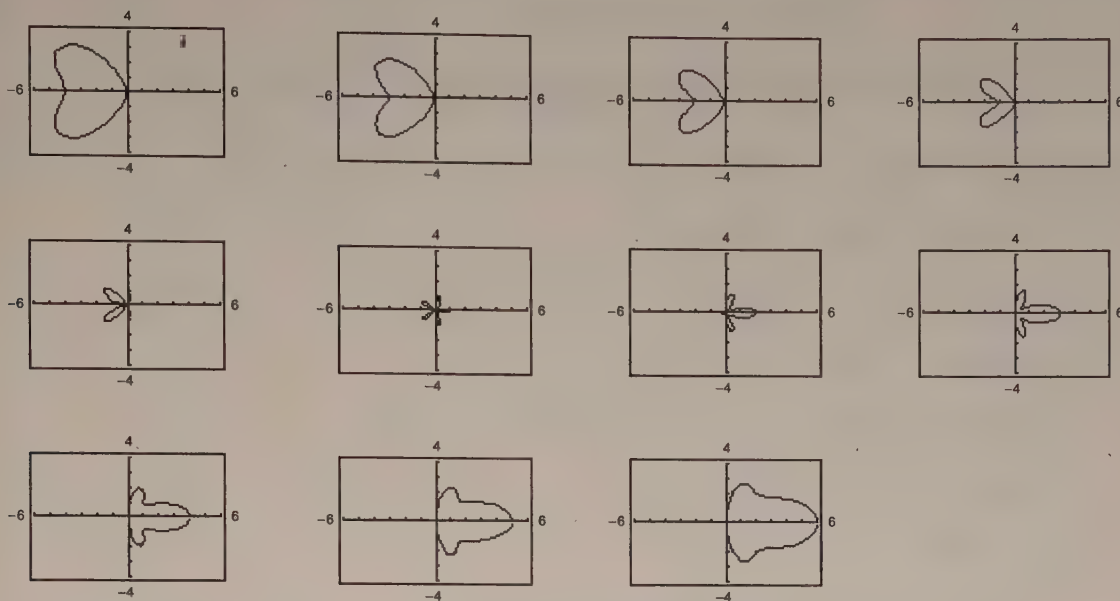
$$\Rightarrow \tan \alpha = \int_0^\alpha \sec^2 \theta d\theta$$

(c) Differentiating, $\frac{d}{d\alpha}(\tan \alpha) = \sec^2 \alpha$.



The minimum distance is 7.59 miles when $t = 0.4145$.

17.



$n = 1, 2, 3, 4, 5$ produce "bells"; $n = -1, -2, -3, -4, -5$ produce "hearts".

C H A P T E R 1 0

Vectors and the Geometry of Space

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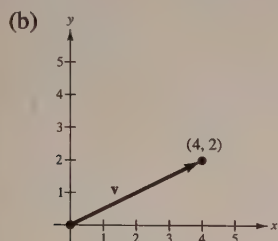
CHAPTER 10

Vectors and the Geometry of Space

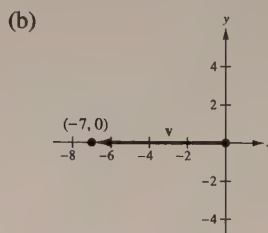
Section 10.1 Vectors in the Plane

Solutions to Odd-Numbered Exercises

1. (a) $\mathbf{v} = \langle 5 - 1, 3 - 1 \rangle = \langle 4, 2 \rangle$



3. (a) $\mathbf{v} = \langle -4 - 3, -2 - (-2) \rangle = \langle -7, 0 \rangle$



5. $\mathbf{u} = \langle 5 - 3, 6 - 2 \rangle = \langle 2, 4 \rangle$

$\mathbf{v} = \langle 1 - (-1), 8 - 4 \rangle = \langle 2, 4 \rangle$

$\mathbf{u} = \mathbf{v}$

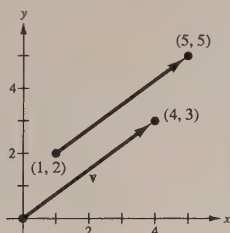
7. $\mathbf{u} = \langle 6 - 0, -2 - 3 \rangle = \langle 6, -5 \rangle$

$\mathbf{v} = \langle 9 - 3, 5 - 10 \rangle = \langle 6, -5 \rangle$

$\mathbf{u} = \mathbf{v}$

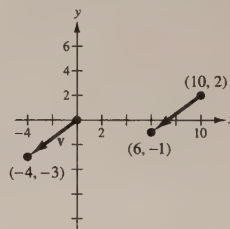
9. (b) $\mathbf{v} = \langle 5 - 1, 5 - 2 \rangle = \langle 4, 3 \rangle$

(a) and (c).



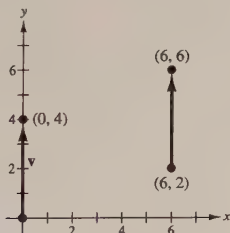
11. (b) $\mathbf{v} = \langle 6 - 10, -1 - 2 \rangle = \langle -4, -3 \rangle$

(a) and (c).



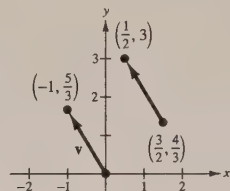
13. (b) $\mathbf{v} = \langle 6 - 6, 6 - 2 \rangle = \langle 0, 4 \rangle$

(a) and (c).

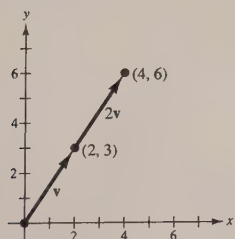


15. (b) $\mathbf{v} = \langle \frac{1}{2} - \frac{3}{2}, 3 - \frac{4}{3} \rangle = \langle -1, \frac{5}{3} \rangle$

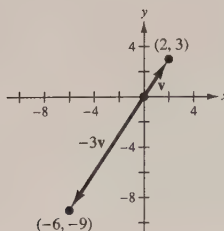
(a) and (c).



17. (a) $2\mathbf{v} = \langle 4, 6 \rangle$



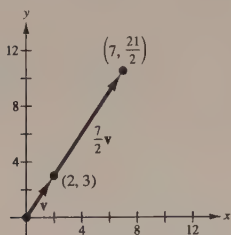
(b) $-3\mathbf{v} = \langle -6, -9 \rangle$



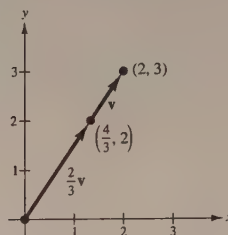
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17. —CONTINUED—

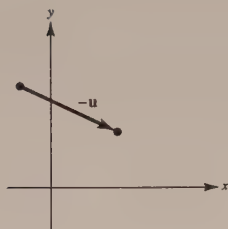
(c) $\frac{7}{2}\mathbf{v} = \left\langle 7, \frac{21}{2} \right\rangle$



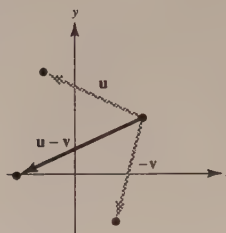
(d) $\frac{2}{3}\mathbf{v} = \left\langle \frac{4}{3}, 2 \right\rangle$



19.



21.

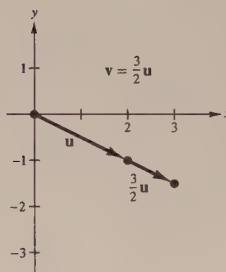


23. (a) $\frac{2}{3}\mathbf{u} = \frac{2}{3}\langle 4, 9 \rangle = \left\langle \frac{8}{3}, 6 \right\rangle$

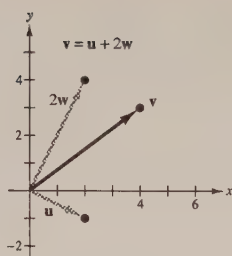
(b) $\mathbf{v} - \mathbf{u} = \langle 2, -5 \rangle - \langle 4, 9 \rangle = \langle -2, -14 \rangle$

(c) $2\mathbf{u} + 5\mathbf{v} = 2\langle 4, 9 \rangle + 5\langle 2, -5 \rangle = \langle 18, -7 \rangle$

25. $\mathbf{v} = \frac{3}{2}(2\mathbf{i} - \mathbf{j}) = 3\mathbf{i} - \frac{3}{2}\mathbf{j}$
 $= \left\langle 3, -\frac{3}{2} \right\rangle$



27. $\mathbf{v} = (2\mathbf{i} - \mathbf{j}) + 2(\mathbf{i} + 2\mathbf{j})$
 $= 4\mathbf{i} + 3\mathbf{j} = \langle 4, 3 \rangle$



29. $u_1 - 4 = -1$

$u_2 - 2 = 3$

$u_1 = 3$

$u_2 = 5$

$Q = (3, 5)$

31. $\|\mathbf{v}\| = \sqrt{16 + 9} = 5$

33. $\|\mathbf{v}\| = \sqrt{36 + 25} = \sqrt{61}$

35. $\|\mathbf{v}\| = \sqrt{0 + 16} = 4$

37. $\|\mathbf{u}\| = \sqrt{3^2 + 12^2} = \sqrt{153}$

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle 3, 12 \rangle}{\sqrt{153}} = \left\langle \frac{3}{\sqrt{153}}, \frac{12}{\sqrt{153}} \right\rangle$$

$$= \left\langle \frac{\sqrt{17}}{17}, \frac{4\sqrt{17}}{17} \right\rangle \text{ unit vector}$$

39. $\|\mathbf{u}\| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \frac{\sqrt{34}}{2}$

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle (3/2), (5/2) \rangle}{\sqrt{34}/2} = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$$

$$= \left\langle \frac{3\sqrt{34}}{34}, \frac{5\sqrt{34}}{34} \right\rangle \text{ unit vector}$$

$$41. \|\mathbf{u}\| = \langle 1, -1 \rangle, \mathbf{v} = \langle -1, 2 \rangle$$

$$(a) \|\mathbf{u}\| = \sqrt{1+1} = \sqrt{2}$$

$$(b) \|\mathbf{v}\| = \sqrt{1+4} = \sqrt{5}$$

$$(c) \mathbf{u} + \mathbf{v} = \langle 0, 1 \rangle$$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{0+1} = 1$$

$$(d) \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$$

$$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$$

$$(e) \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{5}} \langle -1, 2 \rangle$$

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$$

$$(f) \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \langle 0, 1 \rangle$$

$$\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$$

$$45. \mathbf{u} = \langle 2, 1 \rangle$$

$$\|\mathbf{u}\| = \sqrt{5} \approx 2.236$$

$$\mathbf{v} = \langle 5, 4 \rangle$$

$$\|\mathbf{v}\| = \sqrt{41} \approx 6.403$$

$$\mathbf{u} + \mathbf{v} = \langle 7, 5 \rangle$$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{74} \approx 8.602$$

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

$$49. \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{2\sqrt{3}} \langle \sqrt{3}, 3 \rangle$$

$$2\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = \frac{1}{\sqrt{3}} \langle \sqrt{3}, 3 \rangle$$

$$\mathbf{v} = \langle 1, \sqrt{3} \rangle$$

$$53. \mathbf{v} = 2[(\cos 150^\circ)\mathbf{i} + (\sin 150^\circ)\mathbf{j}]$$

$$= -\sqrt{3}\mathbf{i} + \mathbf{j} = \langle -\sqrt{3}, 1 \rangle$$

$$43. \mathbf{u} = \left\langle 1, \frac{1}{2} \right\rangle, \mathbf{v} = \langle 2, 3 \rangle$$

$$(a) \|\mathbf{u}\| = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$$

$$(b) \|\mathbf{v}\| = \sqrt{4+9} = \sqrt{13}$$

$$(c) \mathbf{u} + \mathbf{v} = \left\langle 3, \frac{7}{2} \right\rangle$$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{9 + \frac{49}{4}} = \frac{\sqrt{85}}{2}$$

$$(d) \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{2}{\sqrt{5}} \left\langle 1, \frac{1}{2} \right\rangle$$

$$\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1$$

$$(e) \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{13}} \langle 2, 3 \rangle$$

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = 1$$

$$(f) \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} = \frac{2}{\sqrt{85}} \left\langle 3, \frac{7}{2} \right\rangle$$

$$\left\| \frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\| = 1$$

$$47. \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$$

$$4\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = 2\sqrt{2} \langle 1, 1 \rangle$$

$$\mathbf{v} = \langle 2\sqrt{2}, 2\sqrt{2} \rangle$$

$$51. \mathbf{v} = 3[(\cos 0^\circ)\mathbf{i} + (\sin 0^\circ)\mathbf{j}] = 3\mathbf{i} = \langle 3, 0 \rangle$$

$$55. \mathbf{u} = \mathbf{i}$$

$$\mathbf{v} = \frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{u} + \mathbf{v} = \left(\frac{2 + 3\sqrt{2}}{2}\right)\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$$

57. $\mathbf{u} = 2(\cos 4)\mathbf{i} + 2(\sin 4)\mathbf{j}$

$\mathbf{v} = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j}$

$\mathbf{u} + \mathbf{v} = (2 \cos 4 + \cos 2)\mathbf{i} + (2 \sin 4 + \sin 2)\mathbf{j}$

61. To normalize $\mathbf{v} \neq 0$, you find a unit vector \mathbf{u} in the direction of \mathbf{v} :

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

For Exercises 63–67, $a\mathbf{u} + b\mathbf{w} = a(\mathbf{i} + 2\mathbf{j}) + b(\mathbf{i} - \mathbf{j}) = (a + b)\mathbf{i} + (2a - b)\mathbf{j}$.

63. $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$. Therefore, $a + b = 2$, $2a - b = 1$. Solving simultaneously, we have $a = 1$, $b = 1$.

67. $\mathbf{v} = \mathbf{i} + \mathbf{j}$. Therefore, $a + b = 1$, $2a - b = 1$. Solving simultaneously, we have $a = \frac{2}{3}$, $b = \frac{1}{3}$.

69. $y = x^3$, $y' = 3x^2 = 3$ at $x = 1$.

(a) $m = 3$. Let $\mathbf{w} = \langle 1, 3 \rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{10}} \langle 1, 3 \rangle.$$

(b) $m = -\frac{1}{3}$. Let $\mathbf{w} = \langle 3, -1 \rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{\sqrt{10}} \langle 3, -1 \rangle.$$

73. $\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$\mathbf{u} + \mathbf{v} = \sqrt{2}\mathbf{j}$

$\mathbf{v} = (\mathbf{u} + \mathbf{v}) - \mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

77. $\|\mathbf{F}_1\| = 2$, $\theta_{\mathbf{F}_1} = 33^\circ$

$\|\mathbf{F}_2\| = 3$, $\theta_{\mathbf{F}_2} = -125^\circ$

$\|\mathbf{F}_3\| = 2.5$, $\theta_{\mathbf{F}_3} = 110^\circ$

$\|\mathbf{R}\| = \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 1.33$

$\theta_{\mathbf{R}} = \theta_{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3} \approx 132.5^\circ$

79. (a) $180(\cos 30^\circ\mathbf{i} + \sin 30^\circ\mathbf{j}) + 275\mathbf{i} \approx 430.88\mathbf{i} + 90\mathbf{j}$

Direction: $\alpha \approx \arctan\left(\frac{90}{430.88}\right) \approx 0.206$ ($\approx 11.8^\circ$)

Magnitude: $\sqrt{430.88^2 + 90^2} \approx 440.18$ newtons

59. A scalar is a real number. A vector is represented by a directed line segment. A vector has both length and direction.

65. $\mathbf{v} = 3\mathbf{i}$. Therefore, $a + b = 3$, $2a - b = 0$. Solving simultaneously, we have $a = 1$, $b = 2$.

71. $f(x) = \sqrt{25 - x^2}$

$$f'(x) = \frac{-x}{\sqrt{25 - x^2}} = \frac{-3}{4} \text{ at } x = 3.$$

(a) $m = -\frac{3}{4}$. Let $\mathbf{w} = \langle -4, 3 \rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{5} \langle -4, 3 \rangle.$$

(b) $m = \frac{4}{3}$. Let $\mathbf{w} = \langle 3, 4 \rangle$, then

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \pm \frac{1}{5} \langle 3, 4 \rangle$$

75. Programs will vary.

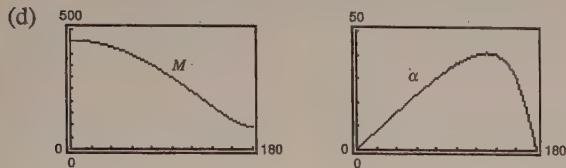
(b) $M = \sqrt{(275 + 180 \cos \theta)^2 + (180 \sin \theta)^2}$

$$\alpha = \arctan\left[\frac{180 \sin \theta}{275 + 180 \cos \theta}\right]$$

79. —CONTINUED—

(c)

θ	0°	30°	60°	90°	120°	150°	180°
M	455	440.2	396.9	328.7	241.9	149.3	95
α	0°	11.8°	23.1°	33.2°	40.1°	37.1°	0°



(e) M decreases because the forces change from acting in the same direction to acting in the opposite direction as θ increases from 0° to 180° .

81. $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (75 \cos 30^\circ \mathbf{i} + 75 \sin 30^\circ \mathbf{j}) + (100 \cos 45^\circ \mathbf{i} + 100 \sin 45^\circ \mathbf{j}) + (125 \cos 120^\circ \mathbf{i} + 125 \sin 120^\circ \mathbf{j})$

$$= \left(\frac{75}{2} \sqrt{3} + 50\sqrt{2} - \frac{125}{2} \right) \mathbf{i} + \left(\frac{75}{2} + 50\sqrt{2} + \frac{125}{2} \sqrt{3} \right) \mathbf{j}$$

$$\|\mathbf{R}\| = \|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| \approx 228.5 \text{ lb}$$

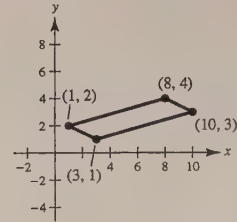
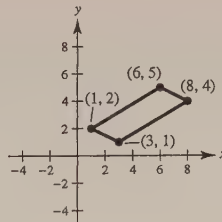
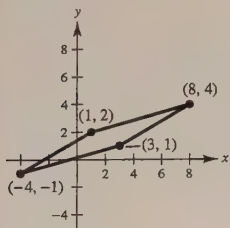
$$\theta_R = \theta_{\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3} \approx 71.3^\circ$$

83. (a) The forces act along the same direction. $\theta = 0^\circ$.

(b) The forces cancel out each other. $\theta = 180^\circ$.

(c) No, the magnitude of the resultant can not be greater than the sum.

85. $(-4, -1), (6, 5), (10, 3)$



87. $\mathbf{u} = \overrightarrow{CB} = \|\mathbf{u}\|(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j})$

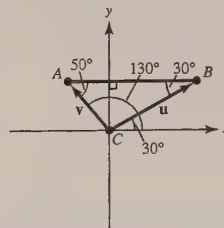
$$\mathbf{v} = \overrightarrow{CA} = \|\mathbf{v}\|(\cos 130^\circ \mathbf{i} + \sin 130^\circ \mathbf{j})$$

Vertical components: $\|\mathbf{u}\| \sin 30^\circ + \|\mathbf{v}\| \sin 130^\circ = 2000$

Horizontal components: $\|\mathbf{u}\| \cos 30^\circ + \|\mathbf{v}\| \cos 130^\circ = 0$

Solving this system, you obtain

$$\|\mathbf{u}\| \approx 1305.5 \text{ pounds and } \|\mathbf{v}\| \approx 1758.8 \text{ pounds.}$$



89. Horizontal component = $\|\mathbf{v}\| \cos \theta = 1200 \cos 6^\circ \approx 1193.43 \text{ ft/sec}$

Vertical component = $\|\mathbf{v}\| \sin \theta = 1200 \sin 6^\circ \approx 125.43 \text{ ft/sec}$

91. $\mathbf{u} = 900[\cos 148^\circ \mathbf{i} + \sin 148^\circ \mathbf{j}]$

$\mathbf{v} = 100[\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j}]$

$\mathbf{u} + \mathbf{v} = [900 \cos 148^\circ + 100 \cos 45^\circ]\mathbf{i} + [900 \sin 148^\circ + 100 \sin 45^\circ]\mathbf{j}$

$\approx -692.53 \mathbf{i} + 547.64 \mathbf{j}$

$\theta \approx \arctan\left(\frac{547.64}{-692.53}\right) \approx -38.34^\circ$ 38.34° North of West.

$\|\mathbf{u} + \mathbf{v}\| \approx \sqrt{(-692.53)^2 + (547.64)^2} \approx 882.9 \text{ km/hr.}$

93. $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$

$-3600\mathbf{j} + T_2(\cos 35^\circ \mathbf{i} - \sin 35^\circ \mathbf{j}) + T_3(\cos 92^\circ \mathbf{i} + \sin 92^\circ \mathbf{j}) = \mathbf{0}$

$T_2 \cos 35^\circ + T_3 \cos 92^\circ = 0$

$-T_2 \sin 35^\circ + T_3 \sin 92^\circ = 3600$

$T_2 = \frac{-T_3 \cos 92^\circ}{\cos 35^\circ} \Rightarrow \frac{T_3 \cos 92^\circ}{\cos 35^\circ} \sin 35^\circ + T_3 \sin 92^\circ = 3600$ and $T_3(0.97495) \approx 3600 \Rightarrow T_3 \approx 3692.48$

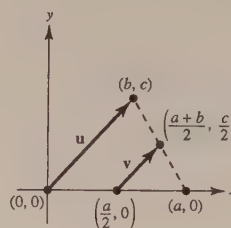
Finally, $T_2 \approx 157.32$

95. Let the triangle have vertices at $(0, 0)$, $(a, 0)$, and (b, c) .

Let \mathbf{u} be the vector joining $(0, 0)$ and (b, c) , as indicated in the figure. Then \mathbf{v} , the vector joining the midpoints, is

$\mathbf{v} = \left(\frac{a+b}{2} - \frac{a}{2}\right)\mathbf{i} + \frac{c}{2}\mathbf{j}$

$= \frac{b}{2}\mathbf{i} + \frac{c}{2}\mathbf{j} = \frac{1}{2}(b\mathbf{i} + c\mathbf{j}) = \frac{1}{2}\mathbf{u}$



97. $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$

$= \|\mathbf{u}\|[\|\mathbf{v}\| \cos \theta_v \mathbf{i} + \|\mathbf{v}\| \sin \theta_v \mathbf{j}] + \|\mathbf{v}\|[\|\mathbf{u}\| \cos \theta_u \mathbf{i} + \|\mathbf{u}\| \sin \theta_u \mathbf{j}] = \|\mathbf{u}\| \|\mathbf{v}\|[(\cos \theta_u + \cos \theta_v)\mathbf{i} + (\sin \theta_u + \sin \theta_v)\mathbf{j}]$

$= 2\|\mathbf{u}\| \|\mathbf{v}\| \left[\cos\left(\frac{\theta_u + \theta_v}{2}\right) \cos\left(\frac{\theta_u - \theta_v}{2}\right) \mathbf{i} + \sin\left(\frac{\theta_u + \theta_v}{2}\right) \cos\left(\frac{\theta_u - \theta_v}{2}\right) \mathbf{j} \right]$

$\tan \theta_w = \frac{\sin\left(\frac{\theta_u + \theta_v}{2}\right) \cos\left(\frac{\theta_u - \theta_v}{2}\right)}{\cos\left(\frac{\theta_u + \theta_v}{2}\right) \cos\left(\frac{\theta_u - \theta_v}{2}\right)} = \tan\left(\frac{\theta_u + \theta_v}{2}\right)$

Thus, $\theta_w = (\theta_u + \theta_v)/2$ and \mathbf{w} bisects the angle between \mathbf{u} and \mathbf{v} .

99. True

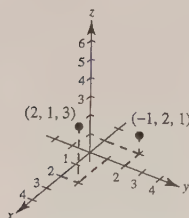
101. True

103. False

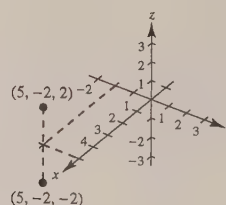
$\|a\mathbf{i} + b\mathbf{j}\| = \sqrt{2}|a|$

Section 10.2 Space Coordinates and Vectors in Space

1.



3.



5. $A(2, 3, 4)$
 $B(-1, -2, 2)$
7. $x = -3, y = 4, z = 5: (-3, 4, 5)$
9. $y = z = 0, x = 10: (10, 0, 0)$

11. The z -coordinate is 0.

13. The point is 6 units above the xy -plane.

15. The point is on the plane parallel to the yz -plane that passes through $x = 4$.

17. The point is to the left of the xz -plane.

19. The point is on or between the planes $y = 3$ and $y = -3$.

21. The point (x, y, z) is 3 units below the xy -plane, and below either quadrant I or III.

23. The point could be above the xy -plane and thus above quadrants II or IV, or below the xy -plane, and thus below quadrants I or III.

$$\begin{aligned} 25. d &= \sqrt{(5-0)^2 + (2-0)^2 + (6-0)^2} \\ &= \sqrt{25 + 4 + 36} = \sqrt{65} \end{aligned}$$

$$\begin{aligned} 27. d &= \sqrt{(6-1)^2 + (-2-(-2))^2 + (-2-4)^2} \\ &= \sqrt{25 + 0 + 36} = \sqrt{61} \end{aligned}$$

29. $A(0, 0, 0), B(2, 2, 1), C(2, -4, 4)$

31. $A(1, -3, -2), B(5, -1, 2), C(-1, 1, 2)$

$$|AB| = \sqrt{4 + 4 + 1} = 3$$

$$|AB| = \sqrt{16 + 4 + 16} = 6$$

$$|AC| = \sqrt{4 + 16 + 16} = 6$$

$$|AC| = \sqrt{4 + 16 + 16} = 6$$

$$|BC| = \sqrt{0 + 36 + 9} = 3\sqrt{5}$$

$$|BC| = \sqrt{36 + 4 + 0} = 2\sqrt{10}$$

$$|BC|^2 = |AB|^2 + |AC|^2$$

Since $|AB| = |AC|$, the triangle is isosceles.

Right triangle

33. The z -coordinate is changed by 5 units:

$$(0, 0, 5), (2, 2, 6), (2, -4, 9)$$

$$35. \left(\frac{5 + (-2)}{2}, \frac{-9 + 3}{2}, \frac{7 + 3}{2} \right) = \left(\frac{3}{2}, -3, 5 \right)$$

37. Center: $(0, 2, 5)$

$$39. \text{Center: } \frac{(2, 0, 0) + (0, 6, 0)}{2} = (1, 3, 0)$$

Radius: 2

Radius: $\sqrt{10}$

$$(x-0)^2 + (y-2)^2 + (z-5)^2 = 4$$

$$(x-1)^2 + (y-3)^2 + (z-0)^2 = 10$$

$$x^2 + y^2 + z^2 - 4y - 10z + 25 = 0$$

$$x^2 + y^2 + z^2 - 2x - 6y = 0$$

$$41. x^2 + y^2 + z^2 - 2x + 6y + 8z + 1 = 0$$

$$(x^2 - 2x + 1) + (y^2 + 6y + 9) + (z^2 + 8z + 16) = -1 + 1 + 9 + 16$$

$$(x-1)^2 + (y+3)^2 + (z+4)^2 = 25$$

Center: $(1, -3, -4)$

Radius: 5

43. $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$

$$x^2 + y^2 + z^2 - \frac{2}{3}x + 2y + \frac{1}{9} = 0$$

$$\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + (y^2 + 2y + 1) + z^2 = -\frac{1}{9} + \frac{1}{9} + 1$$

$$\left(x - \frac{1}{3}\right)^2 + (y + 1)^2 + (z - 0)^2 = 1$$

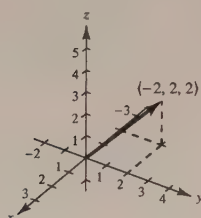
Center: $\left(\frac{1}{3}, -1, 0\right)$

Radius: 1

47. (a) $\mathbf{v} = (2 - 4)\mathbf{i} + (4 - 2)\mathbf{j} + (3 - 1)\mathbf{k}$

$$= -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} = \langle -2, 2, 2 \rangle$$

(b)



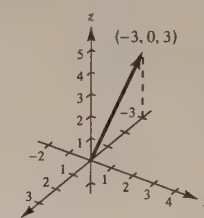
45. $x^2 + y^2 + z^2 \leq 36$

Solid ball of radius 6 centered at origin.

49. (a) $\mathbf{v} = (0 - 3)\mathbf{i} + (3 - 3)\mathbf{j} + (3 - 0)\mathbf{k}$

$$= -3\mathbf{i} + 3\mathbf{k} = \langle -3, 0, 3 \rangle$$

(b)



51. $\langle 4 - 3, 1 - 2, 6 - 0 \rangle = \langle 1, -1, 6 \rangle$

$$\|\langle 1, -1, 6 \rangle\| = \sqrt{1 + 1 + 36} = \sqrt{38}$$

Unit vector: $\frac{\langle 1, -1, 6 \rangle}{\sqrt{38}} = \left\langle \frac{1}{\sqrt{38}}, \frac{-1}{\sqrt{38}}, \frac{6}{\sqrt{38}} \right\rangle$

53. $\langle -5 - (-4), 3 - 3, 0 - 1 \rangle = \langle -1, 0, -1 \rangle$

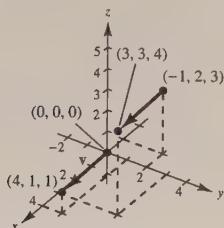
$$\|\langle -1, 0, -1 \rangle\| = \sqrt{1 + 1} = \sqrt{2}$$

Unit vector: $\left\langle \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right\rangle$

55. (b) $\mathbf{v} = (3 + 1)\mathbf{i} + (3 - 2)\mathbf{j} + (4 - 3)\mathbf{k}$

$$= 4\mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 4, 1, 1 \rangle$$

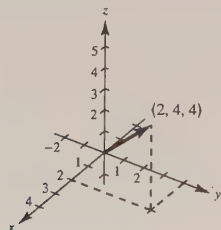
(a) and (c).



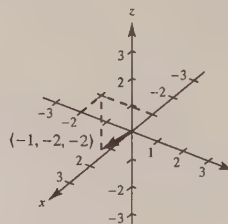
57. $(q_1, q_2, q_3) - (0, 6, 2) = (3, -5, 6)$

$$Q = (3, 1, 8)$$

59. (a) $2\mathbf{v} = \langle 2, 4, 4 \rangle$



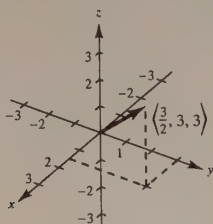
(b) $-\mathbf{v} = \langle -1, -2, -2 \rangle$



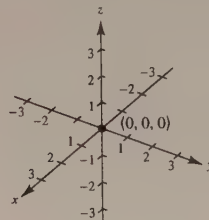
—CONTINUED—

59. —CONTINUED—

(c) $\frac{3}{2}\mathbf{v} = \left\langle \frac{3}{2}, 3, 3 \right\rangle$



(d) $0\mathbf{v} = \langle 0, 0, 0 \rangle$



61. $\mathbf{z} = \mathbf{u} - \mathbf{v} = \langle 1, 2, 3 \rangle - \langle 2, 2, -1 \rangle = \langle -1, 0, 4 \rangle$

63. $\mathbf{z} = 2\mathbf{u} + 4\mathbf{v} - \mathbf{w} = \langle 2, 4, 6 \rangle + \langle 8, 8, -4 \rangle - \langle 4, 0, -4 \rangle = \langle 6, 12, 6 \rangle$

65. $2\mathbf{z} - 3\mathbf{u} = 2\langle z_1, z_2, z_3 \rangle - 3\langle 1, 2, 3 \rangle = \langle 4, 0, -4 \rangle$

$$2z_1 - 3 = 4 \Rightarrow z_1 = \frac{7}{2}$$

$$2z_2 - 6 = 0 \Rightarrow z_2 = 3$$

$$2z_3 - 9 = -4 \Rightarrow z_3 = \frac{5}{2}$$

$$\mathbf{z} = \left\langle \frac{7}{2}, 3, \frac{5}{2} \right\rangle$$

67. (a) and (b) are parallel since $\langle -6, -4, 10 \rangle = -2\langle 3, 2, -5 \rangle$
 and $\left\langle 2, \frac{4}{3}, -\frac{10}{3} \right\rangle = \frac{2}{3}\langle 3, 2, -5 \rangle$.

69. $\mathbf{z} = -3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

(a) is parallel since $-6\mathbf{i} + 8\mathbf{j} + 4\mathbf{k} = 2\mathbf{z}$.

71. $P(0, -2, -5), Q(3, 4, 4), R(2, 2, 1)$

$$\overrightarrow{PQ} = \langle 3, 6, 9 \rangle$$

$$\overrightarrow{PR} = \langle 2, 4, 6 \rangle$$

$$\langle 3, 6, 9 \rangle = \frac{3}{2}\langle 2, 4, 6 \rangle$$

Therefore, \overrightarrow{PQ} and \overrightarrow{PR} are parallel. The points are collinear.

73. $P(1, 2, 4), Q(2, 5, 0), R(0, 1, 5)$

$$\overrightarrow{PQ} = \langle 1, 3, -4 \rangle$$

$$\overrightarrow{PR} = \langle -1, -1, 1 \rangle$$

Since \overrightarrow{PQ} and \overrightarrow{PR} are not parallel, the points are not collinear.

75. $A(2, 9, 1), B(3, 11, 4), C(0, 10, 2), D(1, 12, 5)$

$$\overrightarrow{AB} = \langle 1, 2, 3 \rangle$$

$$\overrightarrow{CD} = \langle 1, 2, 3 \rangle$$

$$\overrightarrow{AC} = \langle -2, 1, 1 \rangle$$

$$\overrightarrow{BD} = \langle -2, 1, 1 \rangle$$

Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the given points form the vertices of a parallelogram.

77. $\|\mathbf{v}\| = 0$

79. $\mathbf{v} = \langle 1, -2, -3 \rangle$

$$\|\mathbf{v}\| = \sqrt{1 + 4 + 9} = \sqrt{14}$$

81. $\mathbf{v} = \langle 0, 3, -5 \rangle$

$$\|\mathbf{v}\| = \sqrt{0 + 9 + 25} = \sqrt{34}$$

83. $\mathbf{u} = \langle 2, -1, 2 \rangle$

$$\|\mathbf{u}\| = \sqrt{4 + 1 + 4} = 3$$

(a) $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{3}\langle 2, -1, 2 \rangle$

(b) $-\frac{\mathbf{u}}{\|\mathbf{u}\|} = -\frac{1}{3}\langle 2, -1, 2 \rangle$

85. $\mathbf{u} = \langle 3, 2, -5 \rangle$

$$\|\mathbf{u}\| = \sqrt{9 + 4 + 25} = \sqrt{38}$$

(a) $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{38}}\langle 3, 2, -5 \rangle$

(b) $-\frac{\mathbf{u}}{\|\mathbf{u}\|} = -\frac{1}{\sqrt{38}}\langle 3, 2, -5 \rangle$

87. Programs will vary.

89. $c\mathbf{v} = \langle 2c, 2c, -c \rangle$

$$\|c\mathbf{v}\| = \sqrt{4c^2 + 4c^2 + c^2} = 5$$

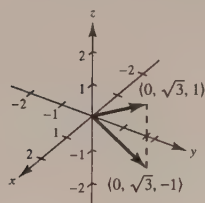
$$9c^2 = 25$$

$$c = \pm \frac{5}{3}$$

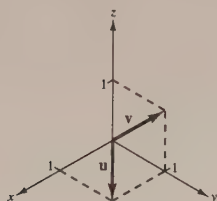
93. $\mathbf{v} = \frac{3}{2} \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{3}{2} \left\langle \frac{2}{3}, \frac{-2}{3}, \frac{1}{3} \right\rangle = \left\langle 1, -1, \frac{1}{2} \right\rangle$

95. $\mathbf{v} = 2[\cos(\pm 30^\circ)\mathbf{j} + \sin(\pm 30^\circ)\mathbf{k}]$

$$= \sqrt{3}\mathbf{j} \pm \mathbf{k} = \langle 0, \sqrt{3}, \pm 1 \rangle$$



99. (a)



(c) $a\mathbf{i} + (a+b)\mathbf{j} + b\mathbf{k} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$$a = 1, b = 1$$

$$\mathbf{w} = \mathbf{u} + \mathbf{v}$$

101. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

105. (a) The height of the right triangle is $h = \sqrt{L^2 - 18^2}$.

The vector \overrightarrow{PQ} is given by

$$\overrightarrow{PQ} = \langle 0, -18, h \rangle.$$

The tension vector \mathbf{T} in each wire is

$$\mathbf{T} = c\langle 0, -18, h \rangle \text{ where } ch = \frac{24}{3} = 8.$$

$$\text{Hence, } \mathbf{T} = \frac{8}{h}\langle 0, -18, h \rangle \text{ and}$$

$$T = \|\mathbf{T}\| = \frac{8}{h}\sqrt{18^2 + h^2} = \frac{8}{\sqrt{L^2 - 18^2}}\sqrt{18^2 + (L^2 - 18^2)} = \frac{8L}{\sqrt{L^2 - 18^2}}$$

(b)	L	20	25	30	35	40	45	50
	T	18.4	11.5	10	9.3	9.0	8.7	8.6

91. $\mathbf{v} = 10 \frac{\mathbf{u}}{\|\mathbf{u}\|} = 10 \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$$= \left\langle 0, \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}} \right\rangle$$

97. $\mathbf{v} = \langle -3, -6, 3 \rangle$

$$\frac{2}{3}\mathbf{v} = \langle -2, -4, 2 \rangle$$

$$(4, 3, 0) + (-2, -4, 2) = (2, -1, 2)$$

(b) $\mathbf{w} = a\mathbf{u} + b\mathbf{v} = a\mathbf{i} + (a+b)\mathbf{j} + b\mathbf{k} = \mathbf{0}$

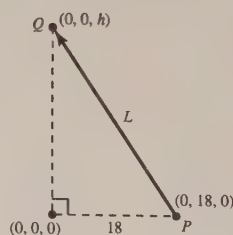
$$a = 0, a + b = 0, b = 0$$

Thus, a and b are both zero.

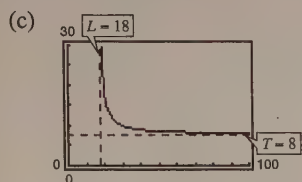
(d) $a\mathbf{i} + (a+b)\mathbf{j} + b\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$$a = 1, a + b = 2, b = 3$$

Not possible

103. Two nonzero vectors \mathbf{u} and \mathbf{v} are parallel if $\mathbf{u} = c\mathbf{v}$ for some scalar c .

105. —CONTINUED—



$x = 18$ is a vertical asymptote and $y = 8$ is a horizontal asymptote.

(d) $\lim_{L \rightarrow 18^+} \frac{8L}{\sqrt{L^2 - 18^2}} = \infty$

$$\lim_{L \rightarrow \infty} \frac{8L}{\sqrt{L^2 - 18^2}} = \lim_{L \rightarrow \infty} \frac{8}{\sqrt{1 - (18/L)^2}} = 8$$

(e) From the table, $T = 10$ implies $L = 30$ inches.

109. $\vec{AB} = \langle 0, 70, 115 \rangle$, $\mathbf{F}_1 = C_1 \langle 0, 70, 115 \rangle$

$$\vec{AC} = \langle -60, 0, 115 \rangle$$
, $\mathbf{F}_2 = C_2 \langle -60, 0, 115 \rangle$

$$\vec{AD} = \langle 45, -65, 115 \rangle$$
, $\mathbf{F}_3 = C_3 \langle 45, -65, 115 \rangle$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \langle 0, 0, 500 \rangle$$

Thus:

$$-60C_2 + 45C_3 = 0$$

$$70C_1 - 65C_3 = 0$$

$$115(C_1 + C_2 + C_3) = 500$$

Solving this system yields $C_1 = \frac{104}{69}$, $C_2 = \frac{28}{23}$, and $C_3 = \frac{112}{69}$. Thus:

$$\|\mathbf{F}_1\| \approx 202.919N$$

$$\|\mathbf{F}_2\| \approx 157.909N$$

$$\|\mathbf{F}_3\| \approx 226.521N$$

111. $d(AP) = 2d(BP)$

$$\sqrt{x^2 + (y+1)^2 + (z-1)^2} = 2\sqrt{(x-1)^2 + (y-2)^2 + z^2}$$

$$x^2 + y^2 + z^2 + 2y - 2z + 2 = 4(x^2 + y^2 + z^2 - 2x - 4y + 5)$$

$$0 = 3x^2 + 3y^2 + 3z^2 - 8x - 18y + 2z + 18$$

$$-6 + \frac{16}{9} + 9 + \frac{1}{9} = \left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) + (y^2 - 6y + 9) + \left(z^2 + \frac{2}{3}z + \frac{1}{9}\right)$$

$$\frac{44}{9} = \left(x - \frac{4}{3}\right)^2 + (y - 3)^2 + \left(z + \frac{1}{3}\right)^2$$

Sphere; center: $\left(\frac{4}{3}, 3, -\frac{1}{3}\right)$, radius: $\frac{2\sqrt{11}}{3}$

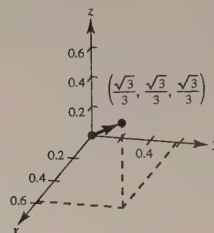
107. Let α be the angle between \mathbf{v} and the coordinate axes.

$$\mathbf{v} = (\cos \alpha)\mathbf{i} + (\cos \alpha)\mathbf{j} + (\cos \alpha)\mathbf{k}$$

$$\|\mathbf{v}\| = \sqrt{3} \cos \alpha = 1$$

$$\cos \alpha = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\mathbf{v} = \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{\sqrt{3}}{3}\langle 1, 1, 1 \rangle$$



Section 10.3 The Dot Product of Two Vectors

1. $\mathbf{u} = \langle 3, 4 \rangle, \mathbf{v} = \langle 2, -3 \rangle$

(a) $\mathbf{u} \cdot \mathbf{v} = 3(2) + 4(-3) = -6$

(b) $\mathbf{u} \cdot \mathbf{u} = 3(3) + 4(4) = 25$

(c) $\|\mathbf{u}\|^2 = 25$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = -6\langle 2, -3 \rangle = \langle -12, 18 \rangle$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-6) = -12$

5. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{v} = \mathbf{i} - \mathbf{k}$

(a) $\mathbf{u} \cdot \mathbf{v} = 2(1) + (-1)(0) + 1(-1) = 1$

(b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-1)(-1) + (1)(1) = 6$

(c) $\|\mathbf{u}\|^2 = 6$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = \mathbf{v} = \mathbf{i} - \mathbf{k}$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2$

9. $\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \cos \theta$

$$\mathbf{u} \cdot \mathbf{v} = (8)(5) \cos \frac{\pi}{3} = 20$$

13. $\mathbf{u} = 3\mathbf{i} + \mathbf{j}, \mathbf{v} = -2\mathbf{i} + 4\mathbf{j}$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-2}{\sqrt{10}\sqrt{20}} = \frac{-1}{5\sqrt{2}}$$

$$\theta = \arccos\left(-\frac{1}{5\sqrt{2}}\right) \approx 98.1^\circ$$

17. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}, \mathbf{v} = -2\mathbf{j} + 3\mathbf{k}$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-8}{5\sqrt{13}} = \frac{-8\sqrt{13}}{65}$$

$$\theta = \arccos\left(-\frac{8\sqrt{13}}{65}\right) \approx 116.3^\circ$$

21. $\mathbf{u} = \langle 4, 3 \rangle, \mathbf{v} = \left\langle \frac{1}{2}, -\frac{2}{3} \right\rangle$

$$\mathbf{u} \neq c\mathbf{v} \Rightarrow \text{not parallel}$$

$$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow \text{orthogonal}$$

25. $\mathbf{u} = \langle 2, -3, 1 \rangle, \mathbf{v} = \langle -1, -1, -1 \rangle$

$$\mathbf{u} \neq c\mathbf{v} \Rightarrow \text{not parallel}$$

$$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow \text{orthogonal}$$

3. $\mathbf{u} = \langle 2, -3, 4 \rangle, \mathbf{v} = \langle 0, 6, 5 \rangle$

(a) $\mathbf{u} \cdot \mathbf{v} = 2(0) + (-3)(6) + (4)(5) = 2$

(b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-3)(-3) + 4(4) = 29$

(c) $\|\mathbf{u}\|^2 = 29$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 2\langle 0, 6, 5 \rangle = \langle 0, 12, 10 \rangle$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(2) = 4$

7. $\mathbf{u} = \langle 3240, 1450, 2235 \rangle$

$$\mathbf{v} = \langle 2.22, 1.85, 3.25 \rangle$$

$$\mathbf{u} \cdot \mathbf{v} = \$17,139.05$$

This gives the total amount that the person earned on his products.

11. $\mathbf{u} = \langle 1, 1 \rangle, \mathbf{v} = \langle 2, -2 \rangle$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{0}{\sqrt{2}\sqrt{8}} = 0$$

$$\theta = \frac{\pi}{2}$$

15. $\mathbf{u} = \langle 1, 1, 1 \rangle, \mathbf{v} = \langle 2, 1, -1 \rangle$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{\sqrt{2}}{3}$$

$$\theta = \arccos \frac{\sqrt{2}}{3} \approx 61.9^\circ$$

19. $\mathbf{u} = \langle 4, 0 \rangle, \mathbf{v} = \langle 1, 1 \rangle$

$$\mathbf{u} \neq c\mathbf{v} \Rightarrow \text{not parallel}$$

$$\mathbf{u} \cdot \mathbf{v} = 4 \neq 0 \Rightarrow \text{not orthogonal}$$

Neither

23. $\mathbf{u} = \mathbf{j} + 6\mathbf{k}, \mathbf{v} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

$$\mathbf{u} \neq c\mathbf{v} \Rightarrow \text{not parallel}$$

$$\mathbf{u} \cdot \mathbf{v} = -8 \neq 0 \Rightarrow \text{not orthogonal}$$

Neither

$$27. \mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}, \|\mathbf{u}\| = 3$$

$$\cos \alpha = \frac{1}{3}$$

$$\cos \beta = \frac{2}{3}$$

$$\cos \gamma = \frac{2}{3}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$$

$$31. \mathbf{u} = \langle 3, 2, -2 \rangle \quad \|\mathbf{u}\| = \sqrt{17}$$

$$\cos \alpha = \frac{3}{\sqrt{17}} \Rightarrow \alpha \approx 0.7560 \text{ or } 43.3^\circ$$

$$\cos \beta = \frac{2}{\sqrt{17}} \Rightarrow \beta \approx 1.0644 \text{ or } 61.0^\circ$$

$$\cos \gamma = \frac{-2}{\sqrt{17}} \Rightarrow \gamma \approx 2.0772 \text{ or } 119.0^\circ$$

$$35. \mathbf{F}_1: C_1 = \frac{50}{\|\mathbf{F}_1\|} \approx 4.3193$$

$$\mathbf{F}_2: C_2 = \frac{80}{\|\mathbf{F}_2\|} \approx 5.4183$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

$$\approx 4.3193\langle 10, 5, 3 \rangle + 5.4183\langle 12, 7, -5 \rangle$$

$$= \langle 108.2126, 59.5246, -14.1336 \rangle$$

$$\|\mathbf{F}\| \approx 124.310 \text{ lb}$$

$$\cos \alpha \approx \frac{108.2126}{\|\mathbf{F}\|} \Rightarrow \alpha \approx 29.48^\circ$$

$$\cos \beta \approx \frac{59.5246}{\|\mathbf{F}\|} \Rightarrow \beta \approx 61.39^\circ$$

$$\cos \gamma \approx \frac{-14.1336}{\|\mathbf{F}\|} \Rightarrow \gamma \approx 96.53^\circ$$

$$39. \overrightarrow{OA} = \langle 0, 10, 10 \rangle$$

$$\cos \alpha = \frac{0}{\sqrt{0^2 + 10^2 + 10^2}} = 0 \Rightarrow \alpha = 90^\circ$$

$$\cos \beta = \cos \gamma = \frac{10}{\sqrt{0^2 + 10^2 + 10^2}}$$

$$= \frac{1}{\sqrt{2}} \Rightarrow \beta = \gamma = 45^\circ$$

$$43. \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 0, 3, 3 \rangle - \langle -2, 2, 2 \rangle = \langle 2, 1, 1 \rangle$$

$$29. \mathbf{u} = \langle 0, 6, -4 \rangle, \|\mathbf{u}\| = \sqrt{52} = 2\sqrt{13}$$

$$\cos \alpha = 0$$

$$\cos \beta = \frac{3}{\sqrt{13}}$$

$$\cos \gamma = -\frac{2}{\sqrt{13}}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 0 + \frac{9}{13} + \frac{4}{13} = 1$$

$$33. \mathbf{u} = \langle -1, 5, 2 \rangle \quad \|\mathbf{u}\| = \sqrt{30}$$

$$\cos \alpha = \frac{-1}{\sqrt{30}} \Rightarrow \alpha \approx 1.7544 \text{ or } 100.5^\circ$$

$$\cos \beta = \frac{5}{\sqrt{30}} \Rightarrow \beta \approx 0.4205 \text{ or } 24.1^\circ$$

$$\cos \gamma = \frac{2}{\sqrt{30}} \Rightarrow \gamma \approx 1.1970 \text{ or } 68.6^\circ$$

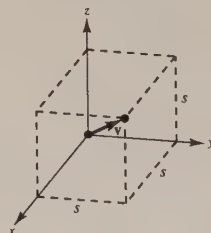
37. Let s = length of a side.

$$\mathbf{v} = \langle s, s, s \rangle$$

$$\|\mathbf{v}\| = s\sqrt{3}$$

$$\cos \alpha = \cos \beta = \cos \gamma = \frac{s}{s\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\alpha = \beta = \gamma = \arccos\left(\frac{1}{\sqrt{3}}\right) \approx 54.7^\circ$$



$$41. \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 6, 7 \rangle - \langle 2, 8 \rangle = \langle 4, -1 \rangle$$

$$45. \mathbf{u} = \langle 2, 3 \rangle, \mathbf{v} = \langle 5, 1 \rangle$$

$$(a) \mathbf{w}_1 = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \frac{13}{26} \langle 5, 1 \rangle = \left\langle \frac{5}{2}, \frac{1}{2} \right\rangle$$

$$(b) \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \left\langle -\frac{1}{2}, \frac{5}{2} \right\rangle$$

47. $\mathbf{u} = \langle 2, 1, 2 \rangle$, $\mathbf{v} = \langle 0, 3, 4 \rangle$

$$\begin{aligned} \text{(a) } \mathbf{w}_1 &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} \\ &= \frac{11}{25} \langle 0, 3, 4 \rangle = \left\langle 0, \frac{33}{25}, \frac{44}{25} \right\rangle \end{aligned}$$

$$\text{(b) } \mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \left\langle 2, -\frac{8}{25}, \frac{6}{25} \right\rangle$$

49. $\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1v_1 + u_2v_2 + u_3v_3$

51. (a) Orthogonal, $\theta = \frac{\pi}{2}$

(b) Acute, $0 < \theta < \frac{\pi}{2}$

(c) Obtuse, $\frac{\pi}{2} < \theta < \pi$

53. See page 738. Direction cosines of $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are

$$\cos \alpha = \frac{v_1}{\|\mathbf{v}\|}, \cos \beta = \frac{v_2}{\|\mathbf{v}\|}, \cos \gamma = \frac{v_3}{\|\mathbf{v}\|}.$$

 α, β , and γ are the direction angles. See Figure 10.26.

55. (a) $\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \mathbf{u} \Rightarrow \mathbf{u} = c\mathbf{v} \Rightarrow \mathbf{u}$ and \mathbf{v} are parallel.

$$\text{(b) } \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal.}$$

57. Programs will vary.

59. Programs will vary.

61. Because \mathbf{u} appears to be perpendicular to \mathbf{v} , the projection of \mathbf{u} onto \mathbf{v} is $\mathbf{0}$. Analytically,

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{\langle 2, -3 \rangle \cdot \langle 6, 4 \rangle}{\|\langle 6, 4 \rangle\|^2} \langle 6, 4 \rangle = 0 \langle 6, 4 \rangle = \mathbf{0}.$$

63. $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j}$. Want $\mathbf{u} \cdot \mathbf{v} = 0$.

$\mathbf{v} = 8\mathbf{i} + 6\mathbf{j}$ and $-\mathbf{v} = -8\mathbf{i} - 6\mathbf{j}$ are orthogonal to \mathbf{u} .

65. $\mathbf{u} = \langle 3, 1, -2 \rangle$. Want $\mathbf{u} \cdot \mathbf{v} = 0$.

$\mathbf{v} = \langle 0, 2, 1 \rangle$ and $-\mathbf{v} = \langle 0, -2, -1 \rangle$ are orthogonal to \mathbf{u} .

67. (a) Gravitational Force $\mathbf{F} = -48,000\mathbf{j}$

$\mathbf{v} = \cos 10^\circ \mathbf{i} + \sin 10^\circ \mathbf{j}$

$$\mathbf{w}_1 = \frac{\mathbf{F} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = (\mathbf{F} \cdot \mathbf{v})\mathbf{v} = (-48,000)(\sin 10^\circ)\mathbf{v}$$

$$\approx -8335.1(\cos 10^\circ \mathbf{i} + \sin 10^\circ \mathbf{j})$$

$\|\mathbf{w}_1\| \approx 8335.1 \text{ lb}$

$$\begin{aligned} \text{(b) } \mathbf{w}_2 &= \mathbf{F} \cdot \mathbf{w}_1 = -48,000\mathbf{j} + 8335.1(\cos 10^\circ \mathbf{i} + \sin 10^\circ \mathbf{j}) \\ &= 8208.5\mathbf{i} - 46,552.6\mathbf{j} \end{aligned}$$

$\|\mathbf{w}_2\| \approx 47,270.8 \text{ lb}$

69. $\mathbf{F} = 85\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right)$

$\mathbf{v} = 10\mathbf{i}$

$W = \mathbf{F} \cdot \mathbf{v} = 425 \text{ ft} \cdot \text{lb}$

71. $\overrightarrow{PQ} = \langle 4, 7, 5 \rangle$

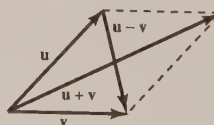
$\mathbf{v} = \langle 1, 4, 8 \rangle$

$W = \overrightarrow{PQ} \cdot \mathbf{v} = 72$

73. False. Let $\mathbf{u} = \langle 2, 4 \rangle$, $\mathbf{v} = \langle 1, 7 \rangle$ and $\mathbf{w} = \langle 5, 5 \rangle$. Then $\mathbf{u} \cdot \mathbf{v} = 2 + 28 = 30$ and $\mathbf{u} \cdot \mathbf{w} = 10 + 20 = 30$.75. In a rhombus, $\|\mathbf{u}\| = \|\mathbf{v}\|$. The diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) &= (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} - (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0 \end{aligned}$$

Therefore, the diagonals are orthogonal.



$$77. \mathbf{u} = \langle \cos \alpha, \sin \alpha, 0 \rangle, \mathbf{v} = \langle \cos \beta, \sin \beta, 0 \rangle$$

The angle between \mathbf{u} and \mathbf{v} is $\alpha - \beta$. (Assuming that $\alpha > \beta$). Also,

$$\cos(\alpha - \beta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{(1)(1)} = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

$$79. \|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$$

$$= (\mathbf{u} - \mathbf{v}) \cdot \mathbf{u} - (\mathbf{u} - \mathbf{v}) \cdot \mathbf{v}$$

$$= \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}$$

$$= \|\mathbf{u}\|^2 - \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$$

$$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}$$

$$81. \|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$$

$$= (\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} + (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}$$

$$= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}$$

$$= \|\mathbf{u}\|^2 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2$$

$$\leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\| \|\mathbf{v}\| + \|\mathbf{v}\|^2 \text{ from Exercise 80}$$

$$\leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$$

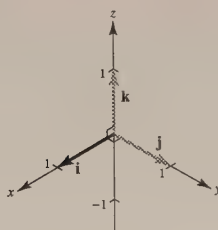
Therefore, $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Section 10.4 The Cross Product of Two Vectors in Space

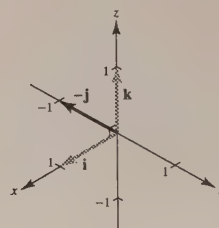
$$1. \mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k}$$



$$3. \mathbf{j} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}$$



$$5. \mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\mathbf{j}$$



$$7. (a) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 4 \\ 3 & 7 & 2 \end{vmatrix} = \langle -22, 16, -23 \rangle$$

$$(b) \mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \langle 22, -16, 23 \rangle$$

$$(c) \mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 7 & 2 \\ 3 & 7 & 2 \end{vmatrix} = \mathbf{0}$$

$$9. (a) \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 2 \\ 1 & -1 & 5 \end{vmatrix} = \langle 17, -33, -10 \rangle$$

$$(b) \mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \langle -17, 33, 10 \rangle$$

$$(c) \mathbf{v} \times \mathbf{v} = \mathbf{0}$$

$$11. \mathbf{u} = \langle 2, -3, 1 \rangle, \mathbf{v} = \langle 1, -2, 1 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k} = \langle -1, -1, -1 \rangle$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 2(-1) + (-3)(-1) + (1)(-1) = 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 1(-1) + (-2)(-1) + (1)(-1) = 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v}$$

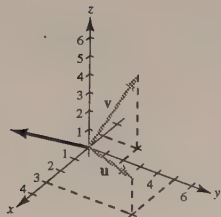
13. $\mathbf{u} = \langle 12, -3, 0 \rangle, \mathbf{v} = \langle -2, 5, 0 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & -3 & 0 \\ -2 & 5 & 0 \end{vmatrix} = 54\mathbf{k} = \langle 0, 0, 54 \rangle$$

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= 12(0) + (-3)(0) + 0(54) \\ &= 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v} \end{aligned}$$

$$\begin{aligned} \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) &= -2(0) + 5(0) + 0(54) \\ &= 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v} \end{aligned}$$

17.



21. $\mathbf{u} = \langle 4, -3.5, 7 \rangle$

$$\mathbf{v} = \langle -1, 8, 4 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \left\langle -70, -23, \frac{57}{2} \right\rangle$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \left\langle \frac{-140}{\sqrt{24,965}}, \frac{-46}{\sqrt{24,965}}, \frac{57}{\sqrt{24,965}} \right\rangle$$

25. Programs will vary.

27. $\mathbf{u} = \mathbf{j}$

$$\mathbf{v} = \mathbf{j} + \mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{i}\| = 1$$

31. $A(1, 1, 1), B(2, 3, 4), C(6, 5, 2), D(7, 7, 5)$

$$\overrightarrow{AB} = \langle 1, 2, 3 \rangle, \overrightarrow{AC} = \langle 5, 4, 1 \rangle, \overrightarrow{CD} = \langle 1, 2, 3 \rangle, \overrightarrow{BD} = \langle 5, 4, 1 \rangle$$

Since $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$, the figure is a parallelogram. \overrightarrow{AB} and \overrightarrow{AC} are adjacent sides and

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 5 & 4 & 1 \end{vmatrix} = -10\mathbf{i} + 14\mathbf{j} - 6\mathbf{k}$$

$$A = \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{332} = 2\sqrt{83}$$

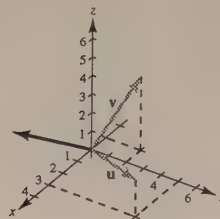
15. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k} = \langle -2, 3, -1 \rangle$$

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= 1(-2) + 1(3) + 1(-1) \\ &= 0 \Rightarrow \mathbf{u} \perp \mathbf{u} \times \mathbf{v} \end{aligned}$$

$$\begin{aligned} \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) &= 2(-2) + 1(3) + (-1)(-1) \\ &= 0 \Rightarrow \mathbf{v} \perp \mathbf{u} \times \mathbf{v} \end{aligned}$$

19. $(-\mathbf{v}) \times \mathbf{u} = -(\mathbf{v} \times \mathbf{u}) = \mathbf{u} \times \mathbf{v}$



23. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$

$$\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{1}{10}\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \left\langle -\frac{71}{20}, -\frac{11}{5}, \frac{5}{4} \right\rangle$$

$$\begin{aligned} \frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} &= \frac{20}{\sqrt{7602}} \left\langle -\frac{71}{20}, -\frac{11}{5}, \frac{5}{4} \right\rangle \\ &= \left\langle -\frac{71}{\sqrt{7602}}, -\frac{44}{\sqrt{7602}}, \frac{25}{\sqrt{7602}} \right\rangle \end{aligned}$$

29. $\mathbf{u} = \langle 3, 2, -1 \rangle$

$$\mathbf{v} = \langle 1, 2, 3 \rangle$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \langle 8, -10, 4 \rangle$$

$$A = \|\mathbf{u} \times \mathbf{v}\| = \|\langle 8, -10, 4 \rangle\| = \sqrt{180} = 6\sqrt{5}$$

33. $A(0, 0, 0), B(1, 2, 3), C(-3, 0, 0)$

$$\overrightarrow{AB} = \langle 1, 2, 3 \rangle, \overrightarrow{AC} = \langle -3, 0, 0 \rangle$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -3 & 0 & 0 \end{vmatrix} = -9\mathbf{j} + 6\mathbf{k}$$

$$A = \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2} \sqrt{117} = \frac{3}{2} \sqrt{13}$$

35. $A(2, -7, 3), B(-1, 5, 8), C(4, 6, -1)$

$$\vec{AB} = \langle -3, 12, 5 \rangle, \vec{AC} = \langle 2, 13, -4 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 12 & 5 \\ 2 & 13 & -4 \end{vmatrix} = \langle -113, -2, -63 \rangle$$

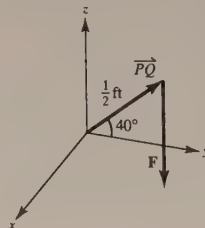
$$\text{Area} = \frac{1}{2} \|\vec{AB} \times \vec{AC}\| = \frac{1}{2} \sqrt{16,742}$$

37. $\mathbf{F} = -20\mathbf{k}$

$$\vec{PQ} = \frac{1}{2}(\cos 40^\circ \mathbf{j} + \sin 40^\circ \mathbf{k})$$

$$\vec{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cos 40^\circ/2 & \sin 40^\circ/2 \\ 0 & 0 & -20 \end{vmatrix} = -10 \cos 40^\circ \mathbf{i}$$

$$\|\vec{PQ} \times \mathbf{F}\| = 10 \cos 40^\circ \approx 7.66 \text{ ft} \cdot \text{lb}$$



39. (a) $\vec{OA} = \frac{3}{2}\mathbf{k}$

$$\mathbf{F} = -60(\sin \theta \mathbf{j} + \cos \theta \mathbf{k})$$

$$\vec{OA} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 3/2 \\ 0 & -60 \sin \theta & -60 \cos \theta \end{vmatrix} = 90 \sin \theta \mathbf{i}$$

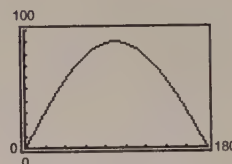
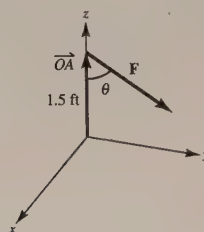
$$\|\vec{OA} \times \mathbf{F}\| = 90 \sin \theta$$

(b) When $\theta = 45^\circ$: $\|\vec{OA} \times \mathbf{F}\| = 90\left(\frac{\sqrt{2}}{2}\right) = 45\sqrt{2} \approx 63.64$

(c) Let $T = 90 \sin \theta$.

$$\frac{dT}{d\theta} = 90 \cos \theta = 0 \text{ when } \theta = 90^\circ.$$

This is what we expected. When $\theta = 90^\circ$ the pipe wrench is horizontal.



41. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

43. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 6$

45. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 2$$

47. $\mathbf{u} = \langle 3, 0, 0 \rangle$

$$\mathbf{v} = \langle 0, 5, 1 \rangle$$

$$\mathbf{w} = \langle 2, 0, 5 \rangle$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 5 & 1 \\ 2 & 0 & 5 \end{vmatrix} = 75$$

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = 75$$

49. $\mathbf{u} \times \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

51. The magnitude of the cross product will increase by a factor of 4.

53. If the vectors are ordered pairs, then the cross product does not exist. False.

55. True

57. $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$

$$\begin{aligned}\mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{vmatrix} \\ &= [u_2(v_3 + w_3) - u_3(v_2 + w_2)]\mathbf{i} - [u_1(v_3 + w_3) - u_3(v_1 + w_1)]\mathbf{j} + [u_1(v_2 + w_2) - u_2(v_1 + w_1)]\mathbf{k} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} + (u_2w_3 - u_3w_2)\mathbf{i} - \\ &\quad (u_1w_3 - u_3w_1)\mathbf{j} + (u_1w_2 - u_2w_1)\mathbf{k} \\ &= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})\end{aligned}$$

59. $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$

$$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = (u_2u_3 - u_3u_2)\mathbf{i} - (u_1u_3 - u_3u_1)\mathbf{j} + (u_1u_2 - u_2u_1)\mathbf{k} = \mathbf{0}$$

61. $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3 = 0$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = (u_2v_3 - u_3v_2)v_1 + (u_3v_1 - u_1v_3)v_2 + (u_1v_2 - u_2v_1)v_3 = 0$$

Thus, $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ and $\mathbf{u} \times \mathbf{v} \perp \mathbf{v}$.

63. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$

If \mathbf{u} and \mathbf{v} are orthogonal, $\theta = \pi/2$ and $\sin \theta = 1$. Therefore, $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\|$.

Section 10.5 Lines and Planes in Space

1. $x = 1 + 3t$, $y = 2 - t$, $z = 2 + 5t$

(a)



(b) When $t = 0$ we have $P = (1, 2, 2)$. When $t = 3$ we have $Q = (10, -1, 17)$.

$$\overrightarrow{PQ} = \langle 9, -3, 15 \rangle$$

The components of the vector and the coefficients of t are proportional since the line is parallel to \overrightarrow{PQ} .

(c) $y = 0$ when $t = 2$. Thus, $x = 7$ and $z = 12$.

Point: $(7, 0, 12)$

$$x = 0 \text{ when } t = -\frac{1}{3}. \text{ Point: } \left(0, \frac{7}{3}, \frac{1}{3}\right)$$

$$z = 0 \text{ when } t = -\frac{2}{5}. \text{ Point: } \left(-\frac{1}{5}, \frac{12}{5}, 0\right)$$

3. Point: $(0, 0, 0)$

Direction vector: $\mathbf{v} = \langle 1, 2, 3 \rangle$

Direction numbers: 1, 2, 3

(a) Parametric: $x = t$, $y = 2t$, $z = 3t$

(b) Symmetric: $x = \frac{y}{2} = \frac{z}{3}$

5. Point: $(-2, 0, 3)$

Direction vector: $\mathbf{v} = \langle 2, 4, -2 \rangle$

Direction numbers: 2, 4, -2

(a) Parametric: $x = -2 + 2t$, $y = 4t$, $z = 3 - 2t$

(b) Symmetric: $\frac{x+2}{2} = \frac{y}{4} = \frac{z-3}{-2}$

7. Point: $(1, 0, 1)$

 Direction vector: $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

Direction numbers: 3, -2, 1

 (a) Parametric: $x = 1 + 3t, y = -2t, z = 1 + t$

 (b) Symmetric: $\frac{x-1}{3} = \frac{y}{-2} = \frac{z-1}{1}$

 11. Points: $(2, 3, 0), (10, 8, 12)$

 Direction vector: $\langle 8, 5, 12 \rangle$

Direction numbers: 8, 5, 12

 (a) Parametric: $x = 2 + 8t, y = 3 + 5t, z = 12t$

 (b) Symmetric: $\frac{x-2}{8} = \frac{y-3}{5} = \frac{z}{12}$

 15. Point: $(-2, 3, 1)$

 Direction vector: $\mathbf{v} = 4\mathbf{i} - \mathbf{k}$

Direction numbers: 4, 0, -1

 Parametric: $x = -2 + 4t, y = 3, z = 1 - t$

 Symmetric: $\frac{x+2}{4} = \frac{z-1}{-1}, y = 3$

 (a) On line ($t = 1$)

 (b) On line ($t = -1$)

 (c) Not on line ($y \neq 3$)

 (d) Not on line $\left(\frac{6+2}{4} \neq \frac{-2-1}{-1}\right)$

 9. Points: $(5, -3, -2), \left(\frac{-2}{3}, \frac{2}{3}, 1\right)$

 Direction vector: $\mathbf{v} = \frac{17}{3}\mathbf{i} - \frac{11}{3}\mathbf{j} - 3\mathbf{k}$

Direction numbers: 17, -11, -9

 (a) Parametric: $x = 5 + 17t, y = -3 - 11t, z = -2 - 9t$

 (b) Symmetric: $\frac{x-5}{17} = \frac{y+3}{-11} = \frac{z+2}{-9}$

 13. Point: $(2, 3, 4)$

 Direction vector: $\mathbf{v} = \mathbf{k}$

Direction numbers: 0, 0, 1

 Parametric: $x = 2, y = 3, z = 4 + t$

 17. $L_1: \mathbf{v} = \langle -3, 2, 4 \rangle$ $(6, -2, 5)$ on line

 $L_2: \mathbf{v} = \langle 6, -4, -8 \rangle$ $(6, -2, 5)$ on line

 $L_3: \mathbf{v} = \langle -6, 4, 8 \rangle$ $(6, -2, 5)$ not on line

 $L_4: \mathbf{v} = \langle 6, 4, -6 \rangle$ not parallel to L_1, L_2 , nor L_3

 Hence, L_1 and L_2 are identical.

 $L_1 = L_2$ and L_3 are parallel.

19. At the point of intersection, the coordinates for one line equal the corresponding coordinates for the other line. Thus,

 (i) $4t + 2 = 2s + 2$, (ii) $3 = 2s + 3$, and (iii) $-t + 1 = s + 1$.

 From (ii), we find that $s = 0$ and consequently, from (iii), $t = 0$. Letting $s = t = 0$, we see that equation (i) is satisfied and therefore the two lines intersect. Substituting zero for s or for t , we obtain the point $(2, 3, 1)$.

 $\mathbf{u} = 4\mathbf{i} - \mathbf{k}$ (First line)

 $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ (Second line)

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{8 - 1}{\sqrt{17}\sqrt{9}} = \frac{7}{3\sqrt{17}} = \frac{7\sqrt{17}}{51}$$

21. Writing the equations of the lines in parametric form we have

$$x = 3t \quad y = 2 - t \quad z = -1 + t$$

$$x = 1 + 4s \quad y = -2 + s \quad z = -3 - 3s$$

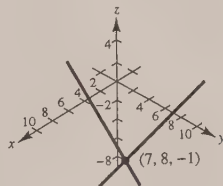
 For the coordinates to be equal, $3t = 1 + 4s$ and $2 - t = -2 + s$. Solving this system yields $t = \frac{17}{7}$ and $s = \frac{11}{7}$. When using these values for s and t , the z coordinates are not equal. The lines do not intersect.

 23. $x = 2t + 3$ $x = -2s + 7$

$$y = 5t - 2 \quad y = s + 8$$

$$z = -t + 1 \quad z = 2s - 1$$

 Point of intersection: $(7, 8, -1)$

 Note: $t = 2$ and $s = 0$


25. $4x - 3y - 6z = 6$

(a) $P = (0, 0, -1), Q = (0, -2, 0), R = (3, 4, -1)$

$\overrightarrow{PQ} = \langle 0, -2, 1 \rangle, \overrightarrow{PR} = \langle 3, 4, 0 \rangle$

(b) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 1 \\ 3 & 4 & 0 \end{vmatrix} = \langle -4, 3, 6 \rangle$

The components of the cross product are proportional to the coefficients of the variables in the equation. The cross product is parallel to the normal vector.

27. Point: $(2, 1, 2)$

$\mathbf{n} = \mathbf{i} = \langle 1, 0, 0 \rangle$

$1(x - 2) + 0(y - 1) + 0(z - 2) = 0$

$x - 2 = 0$

29. Point: $(3, 2, 2)$

Normal vector: $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$2(x - 3) + 3(y - 2) - 1(z - 2) = 0$

$2x + 3y - z = 10$

31. Point: $(0, 0, 6)$

Normal vector: $\mathbf{n} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$-1(x - 0) + 1(y - 0) - 2(z - 6) = 0$

$-x + y - 2z + 12 = 0$

$x - y + 2z = 12$

33. Let \mathbf{u} be the vector from $(0, 0, 0)$ to $(1, 2, 3)$:

$\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

Let \mathbf{v} be the vector from $(0, 0, 0)$ to $(-2, 3, 3)$:

$\mathbf{v} = -2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$

$$\text{Normal vector: } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 3 & 3 \end{vmatrix}$$

$$= -3\mathbf{i} + (-9)\mathbf{j} + 7\mathbf{k}$$

$-3(x - 0) - 9(y - 0) + 7(z - 0) = 0$

$3x + 9y - 7z = 0$

35. Let \mathbf{u} be the vector from $(1, 2, 3)$ to $(3, 2, 1)$: $\mathbf{u} = 2\mathbf{i} - 2\mathbf{k}$

Let \mathbf{v} be the vector from $(1, 2, 3)$ to $(-1, -2, 2)$: $\mathbf{v} = -2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$

$$\text{Normal vector: } \left(\frac{1}{2}\mathbf{u}\right) \times (-\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 2 & 4 & 1 \end{vmatrix} = 4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

$4(x - 1) - 3(y - 2) + 4(z - 3) = 0$

$4x - 3y + 4z = 10$

37. $(1, 2, 3)$, Normal vector: $\mathbf{v} = \mathbf{k}, 1(z - 3) = 0, z = 3$

39. The direction vectors for the lines are $\mathbf{u} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$,
 $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

$$\text{Normal vector: } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix} = -5(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Point of intersection of the lines: $(-1, 5, 1)$

$(x + 1) + (y - 5) + (z - 1) = 0$

$x + y + z = 5$

41. Let \mathbf{v} be the vector from $(-1, 1, -1)$ to $(2, 2, 1)$: $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

Let \mathbf{n} be a vector normal to the plane $2x - 3y + z = 3$: $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

Since \mathbf{v} and \mathbf{n} both lie in the plane P , the normal vector to P is

$$\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 7\mathbf{i} + \mathbf{j} - 11\mathbf{k}$$

$7(x - 2) + 1(y - 2) - 11(z - 1) = 0$

$7x + y - 11z = 5$

43. Let $\mathbf{u} = \mathbf{i}$ and let \mathbf{v} be the vector from $(1, -2, -1)$ to $(2, 5, 6)$: $\mathbf{v} = \mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$

Since \mathbf{u} and \mathbf{v} both lie in the plane P , the normal vector to P is:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 7 & 7 \end{vmatrix} = -7\mathbf{j} + 7\mathbf{k} = -7(\mathbf{j} - \mathbf{k})$$

$$[y - (-2)] - [z - (-1)] = 0$$

$$y - z = -1$$

47. The normal vectors to the planes are

$$\mathbf{n}_1 = \mathbf{i} - 3\mathbf{j} + 6\mathbf{k}, \quad \mathbf{n}_2 = 5\mathbf{i} + \mathbf{j} - \mathbf{k},$$

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{|5 - 3 - 6|}{\sqrt{46}\sqrt{27}} = \frac{4\sqrt{138}}{414}.$$

$$\text{Therefore, } \theta = \arccos\left(\frac{4\sqrt{138}}{414}\right) \approx 83.5^\circ.$$

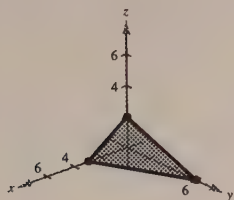
45. The normal vectors to the planes are

$$\mathbf{n}_1 = \langle 5, -3, 1 \rangle, \quad \mathbf{n}_2 = \langle 1, 4, 7 \rangle, \quad \cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = 0.$$

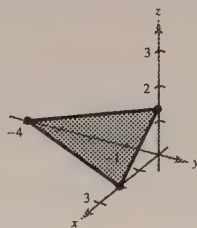
Thus, $\theta = \pi/2$ and the planes are orthogonal.

49. The normal vectors to the planes are $\mathbf{n}_1 = \langle 1, -5, -1 \rangle$ and $\mathbf{n}_2 = \langle 5, -25, -5 \rangle$. Since $\mathbf{n}_2 = 5\mathbf{n}_1$, the planes are parallel, but not equal.

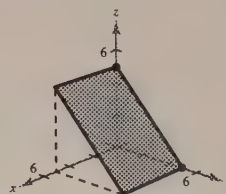
51. $4x + 2y + 6z = 12$



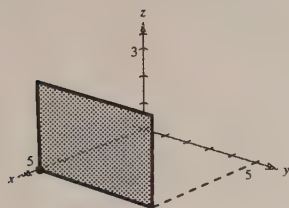
53. $2x - y + 3z = 4$



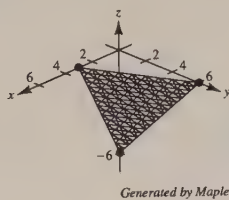
55. $y + z = 5$



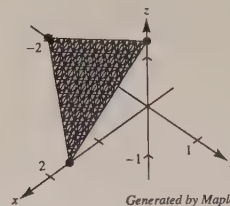
57. $x = 5$



59. $2x + y - z = 6$



61. $-5x + 4y - 6z + 8 = 0$



63. $P_1: \mathbf{n} = \langle 3, -2, 5 \rangle$

$P_2: \mathbf{n} = \langle -6, 4, -10 \rangle$

$P_3: \mathbf{n} = \langle -3, 2, 5 \rangle$

$P_4: \mathbf{n} = \langle 75, -50, 125 \rangle$

P_1 and P_4 are identical.

$P_1 = P_4$ is parallel to P_2 .

$(1, -1, 1)$ on plane

$(1, -1, 1)$ not on plane

$(1, -1, 1)$ on plane

65. Each plane passes through the points

$(c, 0, 0)$, $(0, c, 0)$, and $(0, 0, c)$.

67. The normals to the planes are $\mathbf{n}_1 = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$. The direction vector for the line is

$$\mathbf{n}_2 \times \mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ 3 & 2 & -1 \end{vmatrix} = 7(\mathbf{j} + 2\mathbf{k}).$$

Now find a point of intersection of the planes.

$$6x + 4y - 2z = 14$$

$$x - 4y + 2z = 0$$

$$7x = 14$$

$$x = 2$$

Substituting 2 for x in the second equation, we have $-4y + 2z = -2$ or $z = 2y - 1$. Letting $y = 1$, a point of intersection is $(2, 1, 1)$.

$$x = 2, y = 1 + t, z = 1 + 2t$$

71. Writing the equation of the line in parametric form and substituting into the equation of the plane we have:

$$x = 1 + 3t, y = -1 - 2t, z = 3 + t$$

$$2(1 + 3t) + 3(-1 - 2t) = 10, -1 = 10, \text{contradiction}$$

Therefore, the line does not intersect the plane.

75. Point: $Q(2, 8, 4)$

$$\text{Plane: } 2x + y + z = 5$$

$$\text{Normal to plane: } \mathbf{n} = \langle 2, 1, 1 \rangle$$

$$\text{Point in plane: } P(0, 0, 5)$$

$$\text{Vector: } \overrightarrow{PQ} = \langle 2, 8, -1 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{11}{\sqrt{6}} = \frac{11\sqrt{6}}{6}$$

79. The normal vectors to the planes are $\mathbf{n}_1 = \langle -3, 6, 7 \rangle$ and $\mathbf{n}_2 = \langle 6, -12, -14 \rangle$. Since $\mathbf{n}_2 = -2\mathbf{n}_1$, the planes are parallel. Choose a point in each plane.

$$P = (0, -1, 1) \text{ is a point in } -3x + 6y + 7z = 1.$$

$$Q = \left(\frac{25}{6}, 0, 0\right) \text{ is a point in } 6x - 12y - 14z = 25.$$

$$\overrightarrow{PQ} = \left\langle \frac{25}{6}, 1, -1 \right\rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{|-27/2|}{\sqrt{94}} = \frac{27}{2\sqrt{94}} = \frac{27\sqrt{94}}{188}$$

83. The parametric equations of a line L parallel to $\mathbf{v} = \langle a, b, c \rangle$ and passing through the point $P(x_1, y_1, z_1)$ are

$$x = x_1 + at, y = y_1 + bt, z = z_1 + ct.$$

The symmetric equations are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

69. Writing the equation of the line in parametric form and substituting into the equation of the plane we have:

$$x = \frac{1}{2} + t, y = \frac{-3}{2} - t, z = -1 + 2t$$

$$2\left(\frac{1}{2} + t\right) - 2\left(\frac{-3}{2} - t\right) + (-1 + 2t) = 12, t = \frac{3}{2}$$

Substituting $t = 3/2$ into the parametric equations for the line we have the point of intersection $(2, -3, 2)$. The line does not lie in the plane.

73. Point: $Q(0, 0, 0)$

$$\text{Plane: } 2x + 3y + z - 12 = 0$$

$$\text{Normal to plane: } \mathbf{n} = \langle 2, 3, 1 \rangle$$

$$\text{Point in plane: } P(6, 0, 0)$$

$$\text{Vector } \overrightarrow{PQ} = \langle -6, 0, 0 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|-12|}{\sqrt{14}} = \frac{6\sqrt{14}}{7}$$

77. The normal vectors to the planes are $\mathbf{n}_1 = \langle 1, -3, 4 \rangle$ and $\mathbf{n}_2 = \langle 1, -3, 4 \rangle$. Since $\mathbf{n}_1 = \mathbf{n}_2$, the planes are parallel. Choose a point in each plane.

$$P = (10, 0, 0) \text{ is a point in } x - 3y + 4z = 10.$$

$$Q = (6, 0, 0) \text{ is a point in } x - 3y + 4z = 6.$$

$$\overrightarrow{PQ} = \langle -4, 0, 0 \rangle, D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}_1|}{\|\mathbf{n}_1\|} = \frac{4}{\sqrt{26}} = \frac{2\sqrt{26}}{13}$$

81. $\mathbf{u} = \langle 4, 0, -1 \rangle$ is the direction vector for the line. $Q(1, 5, -2)$ is the given point, and $P(-2, 3, 1)$ is on the line. Hence, $\overrightarrow{PQ} = \langle 3, 2, -3 \rangle$ and

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -3 \\ 4 & 0 & -1 \end{vmatrix} = \langle -2, -9, -8 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{149}}{\sqrt{17}} = \frac{\sqrt{2533}}{17}$$

85. Solve the two linear equations representing the planes to find two points of intersection. Then find the line determined by the two points.

87. (a) Sphere

$$(x - 3)^2 + (y + 2)^2 + (z - 5)^2 = 16$$

$$x^2 + y^2 + z^2 - 6x + 4y - 10z + 22 = 0$$

(b) Parallel planes

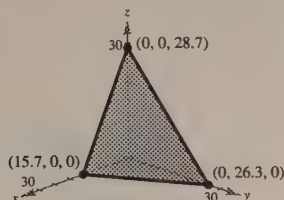
$$4x - 3y + z = 10 \pm 4\|\mathbf{n}\| = 10 \pm 4\sqrt{26}$$

 89. (a) $z = 28.7 - 1.83x - 1.09y$

Year	1980	1985	1990	1994	1995	1996	1997
z (approx.)	16.16	14.23	9.81	8.60	8.42	8.27	8.23

(b) An increase in x or y will cause a decrease in z . In fact, any increase in two variables will cause a decrease in the third.

(c)



91. True

Section 10.6 Surfaces in Space

1. Ellipsoid

Matches graph (c)

3. Hyperboloid of one sheet

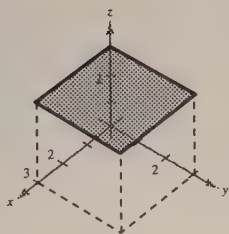
Matches graph (f)

5. Elliptic paraboloid

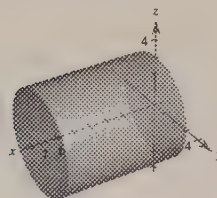
Matches graph (d)

 7. $z = 3$

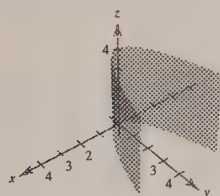
Plane parallel to the xy -coordinate plane


 9. $y^2 + z^2 = 9$

The x -coordinate is missing so we have a cylindrical surface with rulings parallel to the x -axis. The generating curve is a circle.

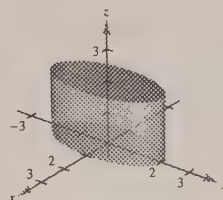

 11. $y = x^2$

The z -coordinate is missing so we have a cylindrical surface with rulings parallel to the z -axis. The generating curve is a parabola.


 13. $4x^2 + y^2 = 4$

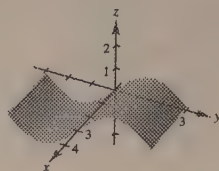
$$\frac{x^2}{1} + \frac{y^2}{4} = 1$$

The z -coordinate is missing so we have a cylindrical surface with rulings parallel to the z -axis. The generating curve is an ellipse.



15. $z = \sin y$

The x -coordinate is missing so we have a cylindrical surface with rulings parallel to the x -axis. The generating curve is the sine curve.



17. $z = x^2 + y^2$

- (a) You are viewing the paraboloid from the x -axis: $(20, 0, 0)$
 (b) You are viewing the paraboloid from above, but not on the z -axis: $(10, 10, 20)$
 (c) You are viewing the paraboloid from the z -axis: $(0, 0, 20)$
 (d) You are viewing the paraboloid from the y -axis: $(0, 20, 0)$

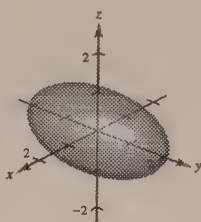
19. $\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{1} = 1$

Ellipsoid

$$xy\text{-trace: } \frac{x^2}{1} + \frac{y^2}{4} = 1 \text{ ellipse}$$

$$xz\text{-trace: } x^2 + z^2 = 1 \text{ circle}$$

$$yz\text{-trace: } \frac{y^2}{4} + \frac{z^2}{1} = 1 \text{ ellipse}$$



21. $16x^2 - y^2 + 16z^2 = 4$

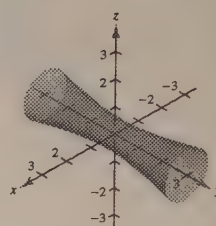
$$4x^2 - \frac{y^2}{4} + 4z^2 = 1$$

Hyperboloid on one sheet

$$xy\text{-trace: } 4x^2 - \frac{y^2}{4} = 1 \text{ hyperbola}$$

$$xz\text{-trace: } 4(x^2 + z^2) = 1 \text{ circle}$$

$$yz\text{-trace: } \frac{-y^2}{4} + 4z^2 = 1 \text{ hyperbola}$$



23. $x^2 - y + z^2 = 0$

Elliptic paraboloid

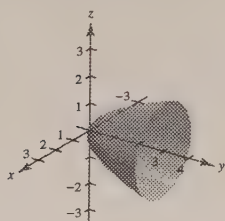
$$xy\text{-trace: } y = x^2$$

$$xz\text{-trace: } x^2 + z^2 = 0,$$

point $(0, 0, 0)$

$$yz\text{-trace: } y = z^2$$

$$y = 1: x^2 + z^2 = 1$$



25. $x^2 - y^2 + z = 0$

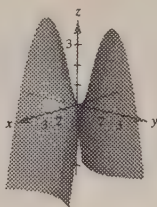
Hyperbolic paraboloid

$$xy\text{-trace: } y = \pm x$$

$$xz\text{-trace: } z = -x^2$$

$$yz\text{-trace: } z = y^2$$

$$y = \pm 1: z = 1 - x^2$$



27. $z^2 = x^2 + \frac{y^2}{4}$

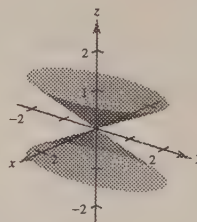
Elliptic Cone

$$xy\text{-trace: point } (0, 0, 0)$$

$$xz\text{-trace: } z = \pm x$$

$$yz\text{-trace: } z = \pm \frac{1}{2}y$$

$$z = \pm 1: x^2 + \frac{y^2}{4} = 1$$



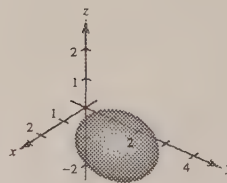
29. $16x^2 + 9y^2 + 16z^2 - 32x - 36y + 36 = 0$

$$16(x^2 - 2x + 1) + 9(y^2 - 4y + 4) + 16z^2 = -36 + 16 + 36$$

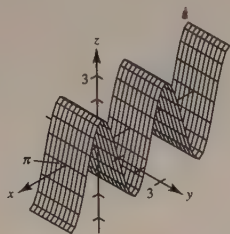
$$16(x - 1)^2 + 9(y - 2)^2 + 16z^2 = 16$$

$$\frac{(x - 1)^2}{1} + \frac{(y - 2)^2}{16/9} + \frac{z^2}{1} = 1$$

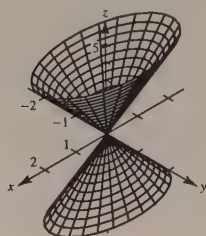
Ellipsoid with center $(1, 2, 0)$.



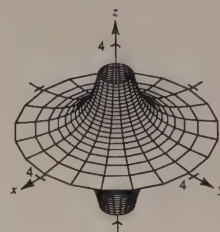
31. $z = 2 \sin x$



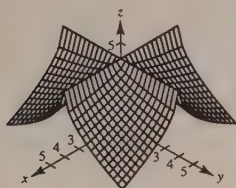
33. $z^2 = x^2 + 4y^2$
 $z = \pm \sqrt{x^2 + 4y^2}$



35. $x^2 + y^2 = \left(\frac{2}{z}\right)^2$
 $y = \pm \sqrt{\frac{4}{z^2} - x^2}$

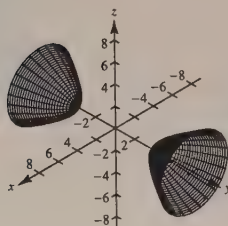


37. $z = 4 - \sqrt{|xy|}$



39. $4x^2 - y^2 + 4z^2 = -16$

$z = \pm \sqrt{\frac{y^2}{4} - x^2 - 4}$

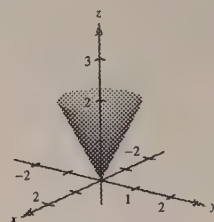


41. $z = 2\sqrt{x^2 + y^2}$

$z = 2$

$2\sqrt{x^2 + y^2} = 2$

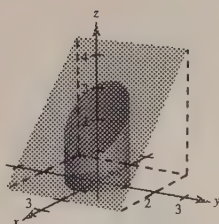
$x^2 + y^2 = 1$



43. $x^2 + y^2 = 1$

$x + z = 2$

$z = 0$



45. $x^2 + z^2 = [r(y)]^2$ and $z = r(y) = \pm 2\sqrt{y}$; therefore,

$x^2 + z^2 = 4y.$

47. $x^2 + y^2 = [r(z)]^2$ and $y = r(z) = \frac{z}{2}$; therefore,

$x^2 + y^2 = \frac{z^2}{4}, 4x^2 + 4y^2 = z^2.$

49. $y^2 + z^2 = [r(x)]^2$ and $y = r(x) = \frac{2}{x}$; therefore,

$y^2 + z^2 = \left(\frac{2}{x}\right)^2, y^2 + z^2 = \frac{4}{x^2}.$

51. $x^2 + y^2 - 2z = 0$

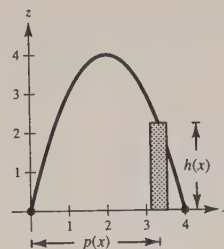
$x^2 + y^2 = (\sqrt{2z})^2$

Equation of generating curve: $y = \sqrt{2z}$ or $x = \sqrt{2z}$

53. Let C be a curve in a plane and let L be a line not in a parallel plane. The set of all lines parallel to L and intersecting C is called a cylinder.

55. See pages 765 and 766.

57. $V = 2\pi \int_0^4 x(4x - x^2) dx$
 $= 2\pi \left[\frac{4x^3}{3} - \frac{x^4}{4} \right]_0^4 = \frac{128\pi}{3}$



59. $z = \frac{x^2}{2} + \frac{y^2}{4}$

(a) When $z = 2$ we have $2 = \frac{x^2}{2} + \frac{y^2}{4}$, or $1 = \frac{x^2}{4} + \frac{y^2}{8}$

Major axis: $2\sqrt{8} = 4\sqrt{2}$

Minor axis: $2\sqrt{4} = 4$

$c^2 = a^2 - b^2$, $c^2 = 4$, $c = 2$

Foci: $(0, \pm 2, 2)$

(b) When $z = 8$ we have $8 = \frac{x^2}{2} + \frac{y^2}{4}$, or $1 = \frac{x^2}{16} + \frac{y^2}{32}$

Major axis: $2\sqrt{32} = 8\sqrt{2}$

Minor axis: $2\sqrt{16} = 8$

$c^2 = 32 - 16 = 16$, $c = 4$

Foci: $(0, \pm 4, 8)$

61. If (x, y, z) is on the surface, then

$$(y + 2)^2 = x^2 + (y - 2)^2 + z^2$$

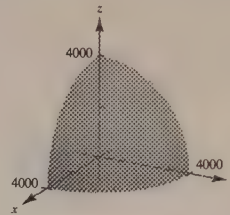
$$y^2 + 4y + 4 = x^2 + y^2 - 4y + 4 + z^2$$

$$x^2 + z^2 = 8y$$

Elliptic paraboloid

Traces parallel to xz -plane are circles.

63. $\frac{x^2}{3963^2} + \frac{y^2}{3963^2} + \frac{z^2}{3942^2} = 1$



65. $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$, $z = bx + ay$

$$bx + ay = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

$$\frac{1}{a^2} \left(x^2 + a^2 bx + \frac{a^4 b^2}{4} \right) = \frac{1}{b^2} \left(y^2 - ab^2 y + \frac{a^2 b^4}{4} \right)$$

$$\frac{\left(x + \frac{a^2 b}{2} \right)^2}{a^2} = \frac{\left(y - \frac{ab^2}{2} \right)^2}{b^2}$$

$$y = \pm \frac{b}{a} \left(x + \frac{a^2 b}{2} \right) + \frac{ab^2}{2}$$

Letting $x = at$, you obtain the two intersecting lines

$$x = at, y = -bt, z = 0 \text{ and } x = at, y = bt + ab^2$$

$$z = 2abt + a^2 b^2.$$

67. The Klein bottle *does not* have both an “inside” and an “outside.” It is formed by inserting the small open end through the side of the bottle and making it contiguous with the top of the bottle.

Section 10.7 Cylindrical and Spherical Coordinates

1. $(5, 0, 2)$, cylindrical

$$x = 5 \cos 0 = 5$$

$$y = 5 \sin 0 = 0$$

$$z = 2$$

$(5, 0, 2)$, rectangular

3. $\left(2, \frac{\pi}{3}, 2 \right)$, cylindrical

$$x = 2 \cos \frac{\pi}{3} = 1$$

$$y = 2 \sin \frac{\pi}{3} = \sqrt{3}$$

$$z = 2$$

$(1, \sqrt{3}, 2)$, rectangular

5. $\left(4, \frac{7\pi}{6}, 3 \right)$, cylindrical

$$x = 4 \cos \frac{7\pi}{6} = -2\sqrt{3}$$

$$y = 4 \sin \frac{7\pi}{6} = -2$$

$$z = 3$$

$(-2\sqrt{3}, -2, 3)$, rectangular

- 7.
- $(0, 5, 1)$
- , rectangular

$$r = \sqrt{(0)^2 + (5)^2} = 5$$

$$\theta = \arctan \frac{5}{0} = \frac{\pi}{2}$$

$$z = 1$$

$$\left(5, \frac{\pi}{2}, 1\right), \text{cylindrical}$$

- 9.
- $(1, \sqrt{3}, 4)$
- , rectangular

$$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\theta = \arctan \sqrt{3} = \frac{\pi}{3}$$

$$z = 4$$

$$\left(2, \frac{\pi}{3}, 4\right), \text{cylindrical}$$

- 11.
- $(2, -2, -4)$
- , rectangular

$$r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$$

$$\theta = \arctan(-1) = -\frac{\pi}{4}$$

$$z = -4$$

$$\left(2\sqrt{2}, -\frac{\pi}{4}, -4\right), \text{cylindrical}$$

- 13.
- $x^2 + y^2 + z^2 = 10$
- rectangular equation

$$r^2 + z^2 = 10 \text{ cylindrical equation}$$

- 15.
- $y = x^2$

rectangular equation

$$r \sin \theta = (r \cos \theta)^2$$

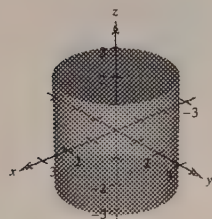
$$\sin \theta = r \cos^2 \theta$$

$$r = \sec \theta \cdot \tan \theta \text{ cylindrical equation}$$

- 17.
- $r = 2$

$$\sqrt{x^2 + y^2} = 2$$

$$x^2 + y^2 = 4$$



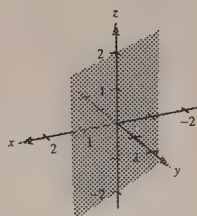
- 19.
- $\theta = \frac{\pi}{6}$

$$\tan \frac{\pi}{6} = \frac{y}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{y}{x}$$

$$x = \sqrt{3}y$$

$$x - \sqrt{3}y = 0$$



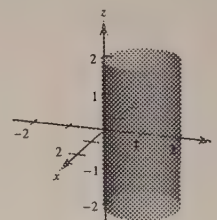
- 21.
- $r = 2 \sin \theta$

$$r^2 = 2r \sin \theta$$

$$x^2 + y^2 = 2y$$

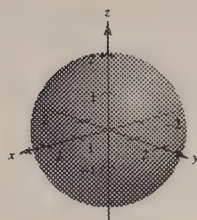
$$x^2 + y^2 - 2y = 0$$

$$x^2 + (y - 1)^2 = 1$$



- 23.
- $r^2 + z^2 = 4$

$$x^2 + y^2 + z^2 = 4$$



- 25.
- $(4, 0, 0)$
- , rectangular

$$\rho = \sqrt{4^2 + 0^2 + 0^2} = 4$$

$$\theta = \arctan 0 = 0$$

$$\phi = \arccos 0 = \frac{\pi}{2}$$

$$\left(4, 0, \frac{\pi}{2}\right), \text{spherical}$$

- 27.
- $(-2, 2\sqrt{3}, 4)$
- , rectangular

$$\rho = \sqrt{(-2)^2 + (2\sqrt{3})^2 + 4^2} = 4\sqrt{2}$$

$$\theta = \arctan(-\sqrt{3}) = \frac{2\pi}{3}$$

$$\phi = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

$$\left(4\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4}\right), \text{spherical}$$

- 29.
- $(\sqrt{3}, 1, 2\sqrt{3})$
- , rectangular

$$\rho = \sqrt{3 + 1 + 12} = 4$$

$$\theta = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

$$\phi = \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

$$\left(4, \frac{\pi}{6}, \frac{\pi}{6}\right), \text{spherical}$$

31. $\left(4, \frac{\pi}{6}, \frac{\pi}{4}\right)$, spherical

$$x = 4 \sin \frac{\pi}{4} \cos \frac{\pi}{6} = \sqrt{6}$$

$$y = 4 \sin \frac{\pi}{4} \sin \frac{\pi}{6} = \sqrt{2}$$

$$z = 4 \cos \frac{\pi}{4} = 2\sqrt{2}$$

$$(\sqrt{6}, \sqrt{2}, 2\sqrt{2}), \text{rectangular}$$

33. $\left(12, \frac{-\pi}{4}, 0\right)$, spherical

$$x = 12 \sin 0 \cos \left(\frac{-\pi}{4}\right) = 0$$

$$y = 12 \sin 0 \sin \left(\frac{-\pi}{4}\right) = 0$$

$$z = 12 \cos 0 = 12$$

$$(0, 0, 12), \text{rectangular}$$

35. $\left(5, \frac{\pi}{4}, \frac{3\pi}{4}\right)$, spherical

$$x = 5 \sin \frac{3\pi}{4} \cos \frac{\pi}{4} = \frac{5}{2}$$

$$y = 5 \sin \frac{3\pi}{4} \sin \frac{\pi}{4} = \frac{5}{2}$$

$$z = 5 \cos \frac{3\pi}{4} = -\frac{5\sqrt{2}}{2}$$

$$\left(\frac{5}{2}, \frac{5}{2}, -\frac{5\sqrt{2}}{2}\right), \text{rectangular}$$

37. (a) Programs will vary.

(b) $(x, y, z) = (3, -4, 2)$

$$(\rho, \theta, \phi) \approx (5.385, -0.927, 1.190)$$

39. $x^2 + y^2 + z^2 = 36$ rectangular equation

$$\rho^2 = 36$$

spherical equation

41. $x^2 + y^2 = 9$ rectangular equation

$$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = 9$$

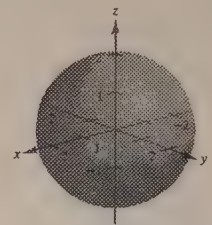
$$\rho^2 \sin^2 \phi = 9$$

$$\rho \sin \phi = 3$$

$$\rho = 3 \csc \phi \text{ spherical equation}$$

43. $\rho = 2$

$$x^2 + y^2 + z^2 = 4$$



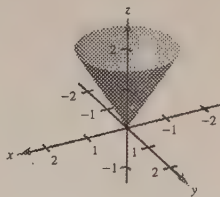
45. $\phi = \frac{\pi}{6}$

$$\cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\sqrt{3}}{2} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{3}{4} = \frac{z^2}{x^2 + y^2 + z^2}$$

$$3x^2 + 3y^2 - z^2 = 0, z \geq 0$$

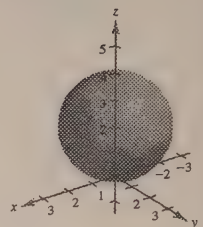


47. $\rho = 4 \cos \phi$

$$\sqrt{x^2 + y^2 + z^2} = \frac{4z}{\sqrt{x^2 + y^2 + z^2}}$$

$$x^2 + y^2 + z^2 - 4z = 0$$

$$x^2 + y^2 + (z - 2)^2 = 4, z \geq 0$$

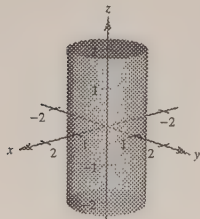


49. $\rho = \csc \phi$

$$\rho \sin \phi = 1$$

$$\sqrt{x^2 + y^2} = 1$$

$$x^2 + y^2 = 1$$



51. $\left(4, \frac{\pi}{4}, 0\right)$, cylindrical

$$\rho = \sqrt{4^2 + 0^2} = 4$$

$$\theta = \frac{\pi}{4}$$

$$\phi = \arccos 0 = \frac{\pi}{2}$$

$$\left(4, \frac{\pi}{4}, \frac{\pi}{2}\right), \text{spherical}$$

53. $\left(4, \frac{\pi}{2}, 4\right)$, cylindrical

$$\rho = \sqrt{4^2 + 4^2} = 4\sqrt{2}$$

$$\theta = \frac{\pi}{2}$$

$$\phi = \arccos \left(\frac{4}{4\sqrt{2}}\right) = \frac{\pi}{4}$$

$$\left(4\sqrt{2}, \frac{\pi}{2}, \frac{\pi}{4}\right), \text{spherical}$$

55. $\left(4, \frac{-\pi}{6}, 6\right)$, cylindrical

$$\rho = \sqrt{4^2 + 6^2} = 2\sqrt{13}$$

$$\theta = \frac{-\pi}{6}$$

$$\phi = \arccos \frac{3}{\sqrt{13}}$$

$$\left(2\sqrt{13}, \frac{-\pi}{6}, \arccos \frac{3}{\sqrt{13}}\right),$$

spherical

57. $(12, \pi, 5)$, cylindrical

$$\rho = \sqrt{12^2 + 5^2} = 13$$

$$\theta = \pi$$

$$\phi = \arccos \frac{5}{13}$$

$$\left(13, \pi, \arccos \frac{5}{13}\right), \text{ spherical}$$

59. $\left(10, \frac{\pi}{6}, \frac{\pi}{2}\right)$, spherical

$$r = 10 \sin \frac{\pi}{2} = 10$$

$$\theta = \frac{\pi}{6}$$

$$z = 10 \cos \frac{\pi}{2} = 0$$

$$\left(10, \frac{\pi}{6}, 0\right), \text{ cylindrical}$$

61. $\left(36, \pi, \frac{\pi}{2}\right)$, spherical

$$r = \rho \sin \phi = 36 \sin \frac{\pi}{2} = 36$$

$$\theta = \pi$$

$$z = \rho \cos \phi = 36 \cos \frac{\pi}{2} = 0$$

$$(36, \pi, 0), \text{ cylindrical}$$

63. $\left(6, -\frac{\pi}{6}, \frac{\pi}{3}\right)$, spherical

$$r = 6 \sin \frac{\pi}{3} = 3\sqrt{3}$$

$$\theta = -\frac{\pi}{6}$$

$$z = 6 \cos \frac{\pi}{3} = 3$$

$$\left(3\sqrt{3}, -\frac{\pi}{6}, 3\right), \text{ cylindrical}$$

65. $\left(8, \frac{7\pi}{6}, \frac{\pi}{6}\right)$, spherical

$$r = 8 \sin \frac{\pi}{6} = 4$$

$$\theta = \frac{7\pi}{6}$$

$$z = 8 \cos \frac{\pi}{6} = \frac{8\sqrt{3}}{2}$$

$$\left(4, \frac{7\pi}{6}, 4\sqrt{3}\right), \text{ cylindrical}$$

Rectangular

67. $(4, 6, 3)$

69. $(4.698, 1.710, 8)$

71. $(-7.071, 12.247, 14.142)$

73. $(3, -2, 2)$

75. $\left(\frac{5}{2}, \frac{4}{3}, \frac{-3}{2}\right)$

77. $(-3.536, 3.536, -5)$

79. $(2.804, -2.095, 6)$

Cylindrical

(7.211, 0.983, 3)

$$\left(5, \frac{\pi}{9}, 8\right)$$

(14.142, 2.094, 14.142)

(3.606, -0.588, 2)

(2.833, 0.490, -1.5)

$$\left(5, \frac{3\pi}{4}, -5\right)$$

(-3.5, 2.5, 6)

Spherical

(7.810, 0.983, 1.177)

(9.434, 0.349, 0.559)

$$\left(20, \frac{2\pi}{3}, \frac{\pi}{4}\right)$$

(4.123, -0.588, 1.064)

(3.206, 0.490, 2.058)

(7.071, 2.356, 2.356)

(6.946, 5.642, 0.528)

[Note: Use the cylindrical coordinates (3.5, 5.642, 6)]

81. $r = 5$

Cylinder

Matches graph (d)

83. $\rho = 5$

Sphere

Matches graph (c)

85. $r^2 = z, x^2 + y^2 = z$

Paraboloid

Matches graph (f)

87. Rectangular to cylindrical: $r^2 = x^2 + y^2$

$$\tan \theta = \frac{y}{x}$$

$$z = z$$

Cylindrical to rectangular: $x = r \cos \theta$

$$y = r \sin \theta$$

$$z = z$$

89. Rectangular to spherical: $\rho^2 = x^2 + y^2 + z^2$

$$\tan \theta = \frac{y}{x}$$

$$\phi = \arccos \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$$

Spherical to rectangular: $x = \rho \sin \phi \cos \theta$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

91. $x^2 + y^2 + z^2 = 16$

(a) $r^2 + z^2 = 16$

(b) $\rho^2 = 16, \rho = 4$

95. $x^2 + y^2 = 4y$

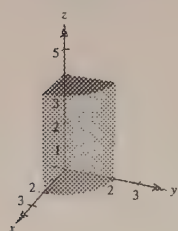
(a) $r^2 = 4r \sin \theta, r = 4 \sin \theta$

(b) $\rho^2 \sin^2 \phi = 4\rho \sin \phi \sin \theta,$
 $\rho \sin \phi (\rho \sin \phi - 4 \sin \theta) = 0,$
 $\rho = \frac{4 \sin \theta}{\sin \phi}, \rho = 4 \sin \theta \csc \phi$

99. $0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq r \leq 2$

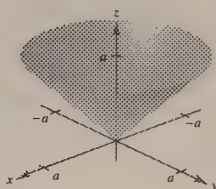
$0 \leq z \leq 4$



101. $0 \leq \theta \leq 2\pi$

$0 \leq r \leq a$

$r \leq z \leq a$



103. $0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \frac{\pi}{6}$

$0 \leq \rho \leq a \sec \phi$

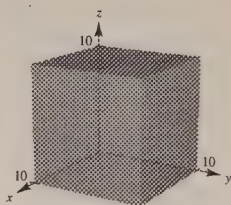


105. Rectangular

$0 \leq x \leq 10$

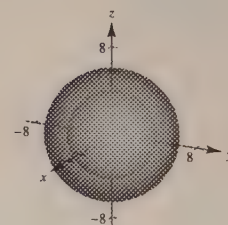
$0 \leq y \leq 10$

$0 \leq z \leq 10$



107. Spherical

$4 \leq \rho \leq 6$



109. $z = \sin \theta, r = 1$

$z = \frac{y}{r} = \frac{y}{1} = y$

The curve of intersection is the ellipse formed by the intersection of the plane $z = y$ and the cylinder $r = 1$.

Review Exercises for Chapter 10

1. $P = (1, 2), Q = (4, 1), R = (5, 4)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle 3, -1 \rangle = 3\mathbf{i} - \mathbf{j},$

$\mathbf{v} = \overrightarrow{PR} = \langle 4, 2 \rangle = 4\mathbf{i} + 2\mathbf{j}$

(b) $\|\mathbf{v}\| = \sqrt{4^2 + 2^2} = 2\sqrt{5}$

(c) $2\mathbf{u} + \mathbf{v} = \langle 6, -2 \rangle + \langle 4, 2 \rangle = \langle 10, 0 \rangle = 10\mathbf{i}$

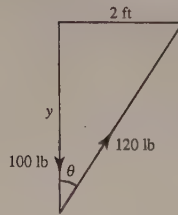
3. $\mathbf{v} = \|\mathbf{v}\| \cos \theta \mathbf{i} + \|\mathbf{v}\| \sin \theta \mathbf{j} = 8 \cos 120^\circ \mathbf{i} + 8 \sin 120^\circ \mathbf{j}$
 $= -4\mathbf{i} + 4\sqrt{3}\mathbf{j}$

5. $120 \cos \theta = 100$

$$\theta = \arccos\left(\frac{5}{6}\right)$$

$$\tan \theta = \frac{2}{y} \Rightarrow y = \frac{2}{\tan \theta}$$

$$y = \frac{2}{\tan[\arccos(5/6)]} = \frac{2}{\sqrt{11}/5} = \frac{10}{\sqrt{11}} \approx 3.015 \text{ ft}$$



7. $z = 0, y = 4, x = -5: (-5, 4, 0)$

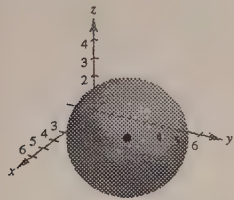
11. $(x - 3)^2 + (y + 2)^2 + (z - 6)^2 = \left(\frac{15}{2}\right)^2$

13. $(x^2 - 4x + 4) + (y^2 - 6y + 9) + z^2 = -4 + 4 + 9$

$$(x - 2)^2 + (y - 3)^2 + z^2 = 9$$

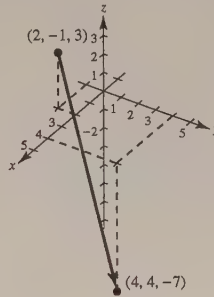
Center: $(2, 3, 0)$

Radius: 3



9. Looking down from the positive x -axis towards the yz -plane, the point is either in the first quadrant ($y > 0, z > 0$) or in the third quadrant ($y < 0, z < 0$). The x -coordinate can be any number.

15. $\mathbf{v} = \langle 4 - 2, 4 + 1, -7 - 3 \rangle = \langle 2, 5, -10 \rangle$



17. $\mathbf{v} = \langle -1 - 3, 6 - 4, 9 + 1 \rangle = \langle -4, 2, 10 \rangle$

$$\mathbf{w} = \langle 5 - 3, 3 - 4, -6 + 1 \rangle = \langle 2, -1, -5 \rangle$$

Since $-2\mathbf{w} = \mathbf{v}$, the points lie in a straight line.

19. Unit vector: $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle 2, 3, 5 \rangle}{\sqrt{38}} = \left\langle \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle$

21. $P = (5, 0, 0), Q = (4, 4, 0), R = (2, 0, 6)$

(a) $\mathbf{u} = \overrightarrow{PQ} = \langle -1, 4, 0 \rangle = -\mathbf{i} + 4\mathbf{j}$,

$$\mathbf{v} = \overrightarrow{PR} = \langle -3, 0, 6 \rangle = -3\mathbf{i} + 6\mathbf{k}$$

(b) $\mathbf{u} \cdot \mathbf{v} = (-1)(-3) + 4(0) + 0(6) = 3$

(c) $\mathbf{v} \cdot \mathbf{v} = 9 + 36 = 45$

23. $\mathbf{u} = \langle 7, -2, 3 \rangle, \mathbf{v} = \langle -1, 4, 5 \rangle$

Since $\mathbf{u} \cdot \mathbf{v} = 0$, the vectors are orthogonal.

25. $\mathbf{u} = 5\left(\cos \frac{3\pi}{4}\mathbf{i} + \sin \frac{3\pi}{4}\mathbf{j}\right) = \frac{5\sqrt{2}}{2}[-\mathbf{i} + \mathbf{j}]$

$$\mathbf{v} = 2\left(\cos \frac{2\pi}{3}\mathbf{i} + \sin \frac{2\pi}{3}\mathbf{j}\right) = -\mathbf{i} + \sqrt{3}\mathbf{j}$$

$$\mathbf{u} \cdot \mathbf{v} = \frac{5\sqrt{2}}{2}(1 + \sqrt{3})$$

$$\|\mathbf{u}\| = 5$$

$$\|\mathbf{v}\| = 2$$

$$\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(5\sqrt{2}/2)(1 + \sqrt{3})}{5(2)} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

$$\theta = \arccos \frac{\sqrt{2} + \sqrt{6}}{4} = 15^\circ \text{ or } \frac{3\pi}{4} - \frac{2\pi}{3} = \frac{\pi}{12} \text{ or } 15^\circ$$

27. $\mathbf{u} = \langle 10, -5, 15 \rangle, \mathbf{v} = \langle -2, 1, -3 \rangle$

$\mathbf{u} = -5\mathbf{v} \Rightarrow \mathbf{u}$ is parallel to \mathbf{v} and in the opposite direction.

$$\theta = \pi$$

29. There are many correct answers. For example: $\mathbf{v} = \pm\langle 6, -5, 0 \rangle$.

In Exercises 31–39, $\mathbf{u} = \langle 3, -2, 1 \rangle$, $\mathbf{v} = \langle 2, -4, -3 \rangle$, $\mathbf{w} = \langle -1, 2, 2 \rangle$.

$$\begin{aligned} 31. \mathbf{u} \cdot \mathbf{u} &= 3(3) + (-2)(-2) + (1)(1) \\ &= 14 = (\sqrt{14})^2 = \|\mathbf{u}\|^2 \end{aligned}$$

$$\begin{aligned} 33. \text{proj}_{\mathbf{u}} \mathbf{w} &= \left(\frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{u}\|^2} \right) \mathbf{u} \\ &= -\frac{5}{14} \langle 3, -2, 1 \rangle \\ &= \left\langle -\frac{15}{14}, \frac{10}{14}, -\frac{5}{14} \right\rangle \\ &= \left\langle -\frac{15}{14}, \frac{5}{7}, -\frac{5}{14} \right\rangle \end{aligned}$$

$$\begin{aligned} 35. \mathbf{n} = \mathbf{v} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -3 \\ -1 & 2 & 2 \end{vmatrix} = -2\mathbf{i} - \mathbf{j} \\ \|\mathbf{n}\| &= \sqrt{5} \\ \frac{\mathbf{n}}{\|\mathbf{n}\|} &= \frac{1}{\sqrt{5}}(-2\mathbf{i} - \mathbf{j}), \text{ unit vector or } \frac{1}{\sqrt{5}}(2\mathbf{i} + \mathbf{j}) \end{aligned}$$

$$\begin{aligned} 37. V &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \\ &= |\langle 3, -2, 1 \rangle \cdot \langle -2, -1, 0 \rangle| = |-4| = 4 \end{aligned}$$

$$\begin{aligned} 39. \text{Area parallelogram} &= \|\mathbf{u} \times \mathbf{v}\| = \|\langle 10, 11, -8 \rangle\| = \sqrt{10^2 + 11^2 + (-8)^2} \quad (\text{See Exercises 36, 38}) \\ &= \sqrt{285} \end{aligned}$$

$$41. \mathbf{F} = c(\cos 20^\circ \mathbf{j} + \sin 20^\circ \mathbf{k})$$

$$\overrightarrow{PQ} = 2\mathbf{k}$$

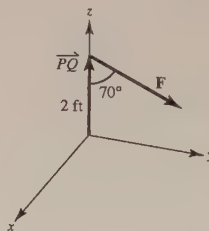
$$\overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0 & c \cos 20^\circ & c \sin 20^\circ \end{vmatrix} = -2c \cos 20^\circ \mathbf{i}$$

$$200 = \|\overrightarrow{PQ} \times \mathbf{F}\| = 2c \cos 20^\circ$$

$$c = \frac{100}{\cos 20^\circ}$$

$$\mathbf{F} = \frac{100}{\cos 20^\circ}(\cos 20^\circ \mathbf{j} + \sin 20^\circ \mathbf{k}) = 100(\mathbf{j} + \tan 20^\circ \mathbf{k})$$

$$\|\mathbf{F}\| = 100\sqrt{1 + \tan^2 20^\circ} = 100 \sec 20^\circ \approx 106.4 \text{ lb}$$



$$43. \mathbf{v} = \mathbf{j}$$

$$(a) x = 1, y = 2 + t, z = 3$$

$$(b) \text{None}$$

$$45. 3x - 3y - 7z = -4, x - y + 2z = 3$$

Solving simultaneously, we have $z = 1$. Substituting $z = 1$ into the second equation we have $y = x - 1$. Substituting for x in this equation we obtain two points on the line of intersection, $(0, -1, 1)$, $(1, 0, 1)$. The direction vector of the line of intersection is $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

$$(a) x = t, y = -1 + t, z = 1$$

$$(b) x = y + 1, z = 1$$

47. The two lines are parallel as they have the same direction numbers, $-2, 1, 1$. Therefore, a vector parallel to the plane is $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. A point on the first line is $(1, 0, -1)$ and a point on the second line is $(-1, 1, 2)$. The vector $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ connecting these two points is also parallel to the plane. Therefore, a normal to the plane is

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 2 & -1 & -3 \end{vmatrix}$$

$$= -2\mathbf{i} - 4\mathbf{j} = -2(\mathbf{i} + 2\mathbf{j}).$$

Equation of the plane: $(x - 1) + 2y = 0$

$$x + 2y = 1$$

51. $Q(3, -2, 4)$ point

$P(5, 0, 0)$ point on plane

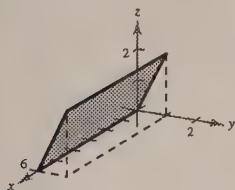
$\mathbf{n} = \langle 2, -5, 1 \rangle$ normal to plane

$$\overrightarrow{PQ} = \langle -2, -2, 4 \rangle$$

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{10}{\sqrt{30}} = \frac{\sqrt{30}}{3}$$

55. $y = \frac{1}{2}z$

Plane with rulings parallel to the x -axis



59. $\frac{x^2}{16} - \frac{y^2}{9} + z^2 = -1$

$$\frac{y^2}{9} - \frac{x^2}{16} - z^2 = 1$$

Hyperboloid of two sheets

$$xy\text{-trace: } \frac{y^2}{9} - \frac{x^2}{16} = 1$$

xz -trace: None

$$yz\text{-trace: } \frac{y^2}{9} - z^2 = 1$$



49. $Q(1, 0, 2)$ point

$$2x - 3y + 6z = 6$$

A point P on the plane is $(3, 0, 0)$.

$$\overrightarrow{PQ} = \langle -2, 0, 2 \rangle$$

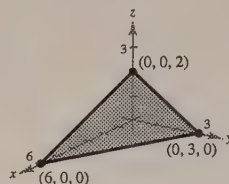
$\mathbf{n} = \langle 2, -3, 6 \rangle$ normal to plane

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{8}{7}$$

53. $x + 2y + 3z = 6$

Plane

Intercepts: $(6, 0, 0)$, $(0, 3, 0)$, $(0, 0, 2)$



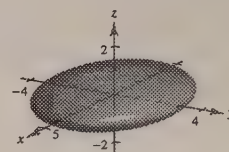
57. $\frac{x^2}{16} + \frac{y^2}{9} + z^2 = 1$

Ellipsoid

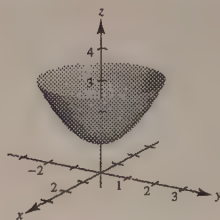
$$xy\text{-trace: } \frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$xz\text{-trace: } \frac{x^2}{16} + z^2 = 1$$

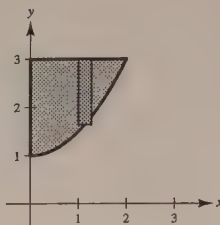
$$yz\text{-trace: } \frac{y^2}{9} + z^2 = 1$$



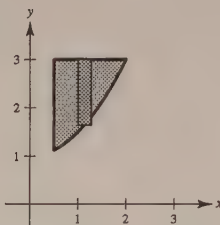
$$\begin{aligned}
 61. \quad (a) \quad x^2 + y^2 &= [r(z)]^2 \\
 &= [\sqrt{2(z-1)}]^2 \\
 x^2 + y^2 - 2z + 2 &= 0
 \end{aligned}$$



$$\begin{aligned}
 (b) \quad V &= 2\pi \int_0^2 x \left[3 - \left(\frac{1}{2}x^2 + 1 \right) \right] dx \\
 &= 2\pi \int_0^2 \left(2x - \frac{1}{2}x^3 \right) dx \\
 &= 2\pi \left[x^2 - \frac{x^4}{8} \right]_0^2 \\
 &= 4\pi \approx 12.6 \text{ cm}^3
 \end{aligned}$$



$$\begin{aligned}
 (c) \quad V &= 2\pi \int_{1/2}^2 x \left[3 - \left(\frac{1}{2}x^2 + 1 \right) \right] dx \\
 &= 2\pi \int_{1/2}^2 \left(2x - \frac{1}{2}x^3 \right) dx \\
 &= 2\pi \left[x^2 - \frac{x^4}{8} \right]_{1/2}^2 \\
 &= 4\pi - \frac{31\pi}{64} = \frac{225\pi}{64} \approx 11.04 \text{ cm}^3
 \end{aligned}$$



$$63. (-2\sqrt{2}, 2\sqrt{2}, 2), \text{ rectangular}$$

$$(a) \quad r = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2} = 4, \quad \theta = \arctan(-1) = \frac{3\pi}{4}, \quad z = 2, \quad \left(4, \frac{3\pi}{4}, 2 \right), \text{ cylindrical}$$

$$(b) \quad \rho = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2 + (2)^2} = 2\sqrt{5}, \quad \theta = \frac{3\pi}{4}, \quad \phi = \arccos \frac{2}{2\sqrt{5}} = \arccos \frac{1}{\sqrt{5}}, \quad \left(2\sqrt{5}, \frac{3\pi}{4}, \arccos \frac{\sqrt{5}}{5} \right), \text{ spherical}$$

$$65. \left(100, -\frac{\pi}{6}, 50 \right), \text{ cylindrical}$$

$$\rho = \sqrt{100^2 + 50^2} = 50\sqrt{5}$$

$$\theta = -\frac{\pi}{6}$$

$$\phi = \arccos \left(\frac{50}{50\sqrt{5}} \right) = \arccos \left(\frac{1}{\sqrt{5}} \right) \approx 63.4^\circ \text{ or } 1.107$$

$$\left(50\sqrt{5}, -\frac{\pi}{6}, 63.4^\circ \right), \text{ spherical}$$

$$67. \left(25, -\frac{\pi}{4}, \frac{3\pi}{4} \right), \text{ spherical}$$

$$r^2 = \left(25 \sin \left(\frac{3\pi}{4} \right) \right)^2 \Rightarrow r = 25 \frac{\sqrt{2}}{2}$$

$$\theta = -\frac{\pi}{4}$$

$$z = \rho \cos \phi = 25 \cos \frac{3\pi}{4} = -25 \frac{\sqrt{2}}{2}$$

$$\left(25 \frac{\sqrt{2}}{2}, -\frac{\pi}{4}, -\frac{25\sqrt{2}}{2} \right), \text{ cylindrical}$$

$$69. x^2 - y^2 = 2z$$

$$(a) \text{ Cylindrical: } r^2 \cos^2 \theta - r^2 \sin^2 \theta = 2z, \quad r^2 \cos 2\theta = 2z$$

$$(b) \text{ Spherical: } \rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta = 2\rho \cos \phi, \quad \rho \sin^2 \phi \cos 2\theta - 2 \cos \phi = 0, \quad \rho = 2 \sec 2\theta \cos \phi \csc^2 \phi$$

Problem Solving for Chapter 10

1. $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

$\mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$

$(\mathbf{b} \times \mathbf{a}) + (\mathbf{b} \times \mathbf{c}) = \mathbf{0}$

$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{b} \times \mathbf{c}\|$

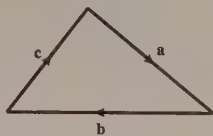
$\|\mathbf{b} \times \mathbf{c}\| = \|\mathbf{b}\| \|\mathbf{c}\| \sin A$

$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin C$

Then,

$$\begin{aligned} \frac{\sin A}{\|\mathbf{a}\|} &= \frac{\|\mathbf{b} \times \mathbf{c}\|}{\|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|} \\ &= \frac{\|\mathbf{a} \times \mathbf{b}\|}{\|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\|} \\ &= \frac{\sin C}{\|\mathbf{c}\|}. \end{aligned}$$

The other case, $\frac{\sin A}{\|\mathbf{a}\|} = \frac{\sin B}{\|\mathbf{b}\|}$ is similar.



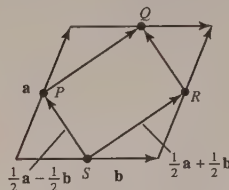
3. Label the figure as indicated.

From the figure, you see that

$$\overrightarrow{SP} = \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} = \overrightarrow{RQ} \text{ and}$$

$$\overrightarrow{SR} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \overrightarrow{PQ}.$$

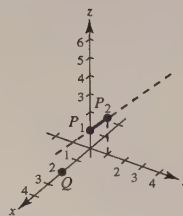
Since $\overrightarrow{SP} = \overrightarrow{RQ}$ and $\overrightarrow{SR} = \overrightarrow{PQ}$, $PSRQ$ is a parallelogram.



5. (a) $\mathbf{u} = \langle 0, 1, 1 \rangle$ direction vector of line determined by P_1 and P_2 .

$$\begin{aligned} D &= \frac{\|\overrightarrow{P_1Q} \times \mathbf{u}\|}{\|\mathbf{u}\|} \\ &= \frac{\|\langle 2, 0, -1 \rangle \times \langle 0, 1, 1 \rangle\|}{\sqrt{2}} \\ &= \frac{\|\langle 1, -2, 2 \rangle\|}{\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \end{aligned}$$

(b) The shortest distance to the line segment is $\|P_1Q\| = \|\langle 2, 0, -1 \rangle\| = \sqrt{5}$.



7. (a) $V = \pi \int_0^1 (\sqrt{z})^2 dz = \left[\pi \frac{z^2}{2} \right]_0^1 = \frac{1}{2} \pi$

Note: $\frac{1}{2}(\text{base})(\text{altitude}) = \frac{1}{2} \pi(1) = \frac{1}{2} \pi$

(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$: (slice at $z = c$)

$$\frac{x^2}{(\sqrt{ca})^2} + \frac{y^2}{(\sqrt{cb})^2} = 1$$

At $z = c$, figure is ellipse of area

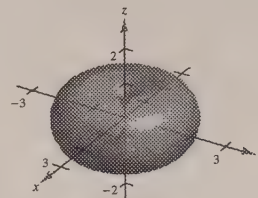
$$\pi(\sqrt{ca})(\sqrt{cb}) = \pi abc.$$

$$V = \int_0^k \pi abc \cdot dc = \left[\frac{\pi abc^2}{2} \right]_0^k = \frac{\pi abk^2}{2}$$

(c) $V = \frac{1}{2}(\pi abk)k = \frac{1}{2}(\text{base})(\text{height})$

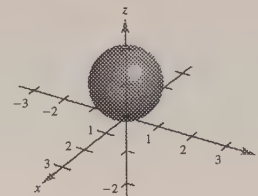
9. (a) $\rho = 2 \sin \phi$

Torus



(b) $\rho = 2 \cos \phi$

Sphere



11. From Exercise 64, Section 10.4, $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{z}) = [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{z}]\mathbf{w} - [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}]\mathbf{z}$.

13. (a) $\mathbf{u} = \|\mathbf{u}\|(\cos 0 \mathbf{i} + \sin 0 \mathbf{j}) = \|\mathbf{u}\|\mathbf{i}$

Downward force $\mathbf{w} = -\mathbf{j}$

$$\mathbf{T} = \|\mathbf{T}\|(\cos(90^\circ + \theta)\mathbf{i} + \sin(90^\circ + \theta)\mathbf{j})$$

$$= \|\mathbf{T}\|(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j})$$

$$\mathbf{0} = \mathbf{u} + \mathbf{w} + \mathbf{T} = \|\mathbf{u}\|\mathbf{i} - \mathbf{j} + \|\mathbf{T}\|(-\sin \theta \mathbf{i} + \cos \theta \mathbf{j})$$

$$\|\mathbf{u}\| = \sin \theta \|\mathbf{T}\|$$

$$1 = \cos \theta \|\mathbf{T}\|$$

$$\text{If } \theta = 30^\circ, \|\mathbf{u}\| = (1/2)\|\mathbf{T}\| \text{ and } 1 = (\sqrt{3}/2)\|\mathbf{T}\|$$

$$\Rightarrow \|\mathbf{T}\| = \frac{2}{\sqrt{3}} \approx 1.1547 \text{ lb}$$

and

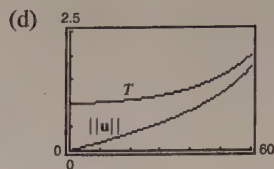
$$\|\mathbf{u}\| = \frac{1}{2} \left(\frac{2}{\sqrt{3}} \right) \approx 0.5774 \text{ lb}$$

(b) From part (a), $\|\mathbf{u}\| = \tan \theta$ and $\|\mathbf{T}\| = \sec \theta$.

Domain: $0 \leq \theta \leq 90^\circ$

(c)

θ	0°	10°	20°	30°	40°	50°	60°
T	1	1.0154	1.0642	1.1547	1.3054	1.5557	2
$\ \mathbf{u}\ $	0	0.1763	0.3640	0.5774	0.8391	1.1918	1.7321



(e) Both are increasing functions.

(f) $\lim_{\theta \rightarrow \pi/2^-} T = \infty$ and $\lim_{\theta \rightarrow \pi/2^-} \|\mathbf{u}\| = \infty$.

15. Let $\theta = \alpha - \beta$, the angle between \mathbf{u} and \mathbf{v} . Then

$$\sin(\alpha - \beta) = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{\|\mathbf{v} \times \mathbf{u}\|}{\|\mathbf{u}\| \|\mathbf{v}\|}.$$

For $\mathbf{u} = \langle \cos \alpha, \sin \alpha, 0 \rangle$ and $\mathbf{v} = \langle \cos \beta, \sin \beta, 0 \rangle$, $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$ and

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \beta & \sin \beta & 0 \\ \cos \alpha & \sin \alpha & 0 \end{vmatrix} = (\sin \alpha \cos \beta - \cos \alpha \sin \beta)\mathbf{k}.$$

Thus, $\sin(\alpha - \beta) = \|\mathbf{v} \times \mathbf{u}\| = \sin \alpha \cos \beta - \cos \alpha \sin \beta$.

17. From Theorem 10.13 and Theorem 10.7 (6) we have

$$\begin{aligned} D &= \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} \\ &= \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{u} \times \mathbf{v}\|}. \end{aligned}$$

19. a_1, b_1, c_1 , and a_2, b_2, c_2 are two sets of direction numbers for the same line. The line is parallel to both $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$. Therefore, \mathbf{u} and \mathbf{v} are parallel, and there exists a scalar d such that $\mathbf{u} = d\mathbf{v}$, $a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k} = d(a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k})$, $a_1 = a_2d$, $b_1 = b_2d$, $c_1 = c_2d$.

CHAPTER 11

Vector-Valued Functions

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CHAPTER 11

Vector-Valued Functions

Section 11.1 Vector-Valued Functions

Solutions to Odd-Numbered Exercises

$$1. \mathbf{r}(t) = 5t\mathbf{i} - 4t\mathbf{j} - \frac{1}{t}\mathbf{k}$$

Component functions: $f(t) = 5t$

$$g(t) = -4t$$

$$h(t) = -\frac{1}{t}$$

Domain: $(-\infty, 0) \cup (0, \infty)$

$$3. \mathbf{r}(t) = \ln t\mathbf{i} - e^t\mathbf{j} - t\mathbf{k}$$

Component functions: $f(t) = \ln t$

$$g(t) = -e^t$$

$$h(t) = -t$$

Domain: $(0, \infty)$

$$5. \mathbf{r}(t) = \mathbf{F}(t) + \mathbf{G}(t) = (\cos t\mathbf{i} - \sin t\mathbf{j} + \sqrt{t}\mathbf{k}) + (\cos t\mathbf{i} + \sin t\mathbf{j}) = 2\cos t\mathbf{i} + \sqrt{t}\mathbf{k}$$

Domain: $[0, \infty)$

$$7. \mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t & \cos t & 0 \\ 0 & \sin t & \cos t \end{vmatrix} = \cos^2 t\mathbf{i} - \sin t \cos t\mathbf{j} + \sin^2 t\mathbf{k}$$

Domain: $(-\infty, \infty)$

$$9. \mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - (t-1)\mathbf{j}$$

$$(a) \mathbf{r}(1) = \frac{1}{2}\mathbf{i}$$

$$(b) \mathbf{r}(0) = \mathbf{j}$$

$$(c) \mathbf{r}(s+1) = \frac{1}{2}(s+1)^2\mathbf{i} - (s+1-1)\mathbf{j} = \frac{1}{2}(s+1)^2\mathbf{i} - s\mathbf{j}$$

$$(d) \mathbf{r}(2+\Delta t) - \mathbf{r}(2) = \frac{1}{2}(2+\Delta t)^2\mathbf{i} - (2+\Delta t-1)\mathbf{j} - (2\mathbf{i} - \mathbf{j})$$

$$= (2 + 2\Delta t + \frac{1}{2}(\Delta t)^2)\mathbf{i} - (1 + \Delta t)\mathbf{j} - 2\mathbf{i} + \mathbf{j}$$

$$= (2\Delta t + \frac{1}{2}(\Delta t)^2)\mathbf{i} - (\Delta t)\mathbf{j}$$

$$11. \mathbf{r}(t) = \ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + 3t\mathbf{k}$$

$$(a) \mathbf{r}(2) = \ln 2\mathbf{i} + \frac{1}{2}\mathbf{j} + 6\mathbf{k}$$

$$(b) \mathbf{r}(-3) \text{ is not defined. } (\ln(-3) \text{ does not exist.})$$

$$(c) \mathbf{r}(t-4) = \ln(t-4)\mathbf{i} + \frac{1}{t-4}\mathbf{j} + 3(t-4)\mathbf{k}$$

$$(d) \mathbf{r}(1+\Delta t) - \mathbf{r}(1) = \ln(1+\Delta t)\mathbf{i} + \frac{1}{1+\Delta t}\mathbf{j} + 3(1+\Delta t)\mathbf{k} - (0\mathbf{i} + \mathbf{j} + 3\mathbf{k})$$

$$= \ln(1+\Delta t)\mathbf{i} + \left(\frac{1}{1+\Delta t} - 1\right)\mathbf{j} + (3\Delta t)\mathbf{k}$$

13. $\mathbf{r}(t) = \sin 3t\mathbf{i} + \cos 3t\mathbf{j} + t\mathbf{k}$

$$\|\mathbf{r}(t)\| = \sqrt{(\sin 3t)^2 + (\cos 3t)^2 + t^2} = \sqrt{1 + t^2}$$

17. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}, -2 \leq t \leq 2$

$$x = t, y = 2t, z = t^2$$

Thus, $z = x^2$. Matches (b)

15. $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3t - 1)(t^2) + \left(\frac{1}{4}t^3\right)(-8) + 4(t^3)$

$$= 3t^3 - t^2 - 2t^3 + 4t^3 = 5t^3 - t^2, \text{ a scalar.}$$

The dot product is a scalar-valued function.

19. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}, -2 \leq t \leq 2$

$$x = t, y = t^2, z = e^{0.75t}$$

Thus, $y = x^2$. Matches (d)

21. (a) View from the negative x -axis: $(-20, 0, 0)$

(c) View from the z -axis: $(0, 0, 20)$

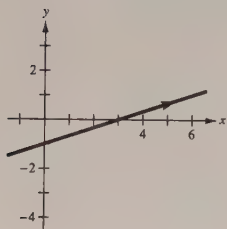
(b) View from above the first octant: $(10, 20, 10)$

(d) View from the positive x -axis: $(20, 0, 0)$

23. $x = 3t$

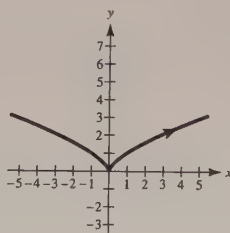
$$y = t - 1$$

$$y = \frac{x}{3} - 1$$



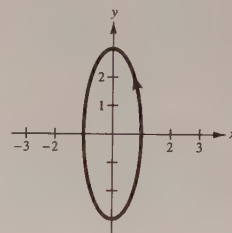
25. $x = t^3, y = t^2$

$$y = x^{2/3}$$



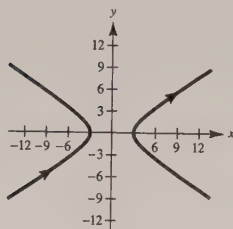
27. $x = \cos \theta, y = 3 \sin \theta$

$$x^2 + \frac{y^2}{9} = 1 \text{ Ellipse}$$



29. $x = 3 \sec \theta, y = 2 \tan \theta$

$$\frac{x^2}{9} = \frac{y^2}{4} + 1 \text{ Hyperbola}$$



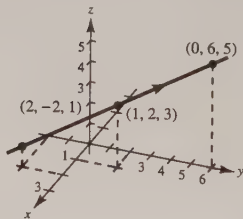
31. $x = -t + 1$

$$y = 4t + 2$$

$$z = 2t + 3$$

Line passing through the points:

$$(0, 6, 5), (1, 2, 3)$$

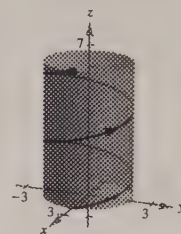


33. $x = 2 \cos t, y = 2 \sin t, z = t$

$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$

$$z = t$$

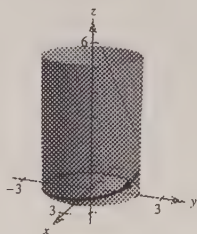
Circular helix



35. $x = 2 \sin t, y = 2 \cos t, z = e^{-t}$

$$x^2 + y^2 = 4$$

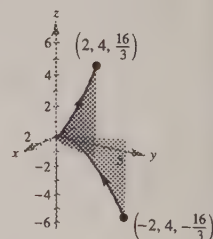
$$z = e^{-t}$$



37. $x = t, y = t^2, z = \frac{2}{3}t^3$

$$y = x^2, z = \frac{2}{3}x^3$$

t	-2	-1	0	1	2
x	-2	-1	0	1	2
y	4	1	0	1	4
z	$-\frac{16}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{16}{3}$



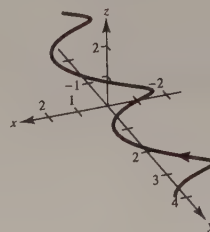
39. $\mathbf{r}(t) = -\frac{1}{2}t^2\mathbf{i} + t\mathbf{j} - \frac{\sqrt{3}}{2}t^2\mathbf{k}$

Parabola

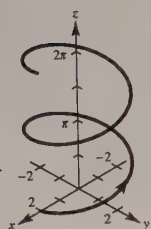


41. $\mathbf{r}(t) = \sin t\mathbf{i} + \left(\frac{\sqrt{3}}{2}\cos t - \frac{1}{2}t\right)\mathbf{j} + \left(\frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\right)\mathbf{k}$

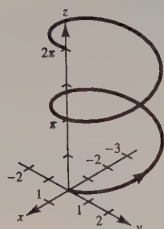
Helix



43.

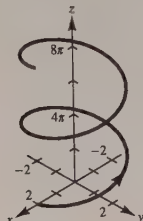


(a)



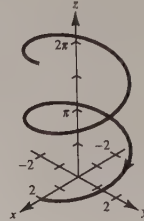
The helix is translated 2 units back on the x -axis.

(b)



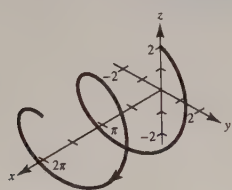
The height of the helix increases at a faster rate.

(c)



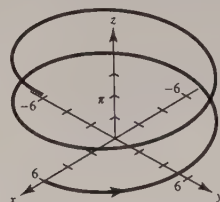
The orientation of the helix is reversed.

(d)



The axis of the helix is the x -axis.

(e)



The radius of the helix is increased from 2 to 6.

45. $y = 4 - x$

Let $x = t$, then $y = 4 - t$.

$$\mathbf{r}(t) = t\mathbf{i} + (4 - t)\mathbf{j}$$

47. $y = (x - 2)^2$

Let $x = t$, then $y = (t - 2)^2$.

$$\mathbf{r}(t) = t\mathbf{i} + (t - 2)^2\mathbf{j}$$

49. $x^2 + y^2 = 25$

Let $x = 5 \cos t$, then $y = 5 \sin t$.

$$\mathbf{r}(t) = 5 \cos t\mathbf{i} + 5 \sin t\mathbf{j}$$

51. $\frac{x^2}{16} - \frac{y^2}{4} = 1$

Let $x = 4 \sec t$, $y = 2 \tan t$.

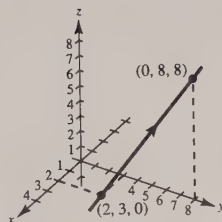
$$\mathbf{r}(t) = 4 \sec t\mathbf{i} + 2 \tan t\mathbf{j}$$

53. The parametric equations for the line are

$$x = 2 - 2t, \quad y = 3 + 5t, \quad z = 8t.$$

One possible answer is

$$\mathbf{r}(t) = (2 - 2t)\mathbf{i} + (3 + 5t)\mathbf{j} + 8t\mathbf{k}.$$



55. $\mathbf{r}_1(t) = t\mathbf{i}, \quad 0 \leq t \leq 4 \quad (\mathbf{r}_1(0) = \mathbf{0}, \mathbf{r}_1(4) = 4\mathbf{i})$

$\mathbf{r}_2(t) = (4 - 4t)\mathbf{i} + 6t\mathbf{j}, \quad 0 \leq t \leq 1 \quad (\mathbf{r}_2(0) = 4\mathbf{i}, \mathbf{r}_2(1) = 6\mathbf{j})$

$\mathbf{r}_3(t) = (6 - t)\mathbf{j}, \quad 0 \leq t \leq 6 \quad (\mathbf{r}_3(0) = 6\mathbf{j}, \mathbf{r}_3(6) = \mathbf{0})$

(Other answers possible)

57. $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \leq t \leq 2 \quad (y = x^2)$

$\mathbf{r}_2(t) = (2 - t)\mathbf{i}, \quad 0 \leq t \leq 2$

$\mathbf{r}_3(t) = (4 - t)\mathbf{j}, \quad 0 \leq t \leq 4$

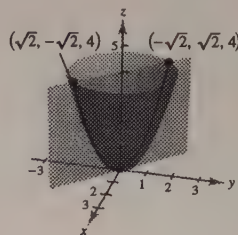
(Other answers possible)

59. $z = x^2 + y^2, \quad x + y = 0$

Let $x = t$, then $y = -x = -t$ and $z = x^2 + y^2 = 2t^2$.
Therefore,

$x = t, \quad y = -t, \quad z = 2t^2.$

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$



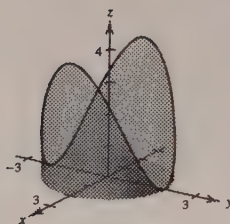
61. $x^2 + y^2 = 4, \quad z = x^2$

$x = 2 \sin t, \quad y = 2 \cos t$

$z = x^2 = 4 \sin^2 t$

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	0	1	$\sqrt{2}$	2	$\sqrt{2}$	0
y	2	$\sqrt{3}$	$\sqrt{2}$	0	$-\sqrt{2}$	-2
z	0	1	2	4	2	0

$\mathbf{r}(t) = 2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + 4 \sin^2 t \mathbf{k}$



63. $x^2 + y^2 + z^2 = 4, \quad x + z = 2$

Let $x = 1 + \sin t$, then $z = 2 - x = 1 - \sin t$ and $x^2 + y^2 + z^2 = 4$.

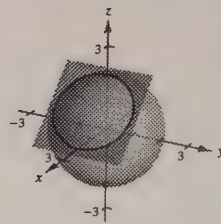
$(1 + \sin t)^2 + y^2 + (1 - \sin t)^2 = 2 + 2 \sin^2 t + y^2 = 4$

$y^2 = 2 \cos^2 t, \quad y = \pm \sqrt{2} \cos t$

$x = 1 + \sin t, \quad y = \pm \sqrt{2} \cos t$

$z = 1 - \sin t$

t	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	0	$\pm \frac{\sqrt{6}}{2}$	$\pm \sqrt{2}$	$\pm \frac{\sqrt{6}}{2}$	0
z	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0



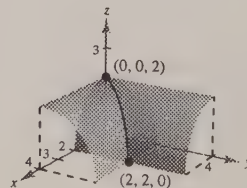
$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + \sqrt{2} \cos t \mathbf{j} - (1 - \sin t)\mathbf{k}$ and

$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} - \sqrt{2} \cos t \mathbf{j} + (1 - \sin t)\mathbf{k}$

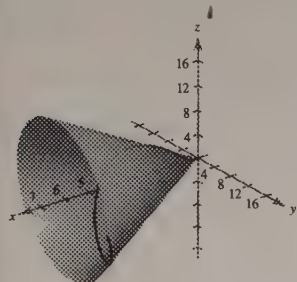
65. $x^2 + z^2 = 4, \quad y^2 + z^2 = 4$

Subtracting, we have $x^2 - y^2 = 0$ or $y = \pm x$.Therefore, in the first octant, if we let $x = t$, then $x = t, \quad y = t, \quad z = \sqrt{4 - t^2}$.

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$



$$67. y^2 + z^2 = (2t \cos t)^2 + (2t \sin t)^2 = 4t^2 = 4x^2$$



$$71. \lim_{t \rightarrow 0} \left[t^2 \mathbf{i} + 3t \mathbf{j} + \frac{1 - \cos t}{t} \mathbf{k} \right] = \mathbf{0}$$

since

$$\lim_{t \rightarrow 0} \frac{1 - \cos t}{t} = \lim_{t \rightarrow 0} \frac{\sin t}{1} = 0. \quad (\text{L'Hôpital's Rule})$$

$$75. \mathbf{r}(t) = t \mathbf{i} + \frac{1}{t} \mathbf{j}$$

Continuous on $(-\infty, 0), (0, \infty)$

$$79. \mathbf{r}(t) = \langle e^{-t}, t^2, \tan t \rangle$$

Discontinuous at $t = \frac{\pi}{2} + n\pi$

Continuous on $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$

$$83. \mathbf{r}(t) = t^2 \mathbf{i} + (t - 3) \mathbf{j} + t \mathbf{k}$$

$$(a) \mathbf{s}(t) = \mathbf{r}(t) + 2\mathbf{k} = t^2 \mathbf{i} + (t - 3) \mathbf{j} + (t + 3) \mathbf{k}$$

$$(b) \mathbf{s}(t) = \mathbf{r}(t) - 2\mathbf{i} = (t^2 - 2) \mathbf{i} + (t - 3) \mathbf{j} + t \mathbf{k}$$

$$(c) \mathbf{s}(t) = \mathbf{r}(t) + 5\mathbf{j} = t^2 \mathbf{i} + (t + 2) \mathbf{j} + t \mathbf{k}$$

$$85. \text{ Let } \mathbf{r}(t) = x_1(t) \mathbf{i} + y_1(t) \mathbf{j} + z_1(t) \mathbf{k} \text{ and } \mathbf{u}(t) = x_2(t) \mathbf{i} + y_2(t) \mathbf{j} + z_2(t) \mathbf{k}. \text{ Then:}$$

$$\begin{aligned} \lim_{t \rightarrow c} [\mathbf{r}(t) \times \mathbf{u}(t)] &= \lim_{t \rightarrow c} \{ [y_1(t)z_2(t) - y_2(t)z_1(t)] \mathbf{i} - [x_1(t)z_2(t) - x_2(t)z_1(t)] \mathbf{j} + [x_1(t)y_2(t) - x_2(t)y_1(t)] \mathbf{k} \} \\ &= \left[\lim_{t \rightarrow c} y_1(t) \lim_{t \rightarrow c} z_2(t) - \lim_{t \rightarrow c} y_2(t) \lim_{t \rightarrow c} z_1(t) \right] \mathbf{i} - \left[\lim_{t \rightarrow c} x_1(t) \lim_{t \rightarrow c} z_2(t) - \lim_{t \rightarrow c} x_2(t) \lim_{t \rightarrow c} z_1(t) \right] \mathbf{j} \\ &\quad + \left[\lim_{t \rightarrow c} x_1(t) \lim_{t \rightarrow c} y_2(t) - \lim_{t \rightarrow c} x_2(t) \lim_{t \rightarrow c} y_1(t) \right] \mathbf{k} \\ &= \left[\lim_{t \rightarrow c} x_1(t) \mathbf{i} + \lim_{t \rightarrow c} y_1(t) \mathbf{j} + \lim_{t \rightarrow c} z_1(t) \mathbf{k} \right] \times \left[\lim_{t \rightarrow c} x_2(t) \mathbf{i} + \lim_{t \rightarrow c} y_2(t) \mathbf{j} + \lim_{t \rightarrow c} z_2(t) \mathbf{k} \right] \\ &= \lim_{t \rightarrow c} \mathbf{r}(t) \times \lim_{t \rightarrow c} \mathbf{u}(t) \end{aligned}$$

$$87. \text{ Let } \mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}. \text{ Since } \mathbf{r} \text{ is continuous at } t = c, \text{ then } \lim_{t \rightarrow c} \mathbf{r}(t) = \mathbf{r}(c).$$

$$\mathbf{r}(c) = x(c) \mathbf{i} + y(c) \mathbf{j} + z(c) \mathbf{k} \Rightarrow x(c), y(c), z(c)$$

are defined at c .

$$\|\mathbf{r}\| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$$

$$\lim_{t \rightarrow c} \|\mathbf{r}\| = \sqrt{(x(c))^2 + (y(c))^2 + (z(c))^2} = \|\mathbf{r}(c)\|$$

Therefore, $\|\mathbf{r}\|$ is continuous at c .

$$69. \lim_{t \rightarrow 2} \left[t \mathbf{i} + \frac{t^2 - 4}{t^2 - 2t} \mathbf{j} + \frac{1}{t} \mathbf{k} \right] = 2 \mathbf{i} + 2 \mathbf{j} + \frac{1}{2} \mathbf{k}$$

since

$$\lim_{t \rightarrow 2} \frac{t^2 - 4}{t^2 - 2t} = \lim_{t \rightarrow 2} \frac{2t}{2t - 2} = 2. \quad (\text{L'Hôpital's Rule})$$

$$73. \lim_{t \rightarrow 0} \left[\frac{1}{t} \mathbf{i} + \cos t \mathbf{j} + \sin t \mathbf{k} \right]$$

does not exist since $\lim_{t \rightarrow 0} \frac{1}{t}$ does not exist.

$$77. \mathbf{r}(t) = t \mathbf{i} + \arcsin t \mathbf{j} + (t - 1) \mathbf{k}$$

Continuous on $[-1, 1]$

81. See the definition on page 786.

89. True

Section 11.2 Differentiation and Integration of Vector-Valued Functions

1. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}, t_0 = 2$

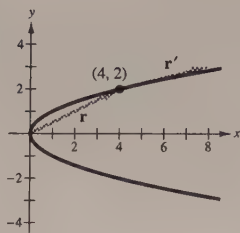
$x(t) = t^2, y(t) = t$

$x = y^2$

$\mathbf{r}(2) = 4\mathbf{i} + 2\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(2) = 4\mathbf{i} + \mathbf{j}$

 $\mathbf{r}'(t_0)$ is tangent to the curve.

3. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, t_0 = \frac{\pi}{2}$

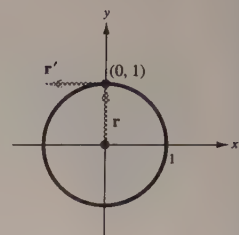
$x(t) = \cos t, y(t) = \sin t$

$x^2 + y^2 = 1$

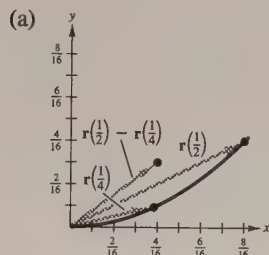
$\mathbf{r}\left(\frac{\pi}{2}\right) = \mathbf{j}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{r}'\left(\frac{\pi}{2}\right) = -\mathbf{i}$

 $\mathbf{r}'(t_0)$ is tangent to the curve.

5. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$



(b) $\mathbf{r}\left(\frac{1}{4}\right) = \frac{1}{4}\mathbf{i} + \frac{1}{16}\mathbf{j}$

$\mathbf{r}\left(\frac{1}{2}\right) = \frac{1}{2}\mathbf{i} + \frac{1}{4}\mathbf{j}$

$\mathbf{r}\left(\frac{1}{2}\right) - \mathbf{r}\left(\frac{1}{4}\right) = \frac{1}{4}\mathbf{i} + \frac{3}{16}\mathbf{j}$

(c) $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{r}'\left(\frac{1}{4}\right) = \mathbf{i} + \frac{1}{2}\mathbf{j}$

$$\frac{\mathbf{r}(1/2) - \mathbf{r}(1/4)}{(1/2) - (1/4)} = \frac{(1/4)\mathbf{i} + (3/16)\mathbf{j}}{1/4} = \mathbf{i} + \frac{3}{4}\mathbf{j}$$

This vector approximates $\mathbf{r}'\left(\frac{1}{4}\right)$.

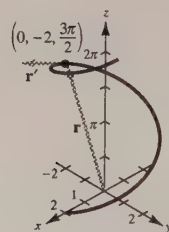
7. $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + t\mathbf{k}, t_0 = \frac{3\pi}{2}$

$x^2 + y^2 = 4, z = t$

$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \mathbf{k}$

$\mathbf{r}\left(\frac{3\pi}{2}\right) = -2\mathbf{j} + \frac{3\pi}{2}\mathbf{k}$

$\mathbf{r}'\left(\frac{3\pi}{2}\right) = 2\mathbf{i} + \mathbf{k}$



9. $\mathbf{r}(t) = 6t\mathbf{i} - 7t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{r}'(t) = 6\mathbf{i} - 14t\mathbf{j} + 3t^2\mathbf{k}$

13. $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j}$

$\mathbf{r}'(t) = -e^{-t}\mathbf{i}$

17. $\mathbf{r}(t) = t^3\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$

(a) $\mathbf{r}'(t) = 3t^2\mathbf{i} + t\mathbf{j}$

$\mathbf{r}''(t) = 6t\mathbf{i} + \mathbf{j}$

11. $\mathbf{r}(t) = a \cos^3 t\mathbf{i} + a \sin^3 t\mathbf{j} + \mathbf{k}$

$\mathbf{r}'(t) = -3a \cos^2 t \sin t\mathbf{i} + 3a \sin^2 t \cos t\mathbf{j}$

15. $\mathbf{r}(t) = \langle t \sin t, t \cos t, t \rangle$

$\mathbf{r}'(t) = \langle \sin t + t \cos t, \cos t - t \sin t, 1 \rangle$

(b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 3t^2(6t) + t = 18t^3 + t$

$$19. \mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j}$$

$$(a) \mathbf{r}'(t) = -4 \sin t \mathbf{i} + 4 \cos t \mathbf{j}$$

$$\mathbf{r}''(t) = -4 \cos t \mathbf{i} - 4 \sin t \mathbf{j}$$

$$(b) \mathbf{r}'(t) \cdot \mathbf{r}''(t) = (-4 \sin t)(-4 \cos t) + 4 \cos t(-4 \sin t) = 0$$

$$21. \mathbf{r}(t) = \frac{1}{2}t^2 \mathbf{i} - t \mathbf{j} + \frac{1}{6}t^3 \mathbf{k}$$

$$(a) \mathbf{r}'(t) = t \mathbf{i} - \mathbf{j} + \frac{1}{2}t^2 \mathbf{k}$$

$$\mathbf{r}''(t) = \mathbf{i} + t \mathbf{k}$$

$$(b) \mathbf{r}'(t) \cdot \mathbf{r}''(t) = t(1) - 1(0) + \frac{1}{2}t^2(t) = t + \frac{t^3}{2}$$

$$23. \mathbf{r}(t) = \langle \cos t + t \sin t, \sin t - t \cos t, t \rangle$$

$$(a) \mathbf{r}'(t) = \langle -\sin t + \sin t + t \cos t, \cos t - \cos t + t \sin t, 1 \rangle = \langle t \cos t, t \sin t, 1 \rangle$$

$$\mathbf{r}''(t) = \langle \cos t - t \sin t, \sin t + t \cos t, 0 \rangle$$

$$(b) \mathbf{r}'(t) \cdot \mathbf{r}''(t) = (t \cos t)(\cos t - t \sin t) + (t \sin t)(\sin t + t \cos t) = t$$

$$25. \mathbf{r}(t) = \cos(\pi t) \mathbf{i} + \sin(\pi t) \mathbf{j} + t^2 \mathbf{k}, t_0 = -\frac{1}{4}$$

$$\mathbf{r}'(t) = -\pi \sin(\pi t) \mathbf{i} + \pi \cos(\pi t) \mathbf{j} + 2t \mathbf{k}$$

$$\mathbf{r}'\left(-\frac{1}{4}\right) = \frac{\sqrt{2}\pi}{2} \mathbf{i} + \frac{\sqrt{2}\pi}{2} \mathbf{j} - \frac{1}{2} \mathbf{k}$$

$$\|\mathbf{r}'\left(\frac{1}{4}\right)\| = \sqrt{\left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\pi^2 + \frac{1}{4}} = \frac{\sqrt{4\pi^2 + 1}}{2}$$

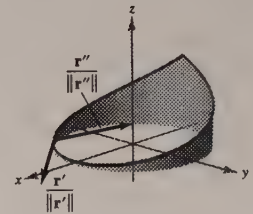
$$\frac{\mathbf{r}'(-1/4)}{\|\mathbf{r}'(-1/4)\|} = \frac{1}{\sqrt{4\pi^2 + 1}} (\sqrt{2}\pi \mathbf{i} + \sqrt{2}\pi \mathbf{j} - \mathbf{k})$$

$$\mathbf{r}''(t) = -\pi^2 \cos(\pi t) \mathbf{i} - \pi^2 \sin(\pi t) \mathbf{j} + 2 \mathbf{k}$$

$$\mathbf{r}''\left(-\frac{1}{4}\right) = -\frac{\sqrt{2}\pi^2}{2} \mathbf{i} + \frac{\sqrt{2}\pi^2}{2} \mathbf{j} + 2 \mathbf{k}$$

$$\|\mathbf{r}''\left(-\frac{1}{4}\right)\| = \sqrt{\left(-\frac{\sqrt{2}\pi^2}{2}\right)^2 + \left(\frac{\sqrt{2}\pi^2}{2}\right)^2 + (2)^2} = \sqrt{\pi^4 + 4}$$

$$\frac{\mathbf{r}''(-1/4)}{\|\mathbf{r}''(-1/4)\|} = \frac{1}{2\sqrt{\pi^4 + 4}} (-\sqrt{2}\pi^2 \mathbf{i} + \sqrt{2}\pi^2 \mathbf{j} + 4 \mathbf{k})$$



$$27. \mathbf{r}(t) = t^2 \mathbf{i} + t^3 \mathbf{j}$$

$$\mathbf{r}'(t) = 2t \mathbf{i} + 3t^2 \mathbf{j}$$

$$\mathbf{r}'(0) = \mathbf{0}$$

Smooth on $(-\infty, 0), (0, \infty)$

$$29. \mathbf{r}(\theta) = 2 \cos^3 \theta \mathbf{i} + 3 \sin^3 \theta \mathbf{j}$$

$$\mathbf{r}'(\theta) = -6 \cos^2 \theta \sin \theta \mathbf{i} + 9 \sin^2 \theta \cos \theta \mathbf{j}$$

$$\mathbf{r}'\left(\frac{n\pi}{2}\right) = \mathbf{0}$$

Smooth on $\left(\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right), n$ any integer.

$$31. \mathbf{r}(\theta) = (\theta - 2 \sin \theta) \mathbf{i} + (1 - 2 \cos \theta) \mathbf{j}$$

$$\mathbf{r}'(\theta) = (1 - 2 \cos \theta) \mathbf{i} + (1 + 2 \sin \theta) \mathbf{j}$$

$$\mathbf{r}'(\theta) \neq \mathbf{0} \text{ for any value of } \theta$$

Smooth on $(-\infty, \infty)$

$$33. \mathbf{r}(t) = (t - 1) \mathbf{i} + \frac{1}{t} \mathbf{j} - t^2 \mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{1}{t^2} \mathbf{j} - 2t \mathbf{k} \neq \mathbf{0}$$

\mathbf{r} is smooth for all $t \neq 0$: $(-\infty, 0), (0, \infty)$

35. $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$

$$\mathbf{r}'(t) = \mathbf{i} - 3\mathbf{j} + \sec^2 t\mathbf{k} \neq \mathbf{0}$$

$$\mathbf{r} \text{ is smooth for all } t \neq \frac{\pi}{2} + n\pi = \frac{2n+1}{2}\pi.$$

$$\text{Smooth on intervals of form } \left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$$

37. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}, \mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$

(b) $\mathbf{r}''(t) = 2\mathbf{k}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 4t^2 + 3t^3 + t^5$

(d) $3\mathbf{r}(t) - \mathbf{u}(t) = -t\mathbf{i} + (9t - t^2)\mathbf{j} + (3t^2 - t^3)\mathbf{k}$

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 8t + 9t^2 + 5t^4$$

$$D_t[3\mathbf{r}(t) - \mathbf{u}(t)] = -\mathbf{i} + (9 - 2t)\mathbf{j} + (6t - 3t^2)\mathbf{k}$$

(e) $\mathbf{r}(t) \times \mathbf{u}(t) = 2t^4\mathbf{i} - (t^4 - 4t^3)\mathbf{j} + (t^3 - 12t^2)\mathbf{k}$

(f) $\|\mathbf{r}(t)\| = \sqrt{10t^2 + t^4} = t\sqrt{10 + t^2}$

$$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = 8t^3\mathbf{i} + (12t^2 - 4t^3)\mathbf{j} + (3t^2 - 24t)\mathbf{k}$$

$$D_t[\|\mathbf{r}(t)\|] = \frac{10 + 2t^2}{\sqrt{10 + t^2}}$$

39. $\mathbf{r}(t) = 3 \sin t \mathbf{i} + 4 \cos t \mathbf{j}$

$$\mathbf{r}'(t) = 3 \cos t \mathbf{i} - 4 \sin t \mathbf{j}$$

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 9 \sin t \cos t - 16 \cos t \sin t = -7 \sin t \cos t$$

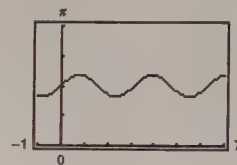
$$\cos \theta = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\| \|\mathbf{r}'(t)\|} = \frac{-7 \sin t \cos t}{\sqrt{9 \sin^2 t + 16 \cos^2 t} \sqrt{9 \cos^2 t + 16 \sin^2 t}}$$

$$\theta = \arccos \left[\frac{-7 \sin t \cos t}{\sqrt{(9 \sin^2 t + 16 \cos^2 t)(9 \cos^2 t + 16 \sin^2 t)}} \right]$$

$$\theta = 1.855 \text{ maximum at } t = 3.927 \left(\frac{5\pi}{4} \right) \text{ and } t = 0.785 \left(\frac{\pi}{4} \right).$$

$$\theta = 1.287 \text{ minimum at } t = 2.356 \left(\frac{3\pi}{4} \right) \text{ and } t = 5.498 \left(\frac{7\pi}{4} \right).$$

$$\theta = \frac{\pi}{2} (1.571) \text{ for } t = \frac{n\pi}{2}, n = 0, 1, 2, 3, \dots$$



41. $\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$

$$= \lim_{\Delta t \rightarrow 0} \frac{[3(t + \Delta t) + 2]\mathbf{i} + [1 - (t + \Delta t)^2]\mathbf{j} - (3t + 2)\mathbf{i} - (1 - t^2)\mathbf{j}}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{(3\Delta t)\mathbf{i} - (2t(\Delta t) + (\Delta t)^2)\mathbf{j}}{\Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} 3\mathbf{i} - (2t + \Delta t)\mathbf{j} = 3\mathbf{i} - 2t\mathbf{j}$$

43. $\int (2t\mathbf{i} + \mathbf{j} + \mathbf{k}) dt = t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

45. $\int \left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\mathbf{k} \right) dt = \ln t \mathbf{i} + t\mathbf{j} - \frac{2}{5}t^{5/2}\mathbf{k} + \mathbf{C}$

47. $\int [(2t - 1)\mathbf{i} + 4t^3\mathbf{j} + 3\sqrt{t}\mathbf{k}] dt = (t^2 - t)\mathbf{i} + t^4\mathbf{j} + 2t^{3/2}\mathbf{k} + \mathbf{C}$

49. $\int \left[\sec^2 t \mathbf{i} + \frac{1}{1 + t^2} \mathbf{j} \right] dt = \tan t \mathbf{i} + \arctan t \mathbf{j} + \mathbf{C}$

$$51. \int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k}) dt = \left[4t^2\mathbf{i} \right]_0^1 + \left[\frac{t^2}{2}\mathbf{j} \right]_0^1 - \left[t\mathbf{k} \right]_0^1 = 4\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$$

$$53. \int_0^{\pi/2} [(a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + \mathbf{k}] dt = \left[a \sin t\mathbf{i} \right]_0^{\pi/2} - \left[a \cos t\mathbf{j} \right]_0^{\pi/2} + \left[t\mathbf{k} \right]_0^{\pi/2} = a\mathbf{i} + a\mathbf{j} + \frac{\pi}{2}\mathbf{k}$$

$$55. \mathbf{r}(t) = \int (4e^{2t}\mathbf{i} + 3e^t\mathbf{j}) dt = 2e^{2t}\mathbf{i} + 3e^t\mathbf{j} + \mathbf{C}$$

$$\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{C} = 2\mathbf{i} \Rightarrow \mathbf{C} = -3\mathbf{j}$$

$$\mathbf{r}(t) = 2e^{2t}\mathbf{i} + 3(e^t - 1)\mathbf{j}$$

$$57. \mathbf{r}'(t) = \int -32\mathbf{j} dt = -32t\mathbf{j} + \mathbf{C}_1$$

$$\mathbf{r}'(0) = \mathbf{C}_1 = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}$$

$$\mathbf{r}'(t) = 600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}$$

$$\mathbf{r}(t) = \int [600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}] dt$$

$$= 600\sqrt{3}t\mathbf{i} + (600t - 16t^2)\mathbf{j} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}$$

$$\mathbf{r}(t) = 600\sqrt{3}t\mathbf{i} + (600t - 16t^2)\mathbf{j}$$

$$59. \mathbf{r}(t) = \int (te^{-t^2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}) dt = -\frac{1}{2}e^{-t^2}\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = -\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{C} = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}(t) = \left(1 - \frac{1}{2}e^{-t^2}\right)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t + 1)\mathbf{k} = \left(\frac{2 - e^{-t^2}}{2}\right)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t + 1)\mathbf{k}$$

61. See "Definition of the Derivative of a Vector-Valued Function" and Figure 11.8 on page 794.

63. At $t = t_0$, the graph of $\mathbf{u}(t)$ is increasing in the x , y , and z directions simultaneously.

65. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $c\mathbf{r}(t) = cx(t)\mathbf{i} + cy(t)\mathbf{j} + cz(t)\mathbf{k}$ and

$$D_t[c\mathbf{r}(t)] = cx'(t)\mathbf{i} + cy'(t)\mathbf{j} + cz'(t)\mathbf{k}$$

$$= c[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] = c\mathbf{r}'(t).$$

67. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then $f(t)\mathbf{r}(t) = f(t)x(t)\mathbf{i} + f(t)y(t)\mathbf{j} + f(t)z(t)\mathbf{k}$.

$$D_t[f(t)\mathbf{r}(t)] = [f(t)x'(t) + f'(t)x(t)]\mathbf{i} + [f(t)y'(t) + f'(t)y(t)]\mathbf{j} + [f(t)z'(t) + f'(t)z(t)]\mathbf{k}$$

$$= f(t)[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] + f'(t)[x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}]$$

$$= f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$$

69. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}(f(t)) = x(f(t))\mathbf{i} + y(f(t))\mathbf{j} + z(f(t))\mathbf{k}$ and

$$D_t[\mathbf{r}(f(t))] = x'(f(t))f'(t)\mathbf{i} + y'(f(t))f'(t)\mathbf{j} + z'(f(t))f'(t)\mathbf{k} \quad (\text{Chain Rule})$$

$$= f'(t)[x'(f(t))\mathbf{i} + y'(f(t))\mathbf{j} + z'(f(t))\mathbf{k}] = f'(t)\mathbf{r}'(f(t)).$$

71. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$, $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$, and $\mathbf{v}(t) = x_3(t)\mathbf{i} + y_3(t)\mathbf{j} + z_3(t)\mathbf{k}$. Then:

$$\begin{aligned}\mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] &= x_1(t)[y_2(t)z_3(t) - z_2(t)y_3(t)] - y_1(t)[x_2(t)z_3(t) - z_2(t)x_3(t)] + z_1(t)[x_2(t)y_3(t) - y_2(t)x_3(t)] \\ D_t[\mathbf{r}(t) \cdot (\mathbf{u}(t) \times \mathbf{v}(t))] &= x_1(t)y_2(t)z_3'(t) + x_1(t)y_2'(t)z_3(t) + x_1'(t)y_2(t)z_3(t) - x_1(t)y_3(t)z_2'(t) - \\ &\quad x_1(t)y_3'(t)z_2(t) - x_1'(t)y_3(t)z_2(t) - y_1(t)x_2(t)z_3'(t) - y_1(t)x_2'(t)z_3(t) - y_1'(t)x_2(t)z_3(t) + \\ &\quad y_1(t)z_2(t)x_3'(t) + y_1(t)z_2'(t)x_3(t) + y_1'(t)z_2(t)x_3(t) + z_1(t)x_2(t)y_3'(t) + z_1(t)x_2'(t)y_3(t) + \\ &\quad z_1'(t)x_2(t)y_3(t) - z_1(t)y_2(t)x_3'(t) - z_1(t)y_2'(t)x_3(t) - z_1'(t)y_2(t)x_3(t) \\ &= \{x_1'(t)[y_2(t)z_3(t) - y_3(t)z_2(t)] + y_1'(t)[-x_2(t)z_3(t) + z_2(t)x_3(t)] + z_1'(t)[x_2(t)y_3(t) - y_2(t)x_3(t)]\} + \\ &\quad \{x_1(t)[y_2'(t)z_3(t) - y_3(t)z_2'(t)] + y_1(t)[-x_2'(t)z_3(t) + z_2'(t)x_3(t)] + z_1(t)[x_2'(t)y_3(t) - y_2'(t)x_3(t)]\} + \\ &\quad \{x_1(t)[y_2(t)z_3'(t) - y_3'(t)z_2(t)] + y_1(t)[-x_2(t)z_3'(t) + z_2(t)x_3'(t)] + z_1(t)[x_2(t)y_3'(t) - y_2(t)x_3'(t)]\} \\ &= \mathbf{r}'(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}'(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}'(t)]\end{aligned}$$

73. False. Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$.

$$\|\mathbf{r}(t)\| = \sqrt{2}$$

$$\frac{d}{dt}[\|\mathbf{r}(t)\|] = 0$$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 1$$

Section 11.3 Velocity and Acceleration

1. $\mathbf{r}(t) = 3t\mathbf{i} + (t-1)\mathbf{j}$

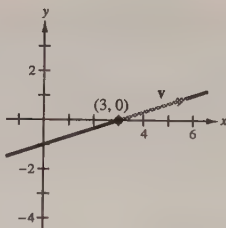
$$\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$$

$$x = 3t, y = t - 1, y = \frac{x}{3} - 1$$

$$\text{At } (3, 0), t = 1.$$

$$\mathbf{v}(1) = 3\mathbf{i} + \mathbf{j}, \mathbf{a}(1) = \mathbf{0}$$



3. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$$

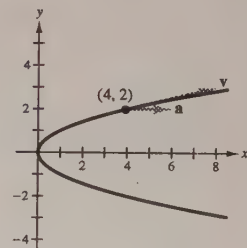
$$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}$$

$$x = t^2, y = t, x = y^2$$

$$\text{At } (4, 2), t = 2.$$

$$\mathbf{v}(2) = 4\mathbf{i} + \mathbf{j}$$

$$\mathbf{a}(2) = 2\mathbf{i}$$



5. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$$

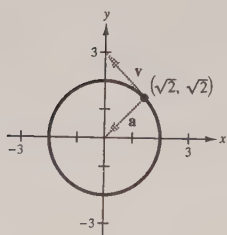
$$\mathbf{a}(t) = \mathbf{r}''(t) = -2\cos t\mathbf{i} - 2\sin t\mathbf{j}$$

$$x = 2\cos t, y = 2\sin t, x^2 + y^2 = 4$$

$$\text{At } (\sqrt{2}, \sqrt{2}), t = \frac{\pi}{4}.$$

$$\mathbf{v}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$$

$$\mathbf{a}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$$



7. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1 - \cos t, \sin t \rangle$$

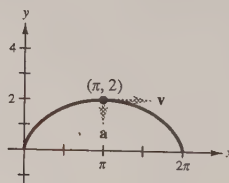
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle \sin t, \cos t \rangle$$

$$x = t - \sin t, y = 1 - \cos t \quad (\text{cycloid})$$

$$\text{At } (\pi, 2), t = \pi.$$

$$\mathbf{v}(\pi) = \langle 2, 0 \rangle = 2\mathbf{i}$$

$$\mathbf{a}(\pi) = \langle 0, -1 \rangle = -\mathbf{j}$$



9. $\mathbf{r}(t) = t\mathbf{i} + (2t - 5)\mathbf{j} + 3t\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

$$s(t) = \|\mathbf{v}(t)\| = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$\mathbf{a}(t) = \mathbf{0}$$

13. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{9 - t^2}\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \frac{t}{\sqrt{9 - t^2}}\mathbf{k}$$

$$s(t) = \sqrt{1 + 1 + \frac{t^2}{9 - t^2}} = \sqrt{\frac{18 - t^2}{9 - t^2}}$$

$$\mathbf{a}(t) = -\frac{9}{(9 - t^2)^{3/2}}\mathbf{k}$$

17. (a) $\mathbf{r}(t) = \left\langle t, -t^2, \frac{t^3}{4} \right\rangle, t_0 = 1$

$$\mathbf{r}'(t) = \left\langle 1, -2t, \frac{3t^2}{4} \right\rangle$$

$$\mathbf{r}'(1) = \left\langle 1, -2, \frac{3}{4} \right\rangle$$

$$x = 1 + t, y = -1 - 2t, z = \frac{1}{4} + \frac{3}{4}t$$

19. $\mathbf{a}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v}(0) = \mathbf{0}, \mathbf{r}(0) = \mathbf{0}$

$$\mathbf{v}(t) = \int (\mathbf{i} + \mathbf{j} + \mathbf{k}) dt = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$$

$$\mathbf{v}(0) = \mathbf{C} = \mathbf{0}, \mathbf{v}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \mathbf{v}(t) = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\mathbf{r}(t) = \int (t\mathbf{i} + t\mathbf{j} + t\mathbf{k}) dt = \frac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}, \mathbf{r}(t) = \frac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}),$$

$$\mathbf{r}(2) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

11. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$s(t) = \sqrt{1 + 4t^2 + t^2} = \sqrt{1 + 5t^2}$$

$$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$$

15. $\mathbf{r}(t) = \langle 4t, 3 \cos t, 3 \sin t \rangle$

$$\mathbf{v}(t) = \langle 4, -3 \sin t, 3 \cos t \rangle = 4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k}$$

$$s(t) = \sqrt{16 + 9 \sin^2 t + 9 \cos^2 t} = 5$$

$$\mathbf{a}(t) = \langle 0, -3 \cos t, -3 \sin t \rangle = -3 \cos t\mathbf{j} - 3 \sin t\mathbf{k}$$

(b) $\mathbf{r}(1 + 0.1) \approx \left\langle 1 + 0.1, -1 - 2(0.1), \frac{1}{4} + \frac{3}{4}(0.1) \right\rangle$
 $= \langle 1.100, -1.200, 0.325 \rangle$

21. $\mathbf{a}(t) = t\mathbf{j} + t\mathbf{k}, \mathbf{v}(1) = 5\mathbf{j}, \mathbf{r}(1) = \mathbf{0}$

$$\mathbf{v}(t) = \int (t\mathbf{j} + t\mathbf{k}) dt = \frac{t^2}{2}\mathbf{j} + \frac{t^2}{2}\mathbf{k} + \mathbf{C}$$

$$\mathbf{v}(1) = \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k} + \mathbf{C} = 5\mathbf{j} \Rightarrow \mathbf{C} = \frac{9}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$$

$$\mathbf{v}(t) = \left(\frac{t^2}{2} + \frac{9}{2}\right)\mathbf{j} + \left(\frac{t^2}{2} - \frac{1}{2}\right)\mathbf{k}$$

$$\mathbf{r}(t) = \int \left[\left(\frac{t^2}{2} + \frac{9}{2}\right)\mathbf{j} + \left(\frac{t^2}{2} - \frac{1}{2}\right)\mathbf{k} \right] dt$$

$$= \left(\frac{t^3}{6} + \frac{9}{2}t\right)\mathbf{j} + \left(\frac{t^3}{6} - \frac{1}{2}t\right)\mathbf{k} + \mathbf{C}$$

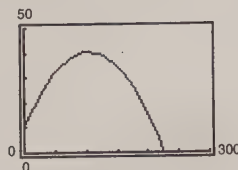
$$\mathbf{r}(1) = \frac{14}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} + \mathbf{C} = \mathbf{0} \Rightarrow \mathbf{C} = -\frac{14}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

$$\mathbf{r}(t) = \left(\frac{t^3}{6} + \frac{9}{2}t - \frac{14}{3}\right)\mathbf{j} + \left(\frac{t^3}{6} - \frac{1}{2}t + \frac{1}{3}\right)\mathbf{k}$$

$$\mathbf{r}(2) = \frac{17}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

23. The velocity of an object involves both magnitude and direction of motion, whereas speed involves only magnitude.

25. $\mathbf{r}(t) = (88 \cos 30^\circ)t\mathbf{i} + [10 + (88 \sin 30^\circ)t - 16t^2]\mathbf{j}$
 $= 44\sqrt{3}t\mathbf{i} + (10 + 44t - 16t^2)\mathbf{j}$



$$27. \mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2 \right]\mathbf{j} = \frac{v_0}{\sqrt{2}}t\mathbf{i} + \left(3 + \frac{v_0}{\sqrt{2}}t - 16t^2 \right)\mathbf{j}$$

$$\frac{v_0}{\sqrt{2}}t = 300 \text{ when } 3 + \frac{v_0}{\sqrt{2}}t - 16t^2 = 3.$$

$$t = \frac{300\sqrt{2}}{v_0}, \quad \frac{v_0}{\sqrt{2}}\left(\frac{300\sqrt{2}}{v_0}\right) - 16\left(\frac{300\sqrt{2}}{v_0}\right)^2 = 0, \quad 300 - \frac{300^2(32)}{v_0^2} = 0$$

$$v_0^2 = 300(32), \quad v_0 = \sqrt{9600} = 40\sqrt{6}, \quad v_0 = 40\sqrt{6} \approx 97.98 \text{ ft/sec}$$

The maximum height is reached when the derivative of the vertical component is zero.

$$y(t) = 3 + \frac{tv_0}{\sqrt{2}} - 16t^2 = 3 + \frac{40\sqrt{6}}{\sqrt{2}}t - 16t^2 = 3 + 40\sqrt{3}t - 16t^2$$

$$y'(t) = 40\sqrt{3} - 32t = 0$$

$$t = \frac{40\sqrt{3}}{32} = \frac{5\sqrt{3}}{4}$$

$$\text{Maximum height: } y\left(\frac{5\sqrt{3}}{4}\right) = 3 + 40\sqrt{3}\left(\frac{5\sqrt{3}}{4}\right) - 16\left(\frac{5\sqrt{3}}{4}\right)^2 = 78 \text{ feet}$$

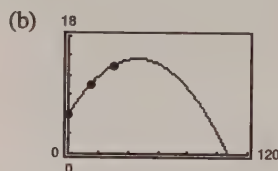
$$29. x(t) = t(v_0 \cos \theta) \text{ or } t = \frac{x}{v_0 \cos \theta}$$

$$y(t) = t(v_0 \sin \theta) - 16t^2 + h$$

$$y = \frac{x}{v_0 \cos \theta}(v_0 \sin \theta) - 16\left(\frac{x^2}{v_0^2 \cos^2 \theta}\right) + h = (\tan \theta)x - \left(\frac{16}{v_0^2} \sec^2 \theta\right)x^2 + h$$

$$31. \mathbf{r}(t) = t\mathbf{i} + (-0.004t^2 + 0.3667t + 6)\mathbf{j}$$

$$(a) y = -0.004x^2 + 0.3667x + 6$$



$$(c) y' = -0.008x + 0.3667 = 0 \Rightarrow x = 45.8375 \text{ and } y(45.8375) \approx 14.4 \text{ feet.}$$

(d) From Exercise 29,

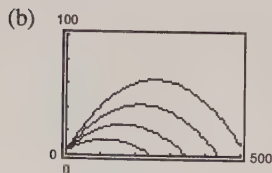
$$\tan \theta = 0.3667 \Rightarrow \theta \approx 20.14^\circ$$

$$\frac{16 \sec^2 \theta}{v_0^2} = 0.004 \Rightarrow v_0^2 = \frac{16 \sec^2 \theta}{0.004} = \frac{4000}{\cos^2 \theta}$$

$$\Rightarrow v_0 \approx 67.4 \text{ ft/sec.}$$

$$33. 100 \text{ mph} = \left(100 \frac{\text{miles}}{\text{hr}}\right) \left(5280 \frac{\text{feet}}{\text{mile}}\right) / (3600 \text{ sec/hour}) = \frac{440}{3} \text{ ft/sec}$$

$$(a) \mathbf{r}(t) = \left(\frac{440}{3} \cos \theta_0\right)t\mathbf{i} + \left[3 + \left(\frac{440}{3} \sin \theta_0\right)t - 16t^2\right]\mathbf{j}$$



Graphing these curves together with $y = 10$ shows that $\theta_0 = 20^\circ$.

—CONTINUED—

33. —CONTINUED—

(c) We want

$$x(t) = \left(\frac{440}{3} \cos \theta\right)t \geq 400 \quad \text{and} \quad y(t) = 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2 \geq 10.$$

From $x(t)$, the minimum angle occurs when $t = 30/(11 \cos \theta)$. Substituting this for t in $y(t)$ yields:

$$3 + \left(\frac{440}{3} \sin \theta\right)\left(\frac{30}{11 \cos \theta}\right) - 16\left(\frac{30}{11 \cos \theta}\right)^2 = 10$$

$$400 \tan \theta - \frac{14,400}{121} \sec^2 \theta = 7$$

$$\frac{14,400}{121}(1 + \tan^2 \theta) - 400 \tan \theta + 7 = 0$$

$$14,400 \tan^2 \theta - 48,400 \tan \theta + 15,247 = 0$$

$$\tan \theta = \frac{48,400 \pm \sqrt{48,400^2 - 4(14,400)(15,247)}}{2(14,400)}$$

$$\theta = \tan^{-1}\left(\frac{48,400 - \sqrt{1,464,332,800}}{28,800}\right) \approx 19.38^\circ$$

$$35. \mathbf{r}(t) = (v \cos \theta)t\mathbf{i} + [(v \sin \theta)t - 16t^2]\mathbf{j}$$

(a) We want to find the minimum initial speed v as a function of the angle θ . Since the bale must be thrown to the position $(16, 8)$, we have

$$16 = (v \cos \theta)t$$

$$8 = (v \sin \theta)t - 16t^2.$$

 $t = 16/(v \cos \theta)$ from the first equation. Substituting into the second equation and solving for v , we obtain:

$$8 = (v \sin \theta)\left(\frac{16}{v \cos \theta}\right) - 16\left(\frac{16}{v \cos \theta}\right)^2$$

$$1 = 2 \frac{\sin \theta}{\cos \theta} - 512\left(\frac{1}{v^2 \cos^2 \theta}\right)$$

$$512 \frac{1}{v^2 \cos^2 \theta} = 2 \frac{\sin \theta}{\cos \theta} - 1$$

$$\frac{1}{v^2} = \left(2 \frac{\sin \theta}{\cos \theta} - 1\right) \frac{\cos^2 \theta}{512} = \frac{2 \sin \theta \cos \theta - \cos^2 \theta}{512}$$

$$v^2 = \frac{512}{2 \sin \theta \cos \theta - \cos^2 \theta}$$

$$\text{We minimize } f(\theta) = \frac{512}{2 \sin \theta \cos \theta - \cos^2 \theta}.$$

$$f'(\theta) = -512 \frac{2 \cos^2 \theta - 2 \sin^2 \theta + 2 \sin \theta \cos \theta}{(2 \sin \theta \cos \theta - \cos^2 \theta)^2}$$

$$f'(\theta) = 0 \Rightarrow 2 \cos(2\theta) + \sin(2\theta) = 0$$

$$\tan(2\theta) = -2$$

$$\theta \approx 1.01722 \approx 58.28^\circ$$

Substituting into the equation for v , $v \approx 28.78$ feet per second.(b) If $\theta = 45^\circ$,

$$16 = (v \cos \theta)t = v \frac{\sqrt{2}}{2}t$$

$$8 = (v \sin \theta)t - 16t^2 = v \frac{\sqrt{2}}{2}t - 16t^2$$

$$\text{From part (a), } v^2 = \frac{512}{2(\sqrt{2}/2)(\sqrt{2}/2) - (\sqrt{2}/2)^2} = \frac{512}{1/2} = 1024 \Rightarrow v = 32 \text{ ft/sec.}$$

37. $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + [(v_0 \sin \theta)t - 16t^2]\mathbf{j}$

$$(v_0 \sin \theta)t - 16t^2 = 0 \text{ when } t = 0 \text{ and } t = \frac{v_0 \sin \theta}{16}.$$

The range is

$$x = (v_0 \cos \theta)t = (v_0 \cos \theta) \frac{v_0 \sin \theta}{16} = \frac{v_0^2}{32} \sin 2\theta.$$

Hence,

$$x = \frac{1200^2}{32} \sin(2\theta) = 3000 \Rightarrow \sin 2\theta = \frac{1}{15} \Rightarrow \theta \approx 1.91^\circ.$$

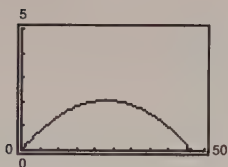
39. (a) $\theta = 10^\circ$, $v_0 = 66$ ft/sec

$$\mathbf{r}(t) = (66 \cos 10^\circ)t\mathbf{i} + [0 + (66 \sin 10^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (65t)\mathbf{i} + (11.46t - 16t^2)\mathbf{j}$$

Maximum height: 2.052 feet

Range: 46.557 feet



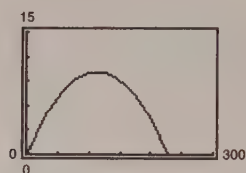
(b) $\theta = 10^\circ$, $v_0 = 146$ ft/sec

$$\mathbf{r}(t) = (146 \cos 10^\circ)t\mathbf{i} + [0 + (146 \sin 10^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (143.78t)\mathbf{i} + (25.35t - 16t^2)\mathbf{j}$$

Maximum height: 10.043 feet

Range: 227.828 feet



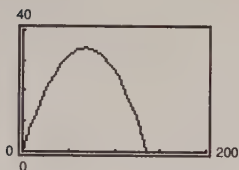
(c) $\theta = 45^\circ$, $v_0 = 66$ ft/sec

$$\mathbf{r}(t) = (66 \cos 45^\circ)t\mathbf{i} + [0 + (66 \sin 45^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (46.67t)\mathbf{i} + (46.67t - 16t^2)\mathbf{j}$$

Maximum height: 34.031 feet

Range: 136.125 feet



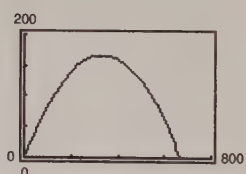
(d) $\theta = 45^\circ$, $v_0 = 146$ ft/sec

$$\mathbf{r}(t) = (146 \cos 45^\circ)t\mathbf{i} + [0 + (146 \sin 45^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (103.24t)\mathbf{i} + (103.24t - 16t^2)\mathbf{j}$$

Maximum height: 166.531 feet

Range: 666.125 feet



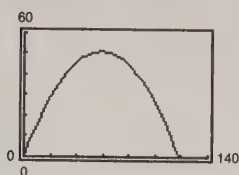
(e) $\theta = 60^\circ$, $v_0 = 66$ ft/sec

$$\mathbf{r}(t) = (66 \cos 60^\circ)t\mathbf{i} + [0 + (66 \sin 60^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (33t)\mathbf{i} + (57.16t - 16t^2)\mathbf{j}$$

Maximum height: 51.074 feet

Range: 117.888 feet



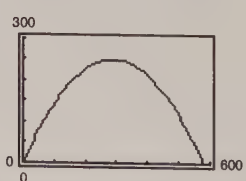
(f) $\theta = 60^\circ$, $v_0 = 146$ ft/sec

$$\mathbf{r}(t) = (146 \cos 60^\circ)t\mathbf{i} + [0 + (146 \sin 60^\circ)t - 16t^2]\mathbf{j}$$

$$\mathbf{r}(t) \approx (73t)\mathbf{i} + (126.44t - 16t^2)\mathbf{j}$$

Maximum height: 249.797 feet

Range: 576.881 feet



$$\begin{aligned}
 41. \mathbf{r}(t) &= (v_0 \cos \theta)t\mathbf{i} + [h + (v_0 \sin \theta)t - 4.9t^2]\mathbf{j} \\
 &= (100 \cos 30^\circ)t\mathbf{i} + [1.5 + (100 \sin 30^\circ)t - 4.9t^2]\mathbf{j}
 \end{aligned}$$

The projectile hits the ground when $-4.9t^2 + 100(\frac{1}{2})t + 1.5 = 0 \Rightarrow t \approx 10.234$ seconds.

The range is therefore $(100 \cos 30^\circ)(10.234) \approx 886.3$ meters.

The maximum height occurs when $dy/dt = 0$.

$$100 \sin 30 = 9.8t \Rightarrow t \approx 5.102 \text{ sec}$$

The maximum height is

$$y = 1.5 + (100 \sin 30^\circ)(5.102) - 4.9(5.102)^2 \approx 129.1 \text{ meters.}$$

$$43. \mathbf{r}(t) = b(\omega t - \sin \omega t)\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$$

$$\mathbf{v}(t) = b(\omega - \omega \cos \omega t)\mathbf{i} + b\omega \sin \omega t \mathbf{j} = b\omega(1 - \cos \omega t)\mathbf{i} + b\omega \sin \omega t \mathbf{j}$$

$$\mathbf{a}(t) = (b\omega^2 \sin \omega t)\mathbf{i} + (b\omega^2 \cos \omega t)\mathbf{j} = b\omega^2[\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}]$$

$$\|\mathbf{v}(t)\| = \sqrt{2} b\omega \sqrt{1 - \cos(\omega t)}$$

$$\|\mathbf{a}(t)\| = b\omega^2$$

$$(a) \|\mathbf{v}(t)\| = 0 \text{ when } \omega t = 0, 2\pi, 4\pi, \dots$$

$$(b) \|\mathbf{v}(t)\| \text{ is maximum when } \omega t = \pi, 3\pi, \dots, \text{ then } \|\mathbf{v}(t)\| = 2b\omega.$$

$$45. \mathbf{v}(t) = -b\omega \sin(\omega t)\mathbf{i} + b\omega \cos(\omega t)\mathbf{j}$$

$$\mathbf{r}(t) \cdot \mathbf{v}(t) = -b^2\omega \sin(\omega t) \cos(\omega t) + b^2\omega \sin(\omega t) \cos(\omega t) = 0$$

Therefore, $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are orthogonal.

$$47. \mathbf{a}(t) = -b\omega^2 \cos(\omega t)\mathbf{i} - b\omega^2 \sin(\omega t)\mathbf{j} = -b\omega^2[\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}] = -\omega^2\mathbf{r}(t)$$

$\mathbf{a}(t)$ is a negative multiple of a unit vector from $(0, 0)$ to $(\cos \omega t, \sin \omega t)$ and thus $\mathbf{a}(t)$ is directed toward the origin.

$$49. \|\mathbf{a}(t)\| = \omega^2 b, b = 2$$

$$1 = m(32)$$

$$F = m(\omega^2 b) = \frac{1}{32}(2\omega^2) = 10$$

$$\omega = 4\sqrt{10} \text{ rad/sec}$$

$$\|\mathbf{v}(t)\| = b\omega = 8\sqrt{10} \text{ ft/sec}$$

$$51. \text{ To find the range, set } y(t) = h + (v_0 \sin \theta)t - \frac{1}{2}gt^2 = 0 \text{ then } 0 = \left(\frac{1}{2}g\right)t^2 - (v_0 \sin \theta)t - h.$$

By the Quadratic Formula, (discount the negative value)

$$t = \frac{v_0 \sin \theta + \sqrt{(-v_0 \sin \theta)^2 - 4[(1/2)g](-h)}}{2[(1/2)g]} = \frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g} \text{ seconds}$$

At this time,

$$\begin{aligned}
 x(t) &= v_0 \cos \theta \left(\frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g} \right) = \frac{v_0 \cos \theta}{g} \left(v_0 \sin \theta + \sqrt{v_0^2 \left(\sin^2 \theta + \frac{2gh}{v_0^2} \right)} \right) \\
 &= \frac{v_0^2 \cos \theta}{g} \left(\sin \theta + \sqrt{\sin^2 \theta + \frac{2gh}{v_0^2}} \right) \text{ feet}
 \end{aligned}$$

53. $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ Position vector $\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ Velocity vector $\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$ Acceleration vector

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

 $= C$, C is a constant.

$$\frac{d}{dt}[x'(t)^2 + y'(t)^2 + z'(t)^2] = 0$$

$$2x'(t)x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t) = 0$$

$$2[x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t)] = 0$$

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$$

Orthogonal

55. $\mathbf{r}(t) = 6 \cos t \mathbf{i} + 3 \sin t \mathbf{j}$

$$(a) \mathbf{v}(t) = \mathbf{r}'(t) = -6 \sin t \mathbf{i} + 3 \cos t \mathbf{j}$$

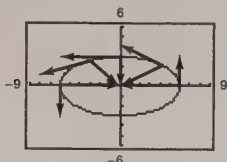
$$\|\mathbf{v}(t)\| = \sqrt{36 \sin^2 t + 9 \cos^2 t}$$

$$= 3\sqrt{4 \sin^2 t + \cos^2 t}$$

$$= 3\sqrt{3 \sin^2 t + 1}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = -6 \cos t \mathbf{i} - 3 \sin t \mathbf{j}$$

(c)



t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
Speed	3	$\frac{3}{2}\sqrt{10}$	6	$\frac{3}{2}\sqrt{13}$	3

(d) The speed is increasing when the angle between \mathbf{v} and \mathbf{a} is in the interval

$$\left[0, \frac{\pi}{2}\right).$$

The speed is decreasing when the angle is in the interval

$$\left(\frac{\pi}{2}, \pi\right].$$

Section 11.4 Tangent Vectors and Normal Vectors

1. $\mathbf{r}(t) = t^2 \mathbf{i} + 2t \mathbf{j}$, $t = 1$

$$\mathbf{r}'(t) = 2t \mathbf{i} + 2 \mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 4} = 2\sqrt{t^2 + 1}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2t \mathbf{i} + 2 \mathbf{j}}{2\sqrt{t^2 + 1}} = \frac{1}{\sqrt{t^2 + 1}}(t \mathbf{i} + \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j}$$

5. $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + t \mathbf{k}$, $p(0, 0, 0)$

$$\mathbf{r}'(t) = \mathbf{i} + 2t \mathbf{j} + \mathbf{k}$$

When $t = 0$, $\mathbf{r}'(0) = \mathbf{i} + \mathbf{k}$, [$t = 0$ at $(0, 0, 0)$].

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})$$

Direction numbers: $a = 1$, $b = 0$, $c = 1$ Parametric equations: $x = t$, $y = 0$, $z = t$ 3. $\mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j}$, $t = \frac{\pi}{4}$

$$\mathbf{r}'(t) = -4 \sin t \mathbf{i} + 4 \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{16 \sin^2 t + 16 \cos^2 t} = 4$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j}$$

7. $\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + t \mathbf{k}$, $P(2, 0, 0)$

$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \mathbf{k}$$

When $t = 0$, $\mathbf{r}'(0) = 2 \mathbf{j} + \mathbf{k}$, [$t = 0$ at $(2, 0, 0)$].

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{5}}{5}(2 \mathbf{j} + \mathbf{k})$$

Direction numbers: $a = 0$, $b = 2$, $c = 1$ Parametric equations: $x = 2$, $y = 2t$, $z = t$

9. $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 4 \rangle$, $P(\sqrt{2}, \sqrt{2}, 4)$

$$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 0 \rangle$$

When $t = \frac{\pi}{4}$, $\mathbf{r}'\left(\frac{\pi}{4}\right) = \langle -\sqrt{2}, \sqrt{2}, 0 \rangle$, $\left[t = \frac{\pi}{4} \text{ at } (\sqrt{2}, \sqrt{2}, 4)\right]$.

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\mathbf{r}'(\pi/4)}{\|\mathbf{r}'(\pi/4)\|} = \frac{1}{2} \langle -\sqrt{2}, \sqrt{2}, 0 \rangle$$

Direction numbers: $a = -\sqrt{2}$, $b = \sqrt{2}$, $c = 0$

Parametric equations: $x = -\sqrt{2}t + \sqrt{2}$, $y = \sqrt{2}t + \sqrt{2}$, $z = 4$

11. $\mathbf{r}(t) = \left\langle t, t^2, \frac{2}{3}t^3 \right\rangle$

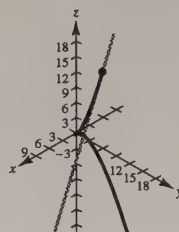
$$\mathbf{r}'(t) = \langle 1, 2t, 2t^2 \rangle$$

When $t = 3$, $\mathbf{r}'(3) = \langle 1, 6, 18 \rangle$, $[t = 3 \text{ at } (3, 9, 18)]$.

$$\mathbf{T}(3) = \frac{\mathbf{r}'(3)}{\|\mathbf{r}'(3)\|} = \frac{1}{19} \langle 1, 6, 18 \rangle$$

Direction numbers: $a = 1$, $b = 6$, $c = 18$

Parametric equations: $x = t + 3$, $y = 6t + 9$, $z = 18t + 18$



13. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \sqrt{t}\mathbf{k}$, $t_0 = 1$

$$\mathbf{r}'(t) = \mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}; \quad \mathbf{r}'(1) = \mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\mathbf{i} + \mathbf{j} + (1/2)\mathbf{k}}{\sqrt{1 + 1 + (1/4)}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

Tangent line: $x = 1 + t$, $y = t$, $z = 1 + \frac{1}{2}t$

$$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(1.1) \approx 1.1\mathbf{i} + 0.1\mathbf{j} + 1.05\mathbf{k}$$

$$= \langle 1.1, 0.1, 1.05 \rangle$$

15. $\mathbf{r}(4) = \langle 2, 16, 2 \rangle$

$$\mathbf{u}(8) = \langle 2, 16, 2 \rangle$$

Hence the curves intersect.

$$\mathbf{r}'(t) = \left\langle 1, 2t, \frac{1}{2} \right\rangle, \quad \mathbf{r}'(4) = \left\langle 1, 8, \frac{1}{2} \right\rangle$$

$$\mathbf{u}'(s) = \left\langle \frac{1}{4}, 2, \frac{1}{3}s^{-2/3} \right\rangle, \quad \mathbf{u}'(8) = \left\langle \frac{1}{4}, 2, \frac{1}{12} \right\rangle$$

$$\cos \theta = \frac{\mathbf{r}'(4) \cdot \mathbf{u}'(8)}{\|\mathbf{r}'(4)\| \|\mathbf{u}'(8)\|} \approx \frac{16.29167}{16.29513} \Rightarrow \theta \approx 1.2^\circ$$

17. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$, $t = 2$

$$\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + t\mathbf{j}}{\sqrt{1 + t^2}}$$

$$\mathbf{T}'(t) = \frac{-t}{(t^2 + 1)^{3/2}}\mathbf{i} + \frac{1}{(t^2 + 1)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(2) = \frac{-2}{5^{3/2}}\mathbf{i} + \frac{1}{5^{3/2}}\mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} = \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j}) = \frac{-2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

19. $\mathbf{r}(t) = 6 \cos t\mathbf{i} + 6 \sin t\mathbf{j} + \mathbf{k}$, $t = \frac{3\pi}{4}$

$$\mathbf{r}'(t) = -6 \sin t\mathbf{i} + 6 \cos t\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}, \quad \|\mathbf{T}'(t)\| = 1$$

$$\mathbf{N}\left(\frac{3\pi}{4}\right) = \frac{\mathbf{T}'(3\pi/4)}{\|\mathbf{T}'(3\pi/4)\|} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

21. $\mathbf{r}(t) = 4t\mathbf{i}$

$\mathbf{v}(t) = 4\mathbf{i}$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{4\mathbf{i}}{4} = \mathbf{i}$

$\mathbf{T}'(t) = \mathbf{0}$

$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is constant.

23. $\mathbf{r}(t) = 4t^2\mathbf{i}$

$\mathbf{v}(t) = 8t\mathbf{i}$

$\mathbf{a}(t) = 8\mathbf{i}$

$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{8t\mathbf{i}}{8t} = \mathbf{i}$

$\mathbf{T}'(t) = \mathbf{0}$

$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is variable.

25. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$, $\mathbf{v}(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$, $\mathbf{v}(1) = \mathbf{i} - \mathbf{j}$,

$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{j}$, $\mathbf{a}(1) = 2\mathbf{j}$

$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{t^2}{\sqrt{t^4 + 1}}\left(\mathbf{i} - \frac{1}{t^2}\mathbf{j}\right) = \frac{1}{\sqrt{t^4 + 1}}(t^2\mathbf{i} - \mathbf{j})$

$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\frac{2t}{(t^4 + 1)^{3/2}}\mathbf{i} + \frac{2t^3}{(t^4 + 1)^{3/2}}\mathbf{j}}{\frac{2t}{(t^4 + 1)}}$$

$$= \frac{1}{\sqrt{t^4 + 1}}(\mathbf{i} + t^2\mathbf{j})$$

$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

$a_T = \mathbf{a} \cdot \mathbf{T} = -\sqrt{2}$

$a_N = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

27. $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j}$

$\mathbf{v}(t) = e^t(\cos t - \sin t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j}$

$\mathbf{a}(t) = e^t(-2 \sin t)\mathbf{i} + e^t(2 \cos t)\mathbf{j}$

At $t = \frac{\pi}{2}$, $\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$.

Motion along \mathbf{r} is counterclockwise. Therefore,

$$\mathbf{N} = \frac{1}{\sqrt{2}}(-\mathbf{i} - \mathbf{j}) = -\frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j}).$$

$a_T = \mathbf{a} \cdot \mathbf{T} = \sqrt{2}e^{\pi/2}$

$a_N = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}e^{\pi/2}$

29. $\mathbf{r}(t_0) = (\cos \omega t_0 + \omega t_0 \sin \omega t_0)\mathbf{i} + (\sin \omega t_0 - \omega t_0 \cos \omega t_0)\mathbf{j}$

$\mathbf{v}(t_0) = (\omega^2 t_0 \cos \omega t_0)\mathbf{i} + (\omega^2 t_0 \sin \omega t_0)\mathbf{j}$

$\mathbf{a}(t_0) = \omega^2[(\cos \omega t_0 - \omega t_0 \sin \omega t_0)\mathbf{i} + (\omega t_0 \cos \omega t_0 + \sin \omega t_0)\mathbf{j}]$

$\mathbf{T}(t_0) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = (\cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}$

Motion along \mathbf{r} is counterclockwise. Therefore

$\mathbf{N}(t_0) = (-\sin \omega t_0)\mathbf{i} + (\cos \omega t_0)\mathbf{j}.$

$a_T = \mathbf{a} \cdot \mathbf{T} = \omega^2$

$a_N = \mathbf{a} \cdot \mathbf{N} = \omega^2(\omega t_0) = \omega^3 t_0$

$$31. \mathbf{r}(t) = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}$$

$$\mathbf{v}(t) = -a\omega \sin \omega t \mathbf{i} + a\omega \cos \omega t \mathbf{j}$$

$$\mathbf{a}(t) = -a\omega^2 \cos \omega t \mathbf{i} - a\omega^2 \sin \omega t \mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = -\sin \omega t \mathbf{i} + \cos \omega t \mathbf{j}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos \omega t \mathbf{i} - \sin \omega t \mathbf{j}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = 0$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = a\omega^2$$

$$35. \mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}, t_0 = 2$$

$$x = t, y = \frac{1}{t} \Rightarrow xy = 1$$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$$

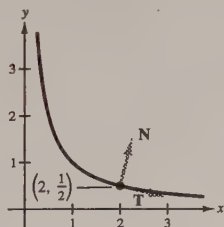
$$\mathbf{T}(t) = \frac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + t^2\mathbf{j}}{\sqrt{t^4 + 1}}$$

$$\mathbf{r}(2) = 2\mathbf{i} + \frac{1}{2}\mathbf{j}$$

$$\mathbf{T}(2) = \frac{\sqrt{17}}{17}(4\mathbf{i} - \mathbf{j})$$

$$\mathbf{N}(2) = \frac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{j})$$



$$39. \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}, t = 1$$

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{1+5t^2}}(\mathbf{i} + 2t\mathbf{j} + t\mathbf{k})$$

$$\mathbf{T}(1) = \frac{\sqrt{6}}{6}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{(1+5t^2)^{3/2}}}{\frac{\sqrt{5}}{1+5t^2}} = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1+5t^2}}$$

$$\mathbf{N}(1) = \frac{\sqrt{30}}{30}(-5\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{5\sqrt{6}}{6}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \frac{\sqrt{30}}{6}$$

$$33. \text{Speed: } \|\mathbf{v}(t)\| = a\omega$$

The speed is constant since $a_T = 0$.

$$37. \mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} - 3t\mathbf{k}, t = 1$$

$$\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = \frac{\sqrt{14}}{14}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \text{ is undefined.}$$

a_T, a_N are not defined.

$$41. \mathbf{r}(t) = 4t\mathbf{i} + 3 \cos t\mathbf{j} + 3 \sin t\mathbf{k}, t = \frac{\pi}{2}$$

$$\mathbf{v}(t) = 4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k}$$

$$\mathbf{v}\left(\frac{\pi}{2}\right) = 4\mathbf{i} - 3\mathbf{j}$$

$$\mathbf{a}(t) = -3 \cos t\mathbf{j} - 3 \sin t\mathbf{k}$$

$$\mathbf{a}\left(\frac{\pi}{2}\right) = -3\mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{5}(4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k})$$

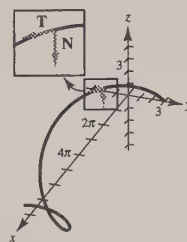
$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = -\cos t\mathbf{j} - \sin t\mathbf{k}$$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{k}$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = 0$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = 3$$



$$43. \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

If $\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$, then $a_{\mathbf{T}}$ is the tangential component of acceleration and $a_{\mathbf{N}}$ is the normal component of acceleration.

45. If $a_{\mathbf{N}} = 0$, then the motion is in a straight line.

$$47. \mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t \rangle$$

The graph is a cycloid.

$$(a) \mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t \rangle$$

$$\mathbf{v}(t) = \langle \pi - \pi \cos \pi t, \pi \sin \pi t \rangle$$

$$\mathbf{a}(t) = \langle \pi^2 \sin \pi t, \pi^2 \cos \pi t \rangle$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \langle 1 - \cos \pi t, \sin \pi t \rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} \langle \sin \pi t, -1 + \cos \pi t \rangle$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} [\pi^2 \sin \pi t (1 - \cos \pi t) + \pi^2 \cos \pi t \sin \pi t] = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{1}{\sqrt{2(1 - \cos \pi t)}} [\pi^2 \sin^2 \pi t + \pi^2 \cos \pi t (-1 + \cos \pi t)] = \frac{\pi^2(1 - \cos \pi t)}{\sqrt{2(1 - \cos \pi t)}} = \frac{\pi^2 \sqrt{2(1 - \cos \pi t)}}{2}$$

$$\text{When } t = \frac{1}{2}: a_{\mathbf{T}} = \frac{\pi^2}{\sqrt{2}} = \frac{\sqrt{2}\pi^2}{2}, a_{\mathbf{N}} = \frac{\sqrt{2}\pi^2}{2}$$

$$\text{When } t = 1: a_{\mathbf{T}} = 0, a_{\mathbf{N}} = \pi^2$$

$$\text{When } t = \frac{3}{2}: a_{\mathbf{T}} = -\frac{\sqrt{2}\pi^2}{2}, a_{\mathbf{N}} = \frac{\sqrt{2}\pi^2}{2}$$

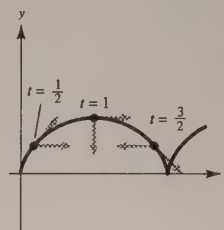
$$(b) \text{ Speed: } s = \|\mathbf{v}(t)\| = \pi \sqrt{2(1 - \cos \pi t)}$$

$$\frac{ds}{dt} = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}} = a_{\mathbf{T}}$$

$$\text{When } t = \frac{1}{2}: a_{\mathbf{T}} = \frac{\sqrt{2}\pi^2}{2} > 0 \Rightarrow \text{the speed is increasing.}$$

$$\text{When } t = 1: a_{\mathbf{T}} = 0 \Rightarrow \text{the height is maximum.}$$

$$\text{When } t = \frac{3}{2}: a_{\mathbf{T}} = -\frac{\sqrt{2}\pi^2}{2} < 0 \Rightarrow \text{the speed is decreasing.}$$



$$49. \quad \mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + \frac{t}{2} \mathbf{k}, \quad t_0 = \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \frac{1}{2} \mathbf{k}$$

$$\mathbf{T}(t) = \frac{2\sqrt{17}}{17} \left(-2 \sin t \mathbf{i} + 2 \cos t \mathbf{j} + \frac{1}{2} \mathbf{k} \right)$$

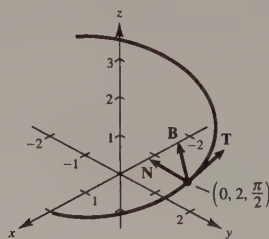
$$\mathbf{N}(t) = -\cos t \mathbf{i} - \sin t \mathbf{j}$$

$$\mathbf{r}\left(\frac{\pi}{2}\right) = 2\mathbf{j} + \frac{\pi}{4} \mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{2\sqrt{17}}{17} \left(-2\mathbf{i} + \frac{1}{2} \mathbf{k} \right) = \frac{\sqrt{17}}{17} (-4\mathbf{i} + \mathbf{k})$$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \mathbf{T}\left(\frac{\pi}{2}\right) \times \mathbf{N}\left(\frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{4\sqrt{17}}{17} & 0 & \frac{\sqrt{17}}{17} \\ 0 & -1 & 0 \end{vmatrix} = \frac{\sqrt{17}}{17} \mathbf{i} + \frac{4\sqrt{17}}{17} \mathbf{k} = \frac{\sqrt{17}}{17} (\mathbf{i} + 4\mathbf{k})$$



51. From Theorem 11.3 we have:

$$\mathbf{r}(t) = (v_0 t \cos \theta) \mathbf{i} + (h + v_0 t \sin \theta - 16t^2) \mathbf{j}$$

$$\mathbf{v}(t) = v_0 \cos \theta \mathbf{i} + (v_0 \sin \theta - 32t) \mathbf{j}$$

$$\mathbf{a}(t) = -32 \mathbf{j}$$

$$\mathbf{T}(t) = \frac{(v_0 \cos \theta) \mathbf{i} + (v_0 \sin \theta - 32t) \mathbf{j}}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}}$$

$$\mathbf{N}(t) = \frac{(v_0 \sin \theta - 32t) \mathbf{i} - v_0 \cos \theta \mathbf{j}}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}} \quad (\text{Motion is clockwise.})$$

$$a_T = \mathbf{a} \cdot \mathbf{T} = \frac{-32(v_0 \sin \theta - 32t)}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}}$$

$$a_N = \mathbf{a} \cdot \mathbf{N} = \frac{32v_0 \cos \theta}{\sqrt{v_0^2 \cos^2 \theta + (v_0 \sin \theta - 32t)^2}}$$

Maximum height when $v_0 \sin \theta - 32t = 0$; (vertical component of velocity)

At maximum height, $a_T = 0$ and $a_N = 32$.

$$53. \quad \mathbf{r}(t) = \langle 10 \cos 10\pi t, 10 \sin 10\pi t, 4 + 4t \rangle, \quad 0 \leq t \leq \frac{1}{20}$$

$$(a) \quad \mathbf{r}'(t) = \langle -100\pi \sin(10\pi t), 100\pi \cos(10\pi t), 4 \rangle$$

$$\begin{aligned} \|\mathbf{r}'(t)\| &= \sqrt{(100\pi)^2 \sin^2(10\pi t) + (100\pi)^2 \cos^2(10\pi t) + 16} \\ &= \sqrt{(100\pi)^2 + 16} = 4\sqrt{625\pi^2 + 1} \approx 314 \text{ mi/hr} \end{aligned}$$

$$(b) \quad a_T = 0 \text{ and } a_N = 1000\pi^2$$

$a_T = 0$ because the speed is constant.

$$55. \quad \mathbf{r}(t) = (a \cos \omega t) \mathbf{i} + (a \sin \omega t) \mathbf{j}$$

From Exercise 31, we know $\mathbf{a} \cdot \mathbf{T} = 0$ and $\mathbf{a} \cdot \mathbf{N} = a\omega^2$.

(a) Let $\omega_0 = 2\omega$. Then

$$\mathbf{a} \cdot \mathbf{N} = a\omega_0^2 = a(2\omega)^2 = 4a\omega^2$$

or the centripetal acceleration is increased by a factor of 4 when the velocity is doubled.

(b) Let $a_0 = a/2$. Then

$$\mathbf{a} \cdot \mathbf{N} = a_0\omega^2 = \left(\frac{a}{2}\right)\omega^2 = \left(\frac{1}{2}\right)a\omega^2$$

or the centripetal acceleration is halved when the radius is halved.

$$57. v = \sqrt{\frac{9.56 \times 10^4}{4100}} \approx 4.83 \text{ mi/sec}$$

$$59. v = \sqrt{\frac{9.56 \times 10^4}{4385}} \approx 4.67 \text{ mi/sec}$$

61. Let $\mathbf{T}(t) = \cos \phi \mathbf{i} + \sin \phi \mathbf{j}$ be the unit tangent vector. Then

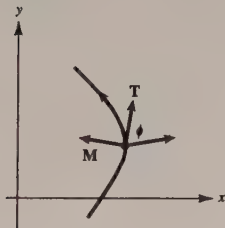
$$\mathbf{T}'(t) = \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{d\phi} \frac{d\phi}{dt} = -(\sin \phi \mathbf{i} - \cos \phi \mathbf{j}) \frac{d\phi}{dt} = \mathbf{M} \frac{d\phi}{dt}.$$

$\mathbf{M} = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j} = \cos[\phi + (\pi/2)] \mathbf{i} + \sin[\phi + (\pi/2)] \mathbf{j}$ and is rotated counterclockwise through an angle of $\pi/2$ from \mathbf{T} .

If $d\phi/dt > 0$, then the curve bends to the left and \mathbf{M} has the same direction as \mathbf{T}' . Thus, \mathbf{M} has the same direction as

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|},$$

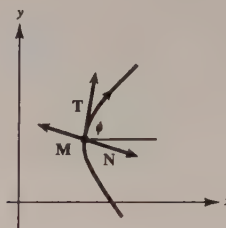
which is toward the concave side of the curve.



If $d\phi/dt < 0$, then the curve bends to the right and \mathbf{M} has the opposite direction as \mathbf{T}' . Thus,

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

again points to the concave side of the curve.



63. Using $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$, $\mathbf{T} \times \mathbf{T} = \mathbf{0}$, and $\|\mathbf{T} \times \mathbf{N}\| = 1$, we have:

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= \|\mathbf{v}\| \mathbf{T} \times (a_T \mathbf{T} + a_N \mathbf{N}) \\ &= \|\mathbf{v}\| a_T (\mathbf{T} \times \mathbf{T}) + \|\mathbf{v}\| a_N (\mathbf{T} \times \mathbf{N}) \\ &= \|\mathbf{v}\| a_N (\mathbf{T} \times \mathbf{N}) \\ \|\mathbf{v} \times \mathbf{a}\| &= \|\mathbf{v}\| a_N \|\mathbf{T} \times \mathbf{N}\| \\ &= \|\mathbf{v}\| a_N \end{aligned}$$

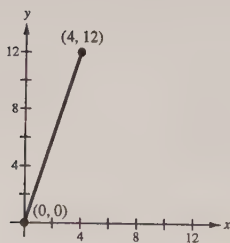
$$\text{Thus, } a_N = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}.$$

Section 11.5 Arc Length and Curvature

1. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j}$

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = 3, \frac{dz}{dt} = 0$$

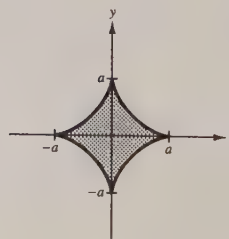
$$\begin{aligned} s &= \int_0^4 \sqrt{1 + 9} \, dt \\ &= \sqrt{10} \int_0^4 dt \\ &= \left[\sqrt{10} t \right]_0^4 = 4\sqrt{10} \end{aligned}$$



3. $\mathbf{r}(t) = a \cos^3 t \mathbf{i} + a \sin^3 t \mathbf{j}$

$$\frac{dx}{dt} = -3a \cos^2 t \sin t, \frac{dy}{dt} = 3a \sin^2 t \cos t$$

$$\begin{aligned} s &= 4 \int_0^{\pi/2} \sqrt{[-3a \cos^2 t \sin t]^2 + [3a \sin^2 t \cos t]^2} \, dt \\ &= 12a \int_0^{\pi/2} \sin t \cos t \, dt \\ &= 3a \int_0^{\pi/2} 2 \sin 2t \, dt = \left[-3a \cos 2t \right]_0^{\pi/2} = 6a \end{aligned}$$



$$\begin{aligned}
 5. \text{ (a) } \mathbf{r}(t) &= (v_0 \cos \theta) \mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2 \right] \mathbf{j} \\
 &= (100 \cos 45^\circ) \mathbf{i} + \left[3 + (100 \sin 45^\circ)t - \frac{1}{2}(32)t^2 \right] \mathbf{j} \\
 &= 50\sqrt{2} \mathbf{i} + [3 + 50\sqrt{2}t - 16t^2] \mathbf{j}
 \end{aligned}$$

$$(b) \mathbf{v}(t) = 50\sqrt{2} \mathbf{i} + (50\sqrt{2} - 32t) \mathbf{j}$$

$$50\sqrt{2} - 32t = 0 \Rightarrow t = \frac{25\sqrt{2}}{16}$$

$$\text{Maximum height: } 3 + 50\sqrt{2} \left(\frac{25\sqrt{2}}{16} \right) - 16 \left(\frac{25\sqrt{2}}{16} \right)^2 = 81.125 \text{ ft}$$

$$(c) 3 + 50\sqrt{2}t - 16t^2 = 0 \Rightarrow t \approx 4.4614$$

$$\text{Range: } 50\sqrt{2}(4.4614) \approx 315.5 \text{ feet}$$

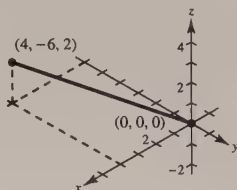
$$(d) s = \int_0^{4.4614} \sqrt{(50\sqrt{2})^2 + (50\sqrt{2} - 32t)^2} dt \approx 362.9 \text{ feet}$$

$$7. \mathbf{r}(t) = 2t \mathbf{i} - 3t \mathbf{j} + t \mathbf{k}$$

$$\frac{dx}{dt} = 2, \frac{dy}{dt} = -3, \frac{dz}{dt} = 1$$

$$s = \int_0^2 \sqrt{2^2 + (-3)^2 + 1^2} dt$$

$$= \int_0^2 \sqrt{14} dt = \left[\sqrt{14}t \right]_0^2 = 2\sqrt{14}$$

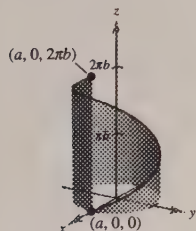


$$9. \mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j} + b t \mathbf{k}$$

$$\frac{dx}{dt} = -a \sin t, \frac{dy}{dt} = a \cos t, \frac{dz}{dt} = b$$

$$s = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} dt$$

$$= \int_0^{2\pi} \sqrt{a^2 + b^2} dt = \left[\sqrt{a^2 + b^2} t \right]_0^{2\pi} = 2\pi \sqrt{a^2 + b^2}$$



$$11. \mathbf{r}(t) = t^2 \mathbf{i} + t \mathbf{j} + \ln t \mathbf{k}$$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 1, \frac{dz}{dt} = \frac{1}{t}$$

$$s = \int_1^3 \sqrt{(2t)^2 + (1)^2 + \left(\frac{1}{t}\right)^2} dt$$

$$= \int_1^3 \sqrt{\frac{4t^4 + t^2 + 1}{t^2}} dt$$

$$= \int_1^3 \frac{\sqrt{4t^4 + t^2 + 1}}{t} dt \approx 8.37$$

$$13. \mathbf{r}(t) = t \mathbf{i} + (4 - t^2) \mathbf{j} + t^3 \mathbf{k}, \quad 0 \leq t \leq 2$$

$$(a) \mathbf{r}(0) = \langle 0, 4, 0 \rangle, \mathbf{r}(2) = \langle 2, 0, 8 \rangle$$

$$\text{distance} = \sqrt{2^2 + 4^2 + 8^2} = \sqrt{84} = 2\sqrt{21} \approx 9.165$$

—CONTINUED—

13. —CONTINUED—

(b) $\mathbf{r}(0) = \langle 0, 4, 0 \rangle$

$\mathbf{r}(0.5) = \langle 0.5, 3.75, 0.125 \rangle$

$\mathbf{r}(1) = \langle 1, 3, 1 \rangle$

$\mathbf{r}(1.5) = \langle 1.5, 1.75, 3.375 \rangle$

$\mathbf{r}(2) = \langle 2, 0, 8 \rangle$

$$\begin{aligned} \text{distance} &\approx \sqrt{(0.5)^2 + (0.25)^2 + (0.125)^2} + \sqrt{(0.5)^2 + (0.75)^2 + (0.875)^2} + \sqrt{(0.5)^2 + (1.25)^2 + (2.375)^2} + \\ &\quad \sqrt{(0.5)^2 + (1.75)^2 + (4.625)^2} \\ &\approx 0.5728 + 1.2562 + 2.7300 + 4.9702 \approx 9.529 \end{aligned}$$

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain 9.57057.

15. $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle$

$$\begin{aligned} \text{(a) } s &= \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2} du \\ &= \int_0^t \sqrt{(-2 \sin u)^2 + (2 \cos u)^2 + (1)^2} du \\ &= \int_0^t \sqrt{5} du = \left[\sqrt{5} u \right]_0^t = \sqrt{5} t \end{aligned}$$

(c) When $s = \sqrt{5}$: $x = 2 \cos 1 \approx 1.081$
 $y = 2 \sin 1 \approx 1.683$
 $z = 1$
 $(1.081, 1.683, 1.000)$

When $s = 4$: $x = 2 \cos \frac{4}{\sqrt{5}} \approx -0.433$
 $y = 2 \sin \frac{4}{\sqrt{5}} \approx 1.953$
 $z = \frac{4}{\sqrt{5}} \approx 1.789$
 $(-0.433, 1.953, 1.789)$

$$\text{(d) } \|\mathbf{r}'(s)\| = \sqrt{\left(-\frac{2}{\sqrt{5}} \sin\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{2}{\sqrt{5}} \cos\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = 1$$

$$\begin{aligned} 17. \quad \mathbf{r}(s) &= \left(1 + \frac{\sqrt{2}}{2}s\right)\mathbf{i} + \left(1 - \frac{\sqrt{2}}{2}s\right)\mathbf{j} \\ \mathbf{r}'(s) &= \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} \quad \text{and} \quad \|\mathbf{r}'(s)\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1 \end{aligned}$$

$$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = \mathbf{r}'(s)$$

$$\mathbf{T}'(s) = \mathbf{0} \Rightarrow K = \|\mathbf{T}'(s)\| = 0 \quad (\text{The curve is a line.})$$

(b) $\frac{s}{\sqrt{5}} = t$

$$x = 2 \cos\left(\frac{s}{\sqrt{5}}\right), \quad y = 2 \sin\left(\frac{s}{\sqrt{5}}\right), \quad z = \frac{s}{\sqrt{5}}$$

$$\mathbf{r}(s) = 2 \cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2 \sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$$

$$19. \quad \mathbf{r}(s) = 2 \cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2 \sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$$

$$\mathbf{T}(s) = \mathbf{r}'(s) = -\frac{2}{\sqrt{5}} \sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + \frac{2}{\sqrt{5}} \cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$$

$$\mathbf{T}'(s) = -\frac{2}{5} \cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} - \frac{2}{5} \sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j}$$

$$K = \|\mathbf{T}'(s)\| = \frac{2}{5}$$

$$21. \mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}$$

$$\mathbf{v}(t) = 4\mathbf{i} - 2\mathbf{j}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0 \quad (\text{The curve is a line.})$$

$$25. \mathbf{r}(t) = 4 \cos 2\pi t \mathbf{i} + 4 \sin 2\pi t \mathbf{j}$$

$$\mathbf{r}'(t) = -8\pi \sin 2\pi t \mathbf{i} + 8\pi \cos 2\pi t \mathbf{j}$$

$$\mathbf{T}(t) = -\sin 2\pi t \mathbf{i} + \cos 2\pi t \mathbf{j}$$

$$\mathbf{T}'(t) = -2\pi \cos 2\pi t \mathbf{i} - 2\pi \sin 2\pi t \mathbf{j}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{2\pi}{8\pi} = \frac{1}{4}$$

$$29. \mathbf{r}(t) = e^t \cos t \mathbf{i} + e^t \sin t \mathbf{j}$$

$$\mathbf{r}'(t) = (-e^t \sin t + e^t \cos t)\mathbf{i} + (e^t \cos t + e^t \sin t)\mathbf{j}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}}[(-\sin t + \cos t)\mathbf{i} + (\cos t + \sin t)\mathbf{j}]$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}[(-\cos t - \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j}]$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{2}e^t} = \frac{\sqrt{2}}{2}e^{-t}$$

$$33. \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1 + 5t^2}}$$

$$\mathbf{T}'(t) = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{(1 + 5t^2)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

$$= \frac{\sqrt{5}}{\sqrt{1 + 5t^2}} = \frac{\sqrt{5}}{(1 + 5t^2)^{3/2}}$$

$$23. \mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$$

$$\mathbf{v}(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$$

$$\mathbf{v}(1) = \mathbf{i} - \mathbf{j}$$

$$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{j}$$

$$\mathbf{a}(1) = 2\mathbf{j}$$

$$\mathbf{T}(t) = \frac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$$

$$\mathbf{N}(t) = \frac{1}{(t^4 + 1)^{1/2}}(\mathbf{i} + t^2\mathbf{j})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$$

$$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{\sqrt{2}}{2}$$

$$27. \mathbf{r}(t) = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}$$

$$\mathbf{r}'(t) = -a\omega \sin \omega t \mathbf{i} + a\omega \cos \omega t \mathbf{j}$$

$$\mathbf{T}(t) = -\sin \omega t \mathbf{i} + \cos \omega t \mathbf{j}$$

$$\mathbf{T}'(t) = -\omega \cos \omega t \mathbf{i} - \omega \sin \omega t \mathbf{j}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\omega}{a\omega} = \frac{1}{a}$$

$$31. \mathbf{r}(t) = \langle \cos \omega t + \omega t \sin \omega t, \sin \omega t - \omega t \cos \omega t \rangle$$

From Exercise 29, Section 11.4, we have:

$$\mathbf{a} \cdot \mathbf{N} = \omega^3 t$$

$$K = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}\|^2} = \frac{\omega^3 t}{\omega^4 t^2} = \frac{1}{\omega t}$$

$$35. \mathbf{r}(t) = 4t\mathbf{i} + 3 \cos t\mathbf{j} + 3 \sin t\mathbf{k}$$

$$\mathbf{r}'(t) = 4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{1}{5}[4\mathbf{i} - 3 \sin t\mathbf{j} + 3 \cos t\mathbf{k}]$$

$$\mathbf{T}'(t) = \frac{1}{5}[-3 \cos t\mathbf{j} - 3 \sin t\mathbf{k}]$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{3/5}{5} = \frac{3}{25}$$

37. $y = 3x - 2$

Since $y'' = 0$, $K = 0$, and the radius of curvature is undefined.

41. $y = \sqrt{a^2 - x^2}$, $x = 0$

$$y' = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$y'' = \frac{a^2}{(a^2 - x^2)^{3/2}}$$

At $x = 0$: $y' = 0$

$$y'' = \frac{1}{a}$$

$$K = \frac{1/a}{(1 + 0^2)^{3/2}} = \frac{1}{a}$$

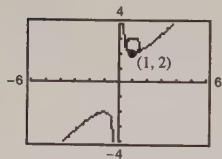
$$\frac{1}{K} = a \quad (\text{radius of curvature})$$

45. $y = x + \frac{1}{x}$, $y' = 1 - \frac{1}{x^2}$, $y'' = \frac{2}{x^3}$

$$K = \frac{2}{(1 + 0^2)^{3/2}} = 2 \text{ at } (1, 2)$$

Radius of curvature = $1/2$. Since the tangent line is horizontal at $(1, 2)$, the normal line is vertical. The center of the circle is $1/2$ unit above the point $(1, 2)$ at $(1, 5/2)$.

Circle: $(x - 1)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{1}{4}$



39. $y = 2x^2 + 3$, $x = -1$

$$y' = 4x$$

$$y'' = 4$$

$$K = \frac{4}{[1 + (-4)^2]^{3/2}} = \frac{4}{17^{3/2}} \approx 0.057$$

$$\frac{1}{K} = \frac{17^{3/2}}{4} \approx 17.523 \quad (\text{radius of curvature})$$

43. (a) Point on circle: $\left(\frac{\pi}{2}, 1\right)$

Center: $\left(\frac{\pi}{2}, 0\right)$

Equation: $\left(x - \frac{\pi}{2}\right)^2 + y^2 = 1$

(b) The circles have different radii since the curvature is different and

$$r = \frac{1}{K}$$

47. $y = e^x$, $x = 0$

$$y' = e^x, \quad y'' = e^x$$

$$y'(0) = 1, \quad y''(0) = 1$$

$$K = \frac{1}{(1 + 1^2)^{3/2}} = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}}, \quad r = \frac{1}{K} = 2\sqrt{2}$$

The slope of the tangent line at $(0, 1)$ is $y'(0) = 1$.

The slope of the normal line is -1 .

Equation of normal line: $y - 1 = -x$ or $y = -x + 1$

The center of the circle is on the normal line $2\sqrt{2}$ units away from the point $(0, 1)$.

$$\sqrt{(0 - x)^2 + (1 - y)^2} = 2\sqrt{2}$$

$$x^2 + x^2 = 8$$

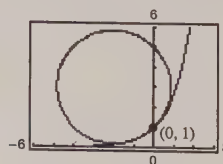
$$x^2 = 4$$

$$x = \pm 2$$

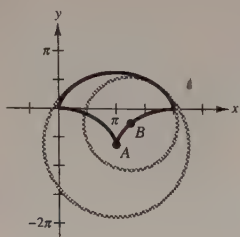
Since the circle is above the curve, $x = -2$ and $y = 3$.

Center of circle: $(-2, 3)$

Equation of circle: $(x + 2)^2 + (y - 3)^2 = 8$



49.



53. $y = x^{2/3}$, $y' = \frac{2}{3}x^{-1/3}$, $y'' = -\frac{2}{9}x^{-4/3}$

$$K = \frac{|(-2/9)x^{-4/3}|}{[1 + (4/9)x^{-2/3}]^{3/2}} = \frac{6}{x^{1/3}(9x^{2/3} + 4)^{3/2}}$$

(a) $K \Rightarrow \infty$ as $x \Rightarrow 0$. No maximum

(b) $\lim_{x \rightarrow \infty} K = 0$

57. $K = \frac{|y''|}{[1 + (y')^2]^{3/2}}$

The curvature is zero when $y'' = 0$.

63. Endpoints of the major axis: $(\pm 2, 0)$

Endpoints of the minor axis: $(0, \pm 1)$

$$x^2 + 4y^2 = 4$$

$$2x + 8yy' = 0$$

$$y' = -\frac{x}{4y}$$

$$y'' = \frac{(4y)(-1) - (-x)(4y')}{16y^2} = \frac{-4y - (x^2/y)}{16y^2} = \frac{-(4y^2 + x^2)}{16y^3} = \frac{-1}{4y^3}$$

$$K = \frac{|-1/4y^3|}{[1 + (-x/4y)^2]^{3/2}} = \frac{|-16|}{(16y^2 + x^2)^{3/2}} = \frac{16}{(12y^2 + 4)^{3/2}} = \frac{16}{(16 - 3x^2)^{3/2}}$$

Therefore, since $-2 \leq x \leq 2$, K is largest when $x = \pm 2$ and smallest when $x = 0$.

65. $f(x) = x^4 - x^2$

(a) $K = \frac{2|6x^2 - 1|}{[16x^6 - 16x^4 + 4x^2 + 1]^{3/2}}$

(b) For $x = 0$, $K = 2$. $f(0) = 0$. At $(0, 0)$, the circle of curvature has radius $\frac{1}{2}$. Using the symmetry of the graph of f , you obtain

$$x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}.$$

For $x = 1$, $K = (2\sqrt{5})/5$. $f(1) = 0$. At $(1, 0)$, the circle of curvature has radius

$$\frac{\sqrt{5}}{2} = \frac{1}{K}.$$

Using the graph of f , you see that the center of curvature is $(0, \frac{1}{2})$. Thus,

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4}.$$

To graph these circles, use

$$y = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - x^2} \quad \text{and} \quad y = \frac{1}{2} \pm \sqrt{\frac{5}{4} - x^2}.$$

51. $y = (x - 1)^2 + 3$, $y' = 2(x - 1)$, $y'' = 2$

$$K = \frac{2}{(1 + [2(x - 1)]^2)^{3/2}} = \frac{2}{[1 + 4(x - 1)^2]^{3/2}}$$

(a) K is maximum when $x = 1$ or at the vertex $(1, 3)$.

(b) $\lim_{x \rightarrow \infty} K = 0$

55. $y = (x - 1)^3 + 3$

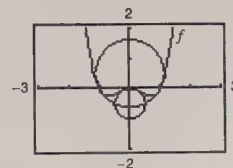
$$y' = 3(x - 1)^2$$

$$y'' = 6(x - 1)$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|6(x - 1)|}{[1 + 9(x - 1)^4]^{3/2}} = 0 \text{ at } x = 1.$$

Curvature is 0 at $(1, 3)$.

61. The curve is a line.

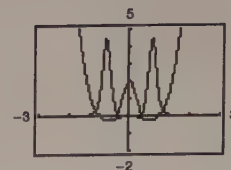


65. —CONTINUED—

- (c) The curvature tends to be greatest near the extrema of f , and K decreases as $x \rightarrow \pm\infty$. However, f and K do not have the same critical numbers.

$$\text{Critical numbers of } f: x = 0, \pm \frac{\sqrt{2}}{2} \approx \pm 0.7071$$

$$\text{Critical numbers of } K: x = 0, \pm 0.7647, \pm 0.4082$$



67. (a) Imagine dropping the circle $x^2 + (y - k)^2 = 16$ into the parabola $y = x^2$. The circle will drop to the point where the tangents to the circle and parabola are equal.

$$y = x^2 \quad \text{and} \quad x^2 + (y - k)^2 = 16 \implies x^2 + (x^2 - k)^2 = 16$$

Taking derivatives, $2x + 2(y - k)y' = 0$ and $y' = 2x$. Hence,

$$(y - k)y' = -x \implies y' = \frac{-x}{y - k}.$$

Thus,

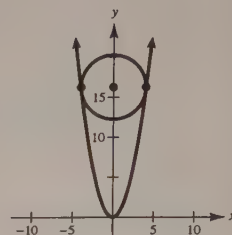
$$\frac{-x}{y - k} = 2x \implies -x = 2x(y - k) \implies -1 = 2(x^2 - k) \implies x^2 - k = -\frac{1}{2}.$$

Thus,

$$x^2 + (x^2 - k)^2 = x^2 + \left(-\frac{1}{2}\right)^2 = 16 \implies x^2 = 15.75.$$

Finally, $k = x^2 + \frac{1}{2} = 16.25$, and the center of the circle is 16.25 units from the vertex of the parabola. Since the radius of the circle is 4, the circle is 12.25 units from the vertex.

- (b) In 2-space, the parabola $z = y^2$ (or $z = x^2$) has a curvature of $K = 2$ at $(0, 0)$. The radius of the largest sphere that will touch the vertex has radius $= 1/K = \frac{1}{2}$.



69. Given $y = f(x)$: $K = \frac{|y''|}{(1 + [y']^2)^{3/2}}$

$$R = \frac{1}{K}$$

The center of the circle is on the normal line at a distance of R from (x, y) .

$$\text{Equation of normal line: } y - y_0 = -\frac{1}{y'}(x - x_0)$$

$$\sqrt{(x - x_0)^2 + \left[-\frac{1}{y'}(x - x_0)\right]^2} = \frac{(1 + [y']^2)^{3/2}}{|y''|}$$

$$(x - x_0)^2 \left[1 + \frac{1}{(y')^2}\right] = \frac{(1 + [y']^2)^3}{(y'')^2}$$

$$(x - x_0)^2 = \frac{(y')^2(1 + [y']^2)^2}{(y'')^2}$$

$$x - x_0 = \frac{y'(1 + [y']^2)}{y''} = y'z$$

$$x_0 = x - y'z$$

$$y - y_0 = -\frac{1}{y'}(x - (x - y'z)) = -z$$

$$y_0 = y + z$$

$$\text{Thus, } (x_0, y_0) = (x - y'z, y + z).$$

$$\text{For } y = e^x, y' = e^x, y'' = e^x, z = \frac{1 + e^{2x}}{e^x} = e^{-x} + e^x.$$

$$\text{When } x = 0: x_0 = x - y'z = 0 - (1)(2) = -2$$

$$y_0 = y + z = 1 + 2 = 3$$

$$\text{Center of curvature: } (-2, 3)$$

(See Exercise 47)

71. $r = 1 + \sin \theta$

$$r' = \cos \theta$$

$$r'' = -\sin \theta$$

$$\begin{aligned}
 K &= \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} \\
 &= \frac{|2\cos^2 \theta - (1 + \sin \theta)(-\sin \theta) + (1 + \sin \theta)^2|}{\sqrt{[\cos^2 \theta + (1 + \sin \theta)^2]^3}} \\
 &= \frac{3(1 + \sin \theta)}{\sqrt{8(1 + \sin \theta)^3}} = \frac{3}{2\sqrt{2}(1 + \sin \theta)}
 \end{aligned}$$

75. $r = e^{a\theta}, a > 0$

$$r' = ae^{a\theta}$$

$$r'' = a^2 e^{a\theta}$$

$$\begin{aligned}
 K &= \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{|2a^2 e^{2a\theta} - a^2 e^{2a\theta} + e^{2a\theta}|}{[a^2 e^{2a\theta} + e^{2a\theta}]^{3/2}} \\
 &= \frac{1}{e^{a\theta} \sqrt{a^2 + 1}}
 \end{aligned}$$

(a) As $\theta \Rightarrow \infty, K \Rightarrow 0$.

(b) As $a \Rightarrow \infty, K \Rightarrow 0$.

79. $x = f(t)$

$$y = g(t)$$

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$\begin{aligned}
 y'' &= \frac{\frac{d}{dt} \left[\frac{g'(t)}{f'(t)} \right]}{\frac{dx}{dt}} = \frac{\frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^2}}{f'(t)} \\
 &= \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3} \\
 K &= \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\left| \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3} \right|}{\left[1 + \left(\frac{g'(t)}{f'(t)} \right)^2 \right]^{3/2}} \\
 &= \frac{\left| \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3} \right|}{\sqrt{\left\{ \frac{[f'(t)]^2 + [g'(t)]^2}{[f'(t)]^2} \right\}^3}} \\
 &= \frac{|f'(t)g''(t) - g'(t)f''(t)|}{([f'(t)]^2 + [g'(t)]^2)^{3/2}}
 \end{aligned}$$

83. $a_N = mK \left(\frac{ds}{dt} \right)^2 = \left(\frac{5500 \text{ lb}}{32 \text{ ft/sec}^2} \right) \left(\frac{1}{100 \text{ ft}} \right) \left(\frac{30(5280 \text{ ft})}{3600 \text{ sec}} \right)^2 = 3327.5 \text{ lb}$

73. $r = a \sin \theta$

$$r' = a \cos \theta$$

$$r'' = -a \sin \theta$$

$$\begin{aligned}
 K &= \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} \\
 &= \frac{|2a^2 \cos^2 \theta + a^2 \sin^2 \theta + a^2 \sin^2 \theta|}{\sqrt{[a^2 \cos^2 \theta + a^2 \sin^2 \theta]^3}} \\
 &= \frac{2a^2}{a^3} = \frac{2}{a}, a > 0
 \end{aligned}$$

77. $r = 4 \sin 2\theta$

$$r' = 8 \cos 2\theta$$

At the pole: $K = \frac{2}{|r'(0)|} = \frac{2}{8} = \frac{1}{4}$

81. $x(\theta) = a(\theta - \sin \theta) \quad y(\theta) = a(1 - \cos \theta)$

$$x'(\theta) = a(1 - \cos \theta) \quad y'(\theta) = a \sin \theta$$

$$x''(\theta) = a \sin \theta \quad y''(\theta) = a \cos \theta$$

$$\begin{aligned}
 K &= \frac{|x'(\theta)y''(\theta) - y'(\theta)x''(\theta)|}{[x'(\theta)^2 + y'(\theta)^2]^{3/2}} \\
 &= \frac{|a^2(1 - \cos \theta) \cos \theta - a^2 \sin^2 \theta|}{[a^2(1 - \cos \theta)^2 + a^2 \sin^2 \theta]^{3/2}} \\
 &= \frac{1}{a} \frac{|\cos \theta - 1|}{[2 - 2 \cos \theta]^{3/2}} \\
 &= \frac{1}{a} \frac{1 - \cos \theta}{2\sqrt{2}[1 - \cos \theta]^{3/2}} \quad (1 - \cos \theta \geq 0) \\
 &= \frac{1}{2a\sqrt{2 - 2 \cos \theta}} = \frac{1}{4a} \csc \left(\frac{\theta}{2} \right)
 \end{aligned}$$

Minimum: $\frac{1}{4a} \quad (\theta = \pi)$

Maximum: none $(K \rightarrow \infty \text{ as } \theta \rightarrow 0)$

85. Let $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $r = \|\mathbf{r}\| = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}$ and $\mathbf{r}' = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$. Then,

$$\begin{aligned} r \left(\frac{d\mathbf{r}}{dt} \right) &= \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2} \left[\frac{1}{2} \{ [x(t)]^2 + [y(t)]^2 + [z(t)]^2 \}^{-1/2} \cdot (2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t)) \right] \\ &= x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = \mathbf{r} \cdot \mathbf{r}'. \end{aligned}$$

87. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where x , y , and z are functions of t , and $r = \|\mathbf{r}\|$.

$$\begin{aligned} \frac{d}{dt} \left[\frac{\mathbf{r}}{r} \right] &= \frac{r\mathbf{r}' - \mathbf{r}(dr/dt)}{r^2} = \frac{r\mathbf{r}' - \mathbf{r}[(\mathbf{r} \cdot \mathbf{r}')/r]}{r^2} = \frac{r^2\mathbf{r}' - (\mathbf{r} \cdot \mathbf{r}')\mathbf{r}}{r^3} \quad (\text{using Exercise 87}) \\ &= \frac{(x^2 + y^2 + z^2)(x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}) - (xx' + yy' + zz')(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{r^3} \\ &= \frac{1}{r^3} [(x'y^2 + x'z^2 - xyy' - xzz')\mathbf{i} + (x^2y' + z^2y' - xx'y - zz'y)\mathbf{j} + (x^2z' + y^2z' - xx'z - yy'z)\mathbf{k}] \\ &= \frac{1}{r^3} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz' - y'z & -(xz' - x'z) & xy' - x'y \\ x & y & z \end{vmatrix} = \frac{1}{r^3} \{ [\mathbf{r} \times \mathbf{r}'] \times \mathbf{r} \} \end{aligned}$$

89. From Exercise 86, we have concluded that planetary motion is planar. Assume that the planet moves in the xy -plane with the sun at the origin. From Exercise 88, we have

$$\mathbf{r}' \times \mathbf{L} = GM \left(\frac{\mathbf{r}}{r} + \mathbf{e} \right).$$

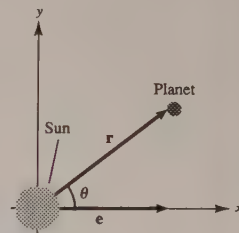
Since $\mathbf{r}' \times \mathbf{L}$ and \mathbf{r} are both perpendicular to \mathbf{L} , so is \mathbf{e} . Thus, \mathbf{e} lies in the xy -plane. Situate the coordinate system so that \mathbf{e} lies along the positive x -axis and θ is the angle between \mathbf{e} and \mathbf{r} . Let $e = \|\mathbf{e}\|$. Then $\mathbf{r} \cdot \mathbf{e} = \|\mathbf{r}\| \|\mathbf{e}\| \cos \theta = re \cos \theta$. Also,

$$\begin{aligned} \|\mathbf{L}\|^2 &= \mathbf{L} \cdot \mathbf{L} = (\mathbf{r} \times \mathbf{r}') \cdot \mathbf{L} \\ &= \mathbf{r} \cdot (\mathbf{r}' \times \mathbf{L}) = \mathbf{r} \cdot \left[GM \left(\mathbf{e} + \frac{\mathbf{r}}{r} \right) \right] = GM \left[\mathbf{r} \cdot \mathbf{e} + \frac{\mathbf{r} \cdot \mathbf{r}}{r} \right] = GM[re \cos \theta + r] \end{aligned}$$

Thus,

$$\frac{\|\mathbf{L}\|^2/GM}{1 + e \cos \theta} = r$$

and the planetary motion is a conic section. Since the planet returns to its initial position periodically, the conic is an ellipse.



91. $A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$

Thus,

$$\frac{dA}{dt} = \frac{dA}{d\theta} \frac{d\theta}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} \|\mathbf{L}\|$$

and \mathbf{r} sweeps out area at a constant rate.

Review Exercises for Chapter 11

1. $\mathbf{r}(t) = t\mathbf{i} + \csc t\mathbf{k}$

(a) Domain: $t \neq n\pi$, n an integer

(b) Continuous except at $t = n\pi$, n an integer

3. $\mathbf{r}(t) = \ln t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$

(a) Domain: $(0, \infty)$

(b) Continuous for all $t > 0$

5. (a) $\mathbf{r}(0) = \mathbf{i}$

(b) $\mathbf{r}(-2) = -3\mathbf{i} + 4\mathbf{j} + \frac{8}{3}\mathbf{k}$

(c) $\mathbf{r}(c-1) = (2(c-1)+1)\mathbf{i} + (c-1)^2\mathbf{j} - \frac{1}{3}(c-1)^3\mathbf{k}$
 $= (2c-1)\mathbf{i} + (c-1)^2\mathbf{j} - \frac{1}{3}(c-1)^3\mathbf{k}$

(d) $\mathbf{r}(1+\Delta t) - \mathbf{r}(1) = ([2(1+\Delta t)+1]\mathbf{i} + [1+\Delta t]^2\mathbf{j} - \frac{1}{3}[1+\Delta t]^3\mathbf{k}) - (3\mathbf{i} + \mathbf{j} - \frac{1}{3}\mathbf{k})$
 $= 2\Delta t\mathbf{i} + \Delta t(\Delta t+2)\mathbf{j} - \frac{1}{3}(\Delta t^3 + 3\Delta t^2 + 3\Delta t)\mathbf{k}$

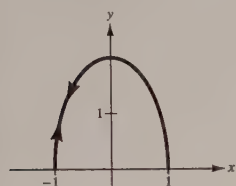
7. $\mathbf{r}(t) = \cos t\mathbf{i} + 2\sin^2 t\mathbf{j}$

$x(t) = \cos t, y(t) = 2\sin^2 t$

$x^2 + \frac{y}{2} = 1$

$y = 2(1-x^2)$

$-1 \leq x \leq 1$

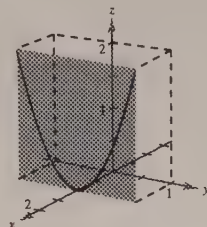


9. $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 1$

$y = t$

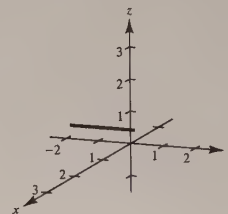
$z = t^2 \Rightarrow z = y^2$



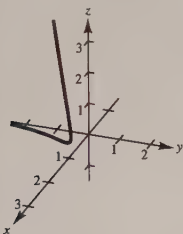
11. $\mathbf{r}(t) = \mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$

$x = 1, y = \sin t, z = 1$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x	1	1	1	1
y	0	1	0	-1
z	1	1	1	1



13. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$



15. One possible answer is:

$\mathbf{r}_1(t) = 4t\mathbf{i} + 3t\mathbf{j}, \quad 0 \leq t \leq 1$

$\mathbf{r}_2(t) = 4\mathbf{i} + (3-t)\mathbf{j}, \quad 0 \leq t \leq 3$

$\mathbf{r}_3(t) = (4-t)\mathbf{i}, \quad 0 \leq t \leq 4$

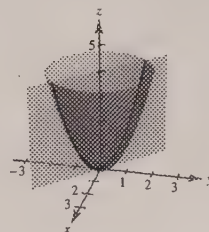
17. The vector joining the points is $\langle 7, 4, -10 \rangle$. One path is

$\mathbf{r}(t) = \langle -2 + 7t, -3 + 4t, 8 - 10t \rangle$.

19. $z = x^2 + y^2, x + y = 0, t = x$

$x = t, y = -t, z = 2t^2$

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$



21. $\lim_{t \rightarrow 2^-} (t^2\mathbf{i} + \sqrt{4-t^2}\mathbf{j} + \mathbf{k}) = 4\mathbf{i} + \mathbf{k}$

$$23. \mathbf{r}(t) = 3t\mathbf{i} + (t-1)\mathbf{j}, \mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$$

$$(a) \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$$

$$(c) \mathbf{r}(t) \cdot \mathbf{u}(t) = 3t^2 + t^2(t-1) = t^3 + 2t^2$$

$$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 3t^2 + 4t$$

$$(e) \|\mathbf{r}(t)\| = \sqrt{10t^2 - 2t + 1}$$

$$D_t[\|\mathbf{r}(t)\|] = \frac{10t - 1}{\sqrt{10t^2 - 2t + 1}}$$

$$(b) \mathbf{r}''(t) = \mathbf{0}$$

$$(d) \mathbf{u}(t) - 2\mathbf{r}(t) = -5t\mathbf{i} + (t^2 - 2t + 2)\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$$

$$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -5\mathbf{i} + (2t - 2)\mathbf{j} + 2t^2\mathbf{k}$$

$$(f) \mathbf{r}(t) \times \mathbf{u}(t) = \frac{2}{3}(t^4 - t^3)\mathbf{i} - 2t^4\mathbf{j} + (3t^3 - t^2 + t)\mathbf{k}$$

$$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left(\frac{8}{3}t^3 - 2t^2\right)\mathbf{i} - 8t^3\mathbf{j} + (9t^2 - 2t + 1)\mathbf{k}$$

25. $x(t)$ and $y(t)$ are increasing functions at $t = t_0$, and $z(t)$ is a decreasing function at $t = t_0$.

$$27. \int (\cos t\mathbf{i} + t \cos t\mathbf{j}) dt = \sin t\mathbf{i} + (t \sin t + \cos t)\mathbf{j} + \mathbf{C}$$

$$29. \int \|\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}\| dt = \int \sqrt{1 + t^2} dt = \frac{1}{2}[t\sqrt{1 + t^2} + \ln|t + \sqrt{1 + t^2}|] + \mathbf{C}$$

$$31. \mathbf{r}(t) = \int (2t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}) dt = t^2\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{j} - \mathbf{k} + \mathbf{C} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

$$\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (e^t + 2)\mathbf{j} - (e^{-t} + 4)\mathbf{k}$$

$$33. \int_{-2}^2 (3t\mathbf{i} + 2t^2\mathbf{j} - t^3\mathbf{k}) dt = \left[\frac{3t^2}{2}\mathbf{i} + \frac{2t^3}{3}\mathbf{j} - \frac{t^4}{4}\mathbf{k}\right]_{-2}^2 = \frac{32}{3}\mathbf{j}$$

$$35. \int_0^2 (e^{t/2}\mathbf{i} - 3t^2\mathbf{j} - \mathbf{k}) dt = \left[2e^{t/2}\mathbf{i} - t^3\mathbf{j} - t\mathbf{k}\right]_0^2 = (2e - 2)\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$$

$$37. \mathbf{r}(t) = \langle \cos^3 t, \sin^3 t, 3t \rangle$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t, 3 \rangle$$

$$\|\mathbf{v}(t)\| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t + 9}$$

$$= 3\sqrt{\cos^2 t \sin^2 t (\cos^2 t + \sin^2 t) + 1}$$

$$= 3\sqrt{\cos^2 t \sin^2 t + 1}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -6\cos t(-\sin^2 t) + (-3\cos^2 t)\cos t, 6\sin t \cos^2 t + 3\sin^2 t(-\sin t), 0 \rangle$$

$$= \langle 3\cos t(2\sin^2 t - \cos^2 t), 3\sin t(2\cos^2 t - \sin^2 t), 0 \rangle$$

$$39. \mathbf{r}(t) = \left\langle \ln(t-3), t^2, \frac{1}{2}t \right\rangle, t_0 = 4$$

$$\mathbf{r}'(t) = \left\langle \frac{1}{t-3}, 2t, \frac{1}{2} \right\rangle$$

$$\mathbf{r}'(4) = \left\langle 1, 8, \frac{1}{2} \right\rangle \text{ direction numbers}$$

Since $\mathbf{r}(4) = \langle 0, 16, 2 \rangle$, the parametric equations are

$$x = t, y = 16 + 8t, z = 2 + \frac{1}{2}t.$$

$$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(4.1) \approx \langle 0.1, 16.8, 2.05 \rangle$$

$$41. \mathbf{r}(t) = \langle v_0 t \cos \theta, v_0 t \sin \theta - \frac{1}{2}gt^2 \rangle$$

$$= \left\langle \frac{75\sqrt{3}}{2}t, \frac{75}{2}t - 16t^2 \right\rangle$$

$$\frac{75}{2}t - 16t^2 \Rightarrow t = \frac{75}{32}$$

$$\text{Range} = \frac{75\sqrt{3}}{2} \left(\frac{75}{32} \right) = \frac{5625}{64}\sqrt{3} \approx 152.2 \text{ feet}$$

$$\text{or, Range} = v_0 \cos \theta \left[\frac{v_0 \sin \theta}{\frac{1}{2}g} \right] = \frac{v_0^2 \sin 2\theta}{g}$$

$$= \frac{75^2 \sin(60^\circ)}{32} \approx 152.2 \text{ feet}$$

$$43. \text{Range} = x = \frac{v_0^2}{9.8} \sin 2\theta = 80 \Rightarrow v_0 = \sqrt{\frac{(80)(9.8)}{\sin 40^\circ}} \approx 34.9 \text{ m/sec (see Exercise 41.)}$$

45. $\mathbf{r}(t) = 5t\mathbf{i}$

$\mathbf{v}(t) = 5\mathbf{i}$

$\|\mathbf{v}(t)\| = 5$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \mathbf{i}$

 $\mathbf{N}(t)$ does not exist.

$\mathbf{a} \cdot \mathbf{T} = 0$

 $\mathbf{a} \cdot \mathbf{N}$ does not exist.

(The curve is a line.)

47. $\mathbf{r}(t) = t\mathbf{i} + \sqrt{t}\mathbf{j}$

$\mathbf{v}(t) = \mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$

$\|\mathbf{v}(t)\| = \frac{\sqrt{4t+1}}{2\sqrt{t}}$

$\mathbf{a}(t) = -\frac{1}{4t\sqrt{t}}\mathbf{j}$

$\mathbf{T}(t) = \frac{\mathbf{i} + (1/2\sqrt{t})\mathbf{j}}{(\sqrt{4t+1})/2\sqrt{t}} = \frac{2\sqrt{t}\mathbf{i} + \mathbf{j}}{\sqrt{4t+1}}$

$\mathbf{N}(t) = \frac{\mathbf{i} - 2\sqrt{t}\mathbf{j}}{\sqrt{4t+1}}$

$\mathbf{a} \cdot \mathbf{T} = \frac{-1}{4t\sqrt{t}\sqrt{4t+1}}$

$\mathbf{a} \cdot \mathbf{N} = \frac{1}{2t\sqrt{4t+1}}$

49. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{e^{2t} + e^{-2t}}$

$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{T}(t) = \frac{e^t\mathbf{i} - e^{-t}\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{N}(t) = \frac{e^{-t}\mathbf{i} + e^t\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{a} \cdot \mathbf{T} = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{a} \cdot \mathbf{N} = \frac{2}{\sqrt{e^{2t} + e^{-2t}}}$

51. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\|\mathbf{v}\| = \sqrt{1 + 5t^2}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1 + 5t^2}}$

$\mathbf{N}(t) = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1 + 5t^2}}$

$\mathbf{a} \cdot \mathbf{T} = \frac{5t}{\sqrt{1 + 5t^2}}$

$\mathbf{a} \cdot \mathbf{N} = \frac{5}{\sqrt{5}\sqrt{1 + 5t^2}} = \frac{\sqrt{5}}{\sqrt{1 + 5t^2}}$

53. $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + t\mathbf{k}, x = 2 \cos t, y = 2 \sin t, z = t$

When $t = \frac{3\pi}{4}$, $x = -\sqrt{2}$, $y = \sqrt{2}$, $z = \frac{3\pi}{4}$.

$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \mathbf{k}$

Direction numbers when $t = \frac{3\pi}{4}$, $a = -\sqrt{2}$, $b = -\sqrt{2}$, $c = 1$

$x = -\sqrt{2}t - \sqrt{2}, y = -\sqrt{2}t + \sqrt{2}, z = t + \frac{3\pi}{4}$

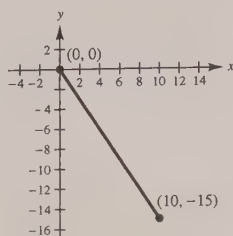
55. $v = \sqrt{\frac{9.56 \times 10^4}{4600}} \approx 4.56 \text{ mi/sec}$ (see Exercise 56, Section 11.4.)

57. $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j}, 0 \leq t \leq 5$

$\mathbf{r}'(t) = 2\mathbf{i} - 3\mathbf{j}$

$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^5 \sqrt{4 + 9} dt$

$= \sqrt{13}t \Big|_0^5 = 5\sqrt{13}$



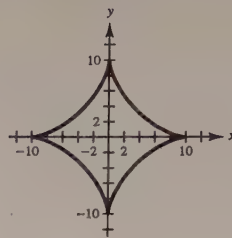
59. $\mathbf{r}(t) = 10 \cos^3 t \mathbf{i} + 10 \sin^3 t \mathbf{j}$

$$\mathbf{r}'(t) = -30 \cos^2 t \sin t \mathbf{i} + 30 \sin^2 t \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 30 \sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t}$$

$$= 30 |\cos t \sin t|$$

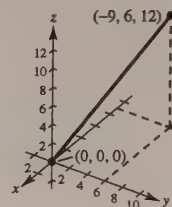
$$s = 4 \int_0^{\pi/2} 30 \cos t \cdot \sin t \, dt = \left[120 \frac{\sin^2 t}{2} \right]_0^{\pi/2} = 60$$



61. $\mathbf{r}(t) = -3t \mathbf{i} + 2t \mathbf{j} + 4t \mathbf{k}, 0 \leq t \leq 3$

$$\mathbf{r}'(t) = -3 \mathbf{i} + 2 \mathbf{j} + 4 \mathbf{k}$$

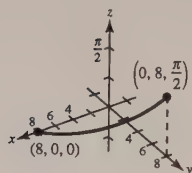
$$s = \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_0^3 \sqrt{9 + 4 + 16} \, dt = \int_0^3 \sqrt{29} \, dt = 3\sqrt{29}$$



63. $\mathbf{r}(t) = \langle 8 \cos t, 8 \sin t, t \rangle, 0 \leq t \leq \frac{\pi}{2}$

$$\mathbf{r}'(t) = \langle -8 \sin t, 8 \cos t, 1 \rangle, \|\mathbf{r}'(t)\| = \sqrt{65}$$

$$s = \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_0^{\pi/2} \sqrt{65} \, dt = \frac{\pi\sqrt{65}}{2}$$



65. $\mathbf{r}(t) = \frac{1}{2}t \mathbf{i} + \sin t \mathbf{j} + \cos t \mathbf{k}, 0 \leq t \leq \pi$

$$\mathbf{r}'(t) = \frac{1}{2} \mathbf{i} + \cos t \mathbf{j} - \sin t \mathbf{k}$$

$$\begin{aligned} s &= \int_0^{\pi} \|\mathbf{r}'(t)\| \, dt \\ &= \int_0^{\pi} \sqrt{\frac{1}{4} + \cos^2 t + \sin^2 t} \, dt \\ &= \frac{\sqrt{5}}{2} \int_0^{\pi} dt = \left[\frac{\sqrt{5}}{2} t \right]_0^{\pi} = \frac{\sqrt{5}}{2} \pi \end{aligned}$$

67. $\mathbf{r}(t) = 3t \mathbf{i} + 2t \mathbf{j}$

Line

$$K = 0$$

69. $\mathbf{r}(t) = 2t \mathbf{i} + \frac{1}{2}t^2 \mathbf{j} + t^2 \mathbf{k}$

$$\mathbf{r}'(t) = 2 \mathbf{i} + t \mathbf{j} + 2t \mathbf{k}, \|\mathbf{r}'\| = \sqrt{5t^2 + 4}$$

$$\mathbf{r}''(t) = \mathbf{j} + 2 \mathbf{k}$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & t & 2t \\ 0 & 1 & 2 \end{vmatrix} = -4 \mathbf{j} + 2 \mathbf{k}, \|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{20}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{20}}{(5t^2 + 4)^{3/2}} = \frac{2\sqrt{5}}{(4 + 5t^2)^{3/2}}$$

71. $y = \frac{1}{2}x^2 + 2$

$$y' = x$$

$$y'' = 1$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1}{(1 + x^2)^{3/2}}$$

$$\text{At } x = 4, K = \frac{1}{17^{3/2}} \text{ and } r = 17^{3/2} = 17\sqrt{17}.$$

73. $y = \ln x$

$$y' = \frac{1}{x}, y'' = -\frac{1}{x^2}$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1/x^2}{[1 + (1/x)^2]^{3/2}}$$

$$\text{At } x = 1, K = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \text{ and } r = 2\sqrt{2}.$$

75. The curvature changes abruptly from zero to a nonzero constant at the points B and C .

Problem Solving for Chapter 11

$$1. \quad x(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du, \quad y(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du$$

$$x'(t) = \cos\left(\frac{\pi t^2}{2}\right), \quad y'(t) = \sin\left(\frac{\pi t^2}{2}\right)$$

$$(a) \quad s = \int_0^a \sqrt{x'(t)^2 + y'(t)^2} dt = \int_0^a dt = a$$

$$(b) \quad x''(t) = -\pi t \sin\left(\frac{\pi t^2}{2}\right), \quad y''(t) = \pi t \cos\left(\frac{\pi t^2}{2}\right)$$

$$K = \frac{\left| \pi t \cos^2\left(\frac{\pi t^2}{2}\right) + \pi t \sin^2\left(\frac{\pi t^2}{2}\right) \right|}{1} = \pi t$$

$$\text{At } t = a, K = \pi a.$$

$$(c) \quad K = \pi a = \pi(\text{length})$$

$$5. \quad x'(\theta) = 1 - \cos \theta, \quad y'(\theta) = \sin \theta, \quad 0 \leq \theta \leq 2\pi$$

$$\begin{aligned} \sqrt{x'(\theta)^2 + y'(\theta)^2} &= \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} \\ &= \sqrt{2 - 2 \cos \theta} = \sqrt{4 \sin^2 \frac{\theta}{2}} \end{aligned}$$

$$s(t) = \int_{\pi}^t 2 \sin \frac{\theta}{2} d\theta = \left[-4 \cos \frac{\theta}{2} \right]_{\pi}^t = -4 \cos \frac{t}{2}$$

$$x''(\theta) = \sin \theta, \quad y''(\theta) = \cos \theta$$

$$\begin{aligned} K &= \frac{|(1 - \cos \theta) \cos \theta - \sin \theta \sin \theta|}{\left(2 \sin \frac{\theta}{2}\right)^3} = \frac{|\cos \theta - 1|}{8 \sin^3 \frac{\theta}{2}} \\ &= \frac{1}{4 \sin \frac{\theta}{2}} \end{aligned}$$

$$\text{Thus, } \rho = \frac{1}{K} = 4 \sin \frac{t}{2} \text{ and}$$

$$s^2 + \rho^2 = 16 \cos^2\left(\frac{t}{2}\right) + 16 \sin^2\left(\frac{t}{2}\right) = 16.$$

$$7. \quad \|\mathbf{r}(t)\|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$$

$$\begin{aligned} \frac{d}{dt} (\|\mathbf{r}(t)\|^2) &= 2\|\mathbf{r}(t)\| \frac{d}{dt} \|\mathbf{r}(t)\| \\ &= \mathbf{r}(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}(t) \Rightarrow \frac{d}{dt} \|\mathbf{r}(t)\| = \frac{\mathbf{r}(t) \cdot \mathbf{r}'(t)}{\|\mathbf{r}(t)\|} \end{aligned}$$

$$3. \quad \text{Bomb: } \mathbf{r}_1(t) = \langle 5000 - 400t, 3200 - 16t^2 \rangle$$

$$\text{Projectile: } \mathbf{r}_2(t) = \langle (v_0 \cos \theta)t, (v_0 \sin \theta)t - 16t^2 \rangle$$

At 1600 feet: Bomb:

$$3200 - 16t^2 = 1600 \Rightarrow t = 10 \text{ seconds.}$$

Projectile will travel 5 seconds:

$$5(v_0 \sin \theta) - 16(25) = 1600$$

$$v_0 \sin \theta = 400.$$

Horizontal position:

$$\text{At } t = 10, \text{ bomb is at } 5000 - 400(10) = 1000.$$

$$\text{At } t = 5, \text{ projectile is at } 5v_0 \cos \theta.$$

$$\text{Thus, } v_0 \cos \theta = 200.$$

$$\text{Combining, } \frac{v_0 \sin \theta}{v_0 \cos \theta} = \frac{400}{200} \Rightarrow \tan \theta = 2 \Rightarrow \theta \approx 63.4^\circ.$$

$$v_0 = \frac{200}{\cos \theta} \approx 447.2 \text{ ft/sec}$$

9. $\mathbf{r}(t) = 4 \cos t \mathbf{i} + 4 \sin t \mathbf{j} + 3t \mathbf{k}, t = \frac{\pi}{2}$

$$\mathbf{r}'(t) = -4 \sin t \mathbf{i} + 4 \cos t \mathbf{j} + 3 \mathbf{k}, \|\mathbf{r}'(t)\| = 5$$

$$\mathbf{r}''(t) = -4 \cos t \mathbf{i} - 4 \sin t \mathbf{j}$$

$$\mathbf{T} = -\frac{4}{5} \sin t \mathbf{i} + \frac{4}{5} \cos t \mathbf{j} + \frac{3}{5} \mathbf{k}$$

$$\mathbf{T}' = -\frac{4}{5} \cos t \mathbf{i} - \frac{4}{5} \sin t \mathbf{j}$$

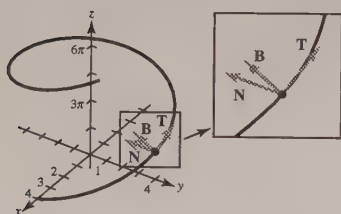
$$\mathbf{N} = -\cos t \mathbf{i} - \sin t \mathbf{j}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{3}{5} \sin t \mathbf{i} - \frac{3}{5} \cos t \mathbf{j} + \frac{4}{5} \mathbf{k}$$

At $t = \frac{\pi}{2}, \mathbf{T}\left(\frac{\pi}{2}\right) = -\frac{4}{5} \mathbf{i} + \frac{3}{5} \mathbf{k}$

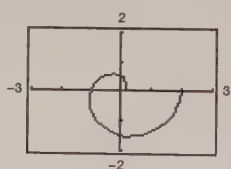
$$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{k}$$



13. $\mathbf{r}(t) = \langle t \cos \pi t, t \sin \pi t \rangle, 0 \leq t \leq 2$

(a)



(c) $K = \frac{\pi(\pi^2 t^2 + 2)}{[\pi^2 t^2 + 1]^{3/2}}$

$$K(0) = 2\pi$$

$$K(1) = \frac{\pi(\pi^2 + 2)}{(\pi^2 + 1)^{3/2}} \approx 1.04$$

$$K(2) \approx 0.51$$

(e) $\lim_{t \rightarrow \infty} K = 0$

11. (a) $\|\mathbf{B}\| = \|\mathbf{T} \times \mathbf{N}\| = 1$ constant length $\Rightarrow \frac{d\mathbf{B}}{ds} \perp \mathbf{B}$

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = (\mathbf{T} \times \mathbf{N}') + (\mathbf{T}' \times \mathbf{N})$$

$$\begin{aligned} \mathbf{T} \cdot \frac{d\mathbf{B}}{ds} &= \mathbf{T} \cdot (\mathbf{T} \times \mathbf{N}') + \mathbf{T} \cdot (\mathbf{T}' \times \mathbf{N}) \\ &= (\mathbf{T} \times \mathbf{T}) \cdot \mathbf{N}' + \mathbf{T} \cdot \left(\mathbf{T}' \times \frac{\mathbf{T}'}{\|\mathbf{T}'\|} \right) = 0 \end{aligned}$$

Hence, $\frac{d\mathbf{B}}{ds} \perp \mathbf{B}$ and $\frac{d\mathbf{B}}{ds} \perp \mathbf{T} \Rightarrow \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}$
for some scalar τ .

(b) $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Using Exercise 10.4, number 64,

$$\begin{aligned} \mathbf{B} \times \mathbf{N} &= (\mathbf{T} \times \mathbf{N}) \times \mathbf{N} = -\mathbf{N} \times (\mathbf{T} \times \mathbf{N}) \\ &= -[(\mathbf{N} \cdot \mathbf{N})\mathbf{T} - (\mathbf{N} \cdot \mathbf{T})\mathbf{N}] \\ &= -\mathbf{T} \end{aligned}$$

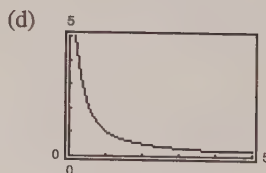
$$\begin{aligned} \mathbf{B} \times \mathbf{T} &= (\mathbf{T} \times \mathbf{N}) \times \mathbf{T} = -\mathbf{T} \times (\mathbf{T} \times \mathbf{N}) \\ &= -[(\mathbf{T} \cdot \mathbf{N})\mathbf{T} - (\mathbf{T} \cdot \mathbf{T})\mathbf{N}] \\ &= \mathbf{N}. \end{aligned}$$

Now, $KN = \left\| \frac{d\mathbf{T}}{ds} \right\| \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \mathbf{T}'(s) = \frac{d\mathbf{T}}{ds}$.

Finally,

$$\begin{aligned} \mathbf{N}'(s) &= \frac{d}{ds}(\mathbf{B} \times \mathbf{T}) = (\mathbf{B} \times \mathbf{T}') + (\mathbf{B}' \times \mathbf{T}) \\ &= (\mathbf{B} \times \mathbf{KN}) + (-\tau \mathbf{N} \times \mathbf{T}) \\ &= -K\mathbf{T} + \tau \mathbf{B}. \end{aligned}$$

(b) Length $= \int_0^2 \|\mathbf{r}'(t)\| dt$
 $= \int_0^2 \sqrt{\pi^2 t^2 + 1} dt \approx 6.766$ (graphing utility)



(f) As $t \rightarrow \infty$, the graph spirals outward and the curvature decreases.

CHAPTER 12

Functions of Several Variables

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CHAPTER 12

Functions of Several Variables

Section 12.1 Introduction to Functions of Several Variables

Solutions to Odd-Numbered Exercises

1. $x^2z + yz - xy = 10$

$$z(x^2 + y) = 10 + xy$$

$$z = \frac{10 + xy}{x^2 + y}$$

Yes, z is a function of x and y .

3. $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$

No, z is not a function of x and y . For example, $(x, y) = (0, 0)$ corresponds to both $z = \pm 1$.

5. $f(x, y) = \frac{x}{y}$

(a) $f(3, 2) = \frac{3}{2}$

(b) $f(-1, 4) = -\frac{1}{4}$

(c) $f(30, 5) = \frac{30}{5} = 6$

(d) $f(5, y) = \frac{5}{y}$

(e) $f(x, 2) = \frac{x}{2}$

(f) $f(5, t) = \frac{5}{t}$

7. $f(x, y) = xe^y$

(a) $f(5, 0) = 5e^0 = 5$

(b) $f(3, 2) = 3e^2$

(c) $f(2, -1) = 2e^{-1} = \frac{2}{e}$

(d) $f(5, y) = 5e^y$

(e) $f(x, 2) = xe^2$

(f) $f(t, t) = te^t$

9. $h(x, y, z) = \frac{xyz}{z}$

(a) $h(2, 3, 9) = \frac{(2)(3)}{9} = \frac{2}{3}$

(b) $h(1, 0, 1) = \frac{(1)(0)}{1} = 0$

11. $f(x, y) = x \sin y$

(a) $f\left(2, \frac{\pi}{4}\right) = 2 \sin \frac{\pi}{4} = \sqrt{2}$

(b) $f(3, 1) = 3 \sin 1$

13. $g(x, y) = \int_x^y (2t - 3) dt$

(a) $g(0, 4) = \int_0^4 (2t - 3) dt = \left[t^2 - 3t\right]_0^4 = 4$

(b) $g(1, 4) = \int_1^4 (2t - 3) dt = \left[t^2 - 3t\right]_1^4 = 6$

15. $f(x, y) = x^2 - 2y$

$$\begin{aligned} \text{(a)} \quad \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} &= \frac{[(x + \Delta x)^2 - 2y] - (x^2 - 2y)}{\Delta x} \\ &= \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - 2y - x^2 + 2y}{\Delta x} = \frac{\Delta x(2x + \Delta x)}{\Delta x} = 2x + \Delta x, \Delta x \neq 0 \end{aligned}$$

$$\text{(b)} \quad \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{[x^2 - 2(y + \Delta y)] - (x^2 - 2y)}{\Delta y} = \frac{x^2 - 2y - 2\Delta y - x^2 + 2y}{\Delta y} = \frac{-2\Delta y}{\Delta y} = -2, \Delta y \neq 0$$

17. $f(x, y) = \sqrt{4 - x^2 - y^2}$

Domain: $4 - x^2 - y^2 \geq 0$

$x^2 + y^2 \leq 4$

$\{(x, y): x^2 + y^2 \leq 4\}$

Range: $0 \leq z \leq 2$

19. $f(x, y) = \arcsin(x + y)$

Domain:

$\{(x, y): -1 \leq x + y \leq 1\}$

Range: $-\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$

21. $f(x, y) = \ln(4 - x - y)$

Domain: $4 - x - y > 0$

$x + y < 4$

$\{(x, y): y < -x + 4\}$

Range: all real numbers

23. $z = \frac{x + y}{xy}$

Domain: $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$

Range: all real numbers

25. $f(x, y) = e^{x/y}$

Domain: $\{(x, y): y \neq 0\}$

Range: $z > 0$

27. $g(x, y) = \frac{1}{xy}$

Domain: $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$

Range: all real numbers except zero

29. $f(x, y) = \frac{-4x}{x^2 + y^2 + 1}$

 (a) View from the positive x -axis: $(20, 0, 0)$

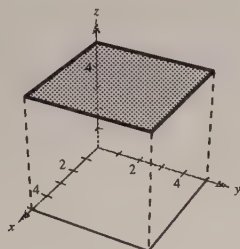
 (c) View from the first octant: $(20, 15, 25)$

 (b) View where x is negative, y and z are positive:
 $(-15, 10, 20)$

 (d) View from the line $y = x$ in the xy -plane: $(20, 20, 0)$

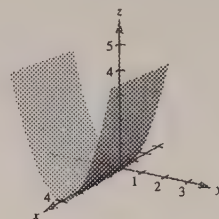
31. $f(x, y) = 5$

Plane: $z = 5$



33. $f(x, y) = y^2$

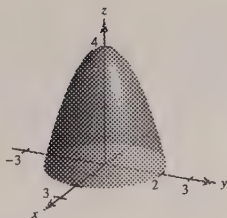
Since the variable x is missing, the surface is a cylinder with rulings parallel to the x -axis. The generating curve is $z = y^2$. The domain is the entire xy -plane and the range is $z \geq 0$.



35. $z = 4 - x^2 - y^2$

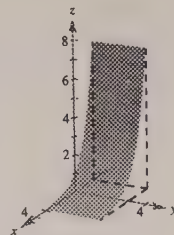
Paraboloid

 Domain: entire xy -plane

 Range: $z \leq 4$


37. $f(x, y) = e^{-x}$

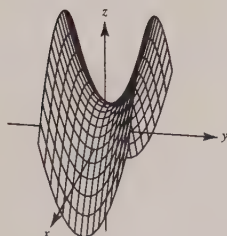
Since the variable y is missing, the surface is a cylinder with rulings parallel to the y -axis. The generating curve is $z = e^{-x}$. The domain is the entire xy -plane and the range is $z > 0$.



39. $z = y^2 - x^2 + 1$

Hyperbolic paraboloid

 Domain: entire xy -plane

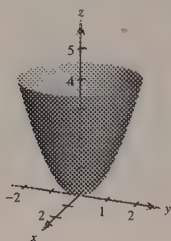
 Range: $-\infty < z < \infty$


41. $f(x, y) = x^2 e^{(-xy/2)}$

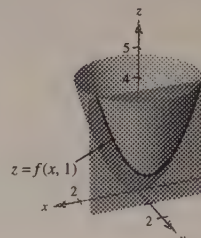
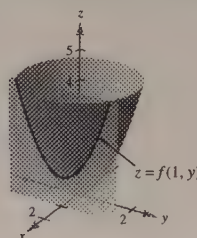


43. $f(x, y) = x^2 + y^2$

(a)

(b) g is a vertical translation of f two units upward(c) g is a horizontal translation of f two units to the right. The vertex moves from $(0, 0, 0)$ to $(0, 2, 0)$.(d) g is a reflection of f in the xy -plane followed by a vertical translation 4 units upward.

(e)



45. $z = e^{1-x^2-y^2}$

Level curves:

$$c = e^{1-x^2-y^2}$$

$$\ln c = 1 - x^2 - y^2$$

$$x^2 + y^2 = 1 - \ln c$$

Circles centered at $(0, 0)$

Matches (c)

47. $z = \ln|y - x^2|$

Level curves:

$$c = \ln|y - x^2|$$

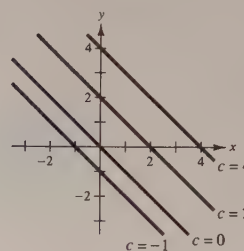
$$\pm e^c = y - x^2$$

$$y = x^2 \pm e^c$$

Parabolas

Matches (b)

49. $z = x + y$

Level curves are parallel lines of the form $x + y = c$.

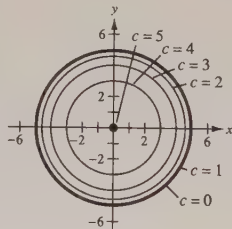
51. $f(x, y) = \sqrt{25 - x^2 - y^2}$

The level curves are of the form

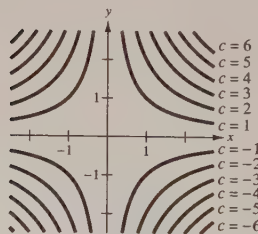
$$c = \sqrt{25 - x^2 - y^2}$$

$$x^2 + y^2 = 25 - c^2$$

Thus, the level curves are circles of radius 5 or less, centered at the origin.



53. $f(x, y) = xy$

The level curves are hyperbolas of the form $xy = c$.

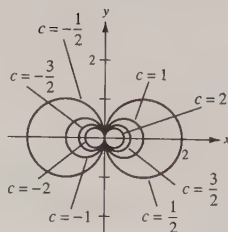
55. $f(x, y) = \frac{x}{x^2 + y^2}$

The level curves are of the form

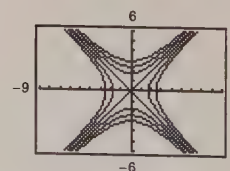
$$c = \frac{x}{x^2 + y^2}$$

$$x^2 - \frac{x}{c} + y^2 = 0$$

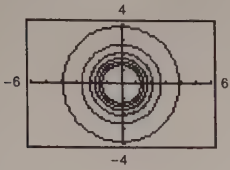
$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \left(\frac{1}{2c}\right)^2$$

Thus, the level curves are circles passing through the origin and centered at $(1/2c, 0)$.

57. $f(x, y) = x^2 - y^2 + 2$



59. $g(x, y) = \frac{8}{1 + x^2 + y^2}$



65. The surface is sloped like a saddle. The graph is not unique. Any vertical translation would have the same level curves.

One possible function is

$$f(x, y) = x^2 - y^2.$$

61. See Definition, page 838.

63. No, The following graphs are not hemispheres.

$$z = e^{-(x^2 + y^2)}$$

$$z = x^2 + y^2$$

67. $V(I, R) = 1000 \left[\frac{1 + 0.10(1 - R)}{1 + I} \right]^{10}$

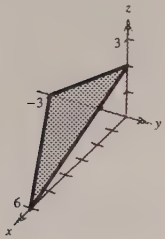
	Inflation Rate		
Tax Rate	0	0.03	0.05
0	2593.74	1929.99	1592.33
0.28	2004.23	1491.34	1230.42
0.35	1877.14	1396.77	1152.40

69. $f(x, y, z) = x - 2y + 3z$

$$c = 6$$

$$6 = x - 2y + 3z$$

Plane

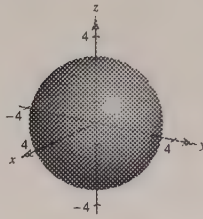


71. $f(x, y, z) = x^2 + y^2 + z^2$

$$c = 9$$

$$9 = x^2 + y^2 + z^2$$

Sphere

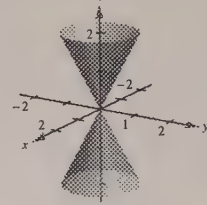


73. $f(x, y, z) = 4x^2 + 4y^2 - z^2$

$$c = 0$$

$$0 = 4x^2 + 4y^2 - z^2$$

Elliptic cone



75. $N(d, L) = \left(\frac{d - 4}{4} \right)^2 L$

(a) $N(22, 12) = \left(\frac{22 - 4}{4} \right)^2 (12) = 243$ board-feet

(b) $N(30, 12) = \left(\frac{30 - 4}{4} \right)^2 (12) = 507$ board-feet

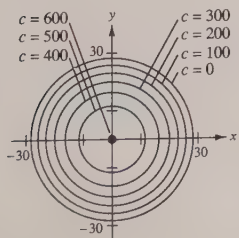
77. $T = 600 - 0.75x^2 - 0.75y^2$

The level curves are of the form

$$c = 600 - 0.75x^2 - 0.75y^2$$

$$x^2 + y^2 = \frac{600 - c}{0.75}.$$

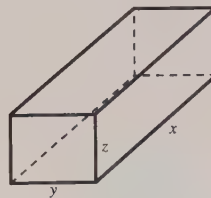
The level curves are circles centered at the origin.



79. $C = 0.75xy + 2(0.40)xz + 2(0.40)yz$

base + front & back + two ends

$$= 0.75xy + 0.80(xz + yz)$$



81. $PV = kT$, $20(2600) = k(300)$

(a) $k = \frac{20(2600)}{300} = \frac{520}{3}$

(b) $P = \frac{kT}{V} = \frac{520}{3} \left(\frac{T}{V} \right)$

The level curves are of the form: $c = \left(\frac{520}{3} \right) \left(\frac{T}{V} \right)$

$$V = \frac{520}{3c} T$$

Thus, the level curves are lines through the origin with slope $\frac{520}{3c}$.

83. (a) Highest pressure at C

(b) Lowest pressure at A

(c) Highest wind velocity at B

85. (a) The boundaries between colors represent level curves

(b) No, the colors represent intervals of different lengths, as indicated in the box

(c) You could use more colors, which means using smaller intervals

87. False. Let

$$f(x, y) = 2xy$$

$$f(1, 2) = f(2, 1), \text{ but } 1 \neq 2$$

89. False. Let

$$f(x, y) = 5.$$

$$\text{Then, } f(2x, 2y) = 5 \neq 2^2 f(x, y).$$

Section 12.2 Limits and Continuity

1. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(x, y) - L| = |y - b| < \varepsilon$

whenever $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$. Take $\delta = \varepsilon$.

Then if $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta = \varepsilon$, we have

$$\sqrt{(y - b)^2} < \varepsilon$$

$$|y - b| < \varepsilon.$$

3. $\lim_{(x, y) \rightarrow (a, b)} [f(x, y) - g(x, y)] = \lim_{(x, y) \rightarrow (a, b)} f(x, y) - \lim_{(x, y) \rightarrow (a, b)} g(x, y) = 5 - 3 = 2$

5. $\lim_{(x, y) \rightarrow (a, b)} [f(x, y)g(x, y)] = \left[\lim_{(x, y) \rightarrow (a, b)} f(x, y) \right] \left[\lim_{(x, y) \rightarrow (a, b)} g(x, y) \right] = 5(3) = 15$

7. $\lim_{(x, y) \rightarrow (2, 1)} (x + 3y^2) = 2 + 3(1)^2 = 5$

Continuous everywhere

9. $\lim_{(x, y) \rightarrow (2, 4)} \frac{x + y}{x - y} = \frac{2 + 4}{2 - 4} = -3$

Continuous for $x \neq y$

11. $\lim_{(x, y) \rightarrow (0, 1)} \frac{\arcsin(x/y)}{1 + xy} = \arcsin 0 = 0$

Continuous for $xy \neq -1$, $y \neq 0$, $|x/y| \leq 1$

13. $\lim_{(x, y) \rightarrow (-1, 2)} e^{xy} = e^{-2} = \frac{1}{e^2}$

Continuous everywhere

15. $\lim_{(x, y, z) \rightarrow (1, 2, 5)} \sqrt{x + y + z} = \sqrt{8} = 2\sqrt{2}$

Continuous for $x + y + z \geq 0$

17. $\lim_{(x, y) \rightarrow (0, 0)} e^{xy} = 1$

Continuous everywhere

19. $\lim_{(x,y) \rightarrow (0,0)} \ln(x^2 + y^2) = \ln(0) = -\infty$

The limit does not exist.

Continuous except at $(0, 0)$

21. $f(x, y) = \frac{xy}{x^2 + y^2}$

Continuous except at $(0, 0)$

Path: $y = 0$

(x, y)	(1, 0)	(0.5, 0)	(0.1, 0)	(0.01, 0)	(0.001, 0)
$f(x, y)$	0	0	0	0	0

Path: $y = x$

(x, y)	(1, 1)	(0.5, 0.5)	(0.1, 0.1)	(0.01, 0.01)	(0.001, 0.001)
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The limit does not exist because along the path $y = 0$ the function equals 0, whereas along the path $y = x$ the function equals $\frac{1}{2}$.

23. $f(x, y) = -\frac{xy^2}{x^2 + y^4}$

Continuous except at $(0, 0)$

Path: $x = y^2$

(x, y)	(1, 1)	(0.25, 0.5)	(0.01, 0.1)	(0.0001, 0.01)	(0.000001, 0.001)
$f(x, y)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Path: $x = -y^2$

(x, y)	(-1, 1)	(-0.25, 0.5)	(-0.01, 0.1)	(-0.0001, 0.01)	(-0.000001, 0.001)
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

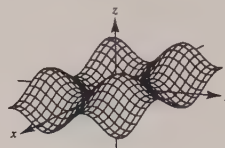
The limit does not exist because along the path $x = y^2$ the function equals $-\frac{1}{2}$, whereas along the path $x = -y^2$ the function equals $\frac{1}{2}$.

25. $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \left(\frac{x^2 + 2xy^2 + y^2}{x^2 + y^2} \right)$
 $= \lim_{(x,y) \rightarrow (0,0)} \left(1 + \frac{2xy^2}{x^2 + y^2} \right) = 1$

(same limit for g)

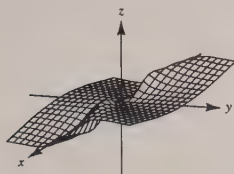
Thus, f is not continuous at $(0, 0)$, whereas g is continuous at $(0, 0)$.

27. $\lim_{(x,y) \rightarrow (0,0)} (\sin x + \sin y) = 0$



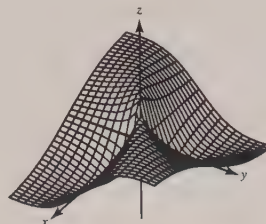
29. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + 4y^2}$

Does not exist



31. $f(x, y) = \frac{10xy}{2x^2 + 3y^2}$

The limit does not exist. Use the paths $x = 0$ and $x = y$.



$$33. \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{\sin r^2}{r^2} = \lim_{r \rightarrow 0} \frac{2r \cos r^2}{2r} = \lim_{r \rightarrow 0} \cos r^2 = 1$$

$$35. \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3(\cos^3 \theta + \sin^3 \theta)}{r^2} = \lim_{r \rightarrow 0} r(\cos^3 \theta + \sin^3 \theta) = 0$$

$$37. f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

Continuous except at (0, 0, 0)

$$39. f(x, y, z) = \frac{\sin z}{e^x + e^y}$$

Continuous everywhere

$$41. f(t) = t^2$$

$$g(x, y) = 3x - 2y$$

$$f(g(x, y)) = f(3x - 2y)$$

$$= (3x - 2y)^2$$

$$= 9x^2 - 12xy + 4y^2$$

Continuous everywhere

$$43. f(t) = \frac{1}{t}$$

$$g(x, y) = 3x - 2y$$

$$f(g(x, y)) = f(3x - 2y) = \frac{1}{3x - 2y}$$

Continuous for $y \neq \frac{3x}{2}$

$$45. f(x, y) = x^2 - 4y$$

$$(a) \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 - 4y] - (x^2 - 4y)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x - \Delta x) = 2x$$

$$(b) \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{[x^2 - 4(y + \Delta y)] - (x^2 - 4y)}{\Delta y}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{-4\Delta y}{\Delta y} = \lim_{\Delta y \rightarrow 0} (-4) = -4$$

$$47. f(x, y) = 2x + xy - 3y$$

$$(a) \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{[2(x + \Delta x) + (x + \Delta x)y - 3y] - (2x + xy - 3y)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2\Delta x + \Delta xy}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2 + y) = 2 + y$$

$$(b) \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{[2x + x(y + \Delta y) - 3(y + \Delta y)] - (2x + xy - 3y)}{\Delta y}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{x\Delta y - 3\Delta y}{\Delta y} = \lim_{\Delta y \rightarrow 0} (x - 3) = x - 3$$

49. See the definition on page 851.

Show that the value of $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ is not the same for two different paths to (x_0, y_0) .

51. No.

The existence of $f(2, 3)$ has no bearing on the existence of the limit as $(x, y) \rightarrow (2, 3)$.

53. Since $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L_1$, then for $\varepsilon/2 > 0$, there corresponds $\delta_1 > 0$ such that $|f(x,y) - L_1| < \varepsilon/2$ whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_1.$$

Since $\lim_{(x,y) \rightarrow (a,b)} g(x,y) = L_2$, then for $\varepsilon/2 > 0$, there corresponds $\delta_2 > 0$ such that $|g(x,y) - L_2| < \varepsilon/2$ whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_2.$$

Let δ be the smaller of δ_1 and δ_2 . By the triangle inequality, whenever $\sqrt{(x-a)^2 + (y-b)^2} < \delta$, we have

$$|f(x,y) + g(x,y) - (L_1 + L_2)| = |(f(x,y) - L_1) + (g(x,y) - L_2)| \leq |f(x,y) - L_1| + |g(x,y) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\lim_{(x,y) \rightarrow (a,b)} [f(x,y) + g(x,y)] = L_1 + L_2$.

55. True. Assuming $f(x, 0)$ exists for $x \neq 0$.

57. False. Let

$$f(x,y) = \begin{cases} \ln(x^2 + y^2), & (x,y) \neq (0,0) \\ 0, & x=0, y=0 \end{cases}$$

See Exercise 19.

Section 12.3 Partial Derivatives

1. $f_x(4, 1) < 0$

3. $f_y(4, 1) > 0$

5. $f(x, y) = 2x - 3y + 5$

$$f_x(x, y) = 2$$

$$f_y(x, y) = -3$$

7. $z = x\sqrt{y}$

$$\frac{\partial z}{\partial x} = \sqrt{y}$$

$$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}}$$

9. $z = x^2 - 5xy + 3y^2$

$$\frac{\partial z}{\partial x} = 2x - 5y$$

$$\frac{\partial z}{\partial y} = -5x + 6y$$

11. $z = x^2 e^{2y}$

$$\frac{\partial z}{\partial x} = 2xe^{2y}$$

$$\frac{\partial z}{\partial y} = 2x^2 e^{2y}$$

13. $z = \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$$

15. $z = \ln\left(\frac{x+y}{x-y}\right) = \ln(x+y) - \ln(x-y)$

$$\frac{\partial z}{\partial x} = \frac{1}{x+y} - \frac{1}{x-y} = -\frac{2y}{x^2 - y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x+y} + \frac{1}{x-y} = \frac{2x}{x^2 - y^2}$$

17. $z = \frac{x^2}{2y} + \frac{4y^2}{x}$

$$\frac{\partial z}{\partial x} = \frac{2x}{2y} - \frac{4y^2}{x^2} = \frac{x^3 - 4y^3}{x^2 y}$$

$$\frac{\partial z}{\partial y} = -\frac{x^2}{2y^2} + \frac{8y}{x} = \frac{-x^3 + 16y^3}{2xy^2}$$

19. $h(x, y) = e^{-(x^2 + y^2)}$

$$h_x(x, y) = -2xe^{-(x^2 + y^2)}$$

$$h_y(x, y) = -2ye^{-(x^2 + y^2)}$$

21. $f(x, y) = \sqrt{x^2 + y^2}$

$$f_x(x, y) = \frac{1}{2}(x^2 + y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2}}$$

$$f_y(x, y) = \frac{1}{2}(x^2 + y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2 + y^2}}$$

23. $z = \tan(2x - y)$

$$\frac{\partial z}{\partial x} = 2 \sec^2(2x - y)$$

$$\frac{\partial z}{\partial y} = -\sec^2(2x - y)$$

25. $z = e^y \sin xy$

$$\frac{\partial z}{\partial x} = ye^y \cos xy$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= e^y \sin xy + xe^y \cos xy \\ &= e^y(x \cos xy + \sin xy)\end{aligned}$$

27. $f(x, y) = \int_x^y (t^2 - 1) dt$

$$= \left[\frac{t^3}{3} - t \right]_x^y = \left(\frac{y^3}{3} - y \right) - \left(\frac{x^3}{3} - x \right)$$

$$f_x(x, y) = -x^2 + 1 = 1 - x^2$$

$$f_y(x, y) = y^2 - 1$$

[You could also use the Second Fundamental Theorem of Calculus.]

29. $f(x, y) = 2x + 3y$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x) + 3y - 2x - 3y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2\Delta x}{\Delta x} = 2$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{2x + 3(y + \Delta y) - 2x - 3y}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{3\Delta y}{\Delta y} = 3$$

31. $f(x, y) = \sqrt{x + y}$

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x + y} - \sqrt{x + y}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(\sqrt{x + \Delta x + y} - \sqrt{x + y})(\sqrt{x + \Delta x + y} + \sqrt{x + y})}{\Delta x(\sqrt{x + \Delta x + y} + \sqrt{x + y})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x + y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\sqrt{x + y + \Delta y} - \sqrt{x + y}}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{(\sqrt{x + y + \Delta y} - \sqrt{x + y})(\sqrt{x + y + \Delta y} + \sqrt{x + y})}{\Delta y(\sqrt{x + y + \Delta y} + \sqrt{x + y})} \\ &= \lim_{\Delta y \rightarrow 0} \frac{1}{\sqrt{x + y + \Delta y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}\end{aligned}$$

33. $g(x, y) = 4 - x^2 - y^2$

$$g_x(x, y) = -2x$$

At (1, 1): $g_x(1, 1) = -2$

$$g_y(x, y) = -2y$$

At (1, 1): $g_y(1, 1) = -2$

35. $z = e^{-x} \cos y$

$$\frac{\partial z}{\partial x} = -e^{-x} \cos y$$

At (0, 0): $\frac{\partial z}{\partial x} = -1$

$$\frac{\partial z}{\partial y} = -e^{-x} \sin y$$

At (0, 0): $\frac{\partial z}{\partial y} = 0$

37. $f(x, y) = \arctan \frac{y}{x}$

$$f_x(x, y) = \frac{1}{1 + (y^2/x^2)} \left(-\frac{y}{x^2} \right) = \frac{-y}{x^2 + y^2}$$

At (2, -2): $f_x(2, -2) = \frac{1}{4}$

$$f_y(x, y) = \frac{1}{1 + (y^2/x^2)} \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2}$$

At (2, -2): $f_y(2, -2) = \frac{1}{4}$

39. $f(x, y) = \frac{xy}{x - y}$

$$f_x(x, y) = \frac{y(x - y) - xy}{(x - y)^2} = \frac{-y^2}{(x - y)^2}$$

At (2, -2): $f_x(2, -2) = -\frac{1}{4}$

$$f_y(x, y) = \frac{x(x - y) + xy}{(x - y)^2} = \frac{x^2}{(x - y)^2}$$

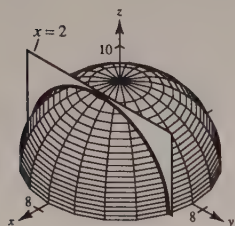
At (2, -2): $f_y(2, -2) = \frac{1}{4}$

41. $z = \sqrt{49 - x^2 - y^2}$, $x = 2$,
(2, 3, 6)

Intersecting curve: $z = \sqrt{45 - y^2}$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{45 - y^2}}$$

At (2, 3, 6): $\frac{\partial z}{\partial y} = \frac{-3}{\sqrt{45 - 9}} = -\frac{1}{2}$



45. $f_x(x, y) = 2x + 4y - 4$, $f_y(x, y) = 4x + 2y + 16$

$$f_x = f_y = 0: 2x + 4y = 4$$

$$4x + 2y = -16$$

Solving for x and y ,

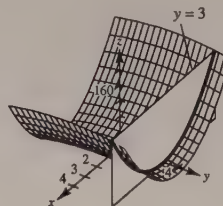
$$x = -6 \text{ and } y = 4.$$

43. $z = 9x^2 - y^2$, $y = 3$, (1, 3, 0)

Intersecting curve: $z = 9x^2 - 9$

$$\frac{\partial z}{\partial x} = 18x$$

At (1, 3, 0): $\frac{\partial z}{\partial x} = 18(1) = 18$



49. (a) The graph is that of f_y .

(b) The graph is that of f_x .

47. $f_x(x, y) = -\frac{1}{x^2} + y$, $f_y(x, y) = -\frac{1}{y^2} + x$

$$f_x = f_y = 0: -\frac{1}{x^2} + y = 0 \text{ and } -\frac{1}{y^2} + x = 0$$

$$y = \frac{1}{x^2} \text{ and } x = \frac{1}{y^2}$$

$$y = y^4 \Rightarrow y = 1 = x$$

Points: (1, 1)

49. (a) The graph is that of f_y .

(b) The graph is that of f_x .

51. $w = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

53. $F(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$

$$= \frac{1}{2} \ln(x^2 + y^2 + z^2)$$

$$F_x(x, y, z) = \frac{x}{x^2 + y^2 + z^2}$$

$$F_y(x, y, z) = \frac{y}{x^2 + y^2 + z^2}$$

$$F_z(x, y, z) = \frac{z}{x^2 + y^2 + z^2}$$

55. $H(x, y, z) = \sin(x + 2y + 3z)$

$$H_x(x, y, z) = \cos(x + 2y + 3z)$$

$$H_y(x, y, z) = 2 \cos(x + 2y + 3z)$$

$$H_z(x, y, z) = 3 \cos(x + 2y + 3z)$$

57. $z = x^2 - 2xy + 3y^2$

$$\frac{\partial z}{\partial x} = 2x - 2y$$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial^2 z}{\partial y \partial x} = -2$$

$$\frac{\partial z}{\partial y} = -2x + 6y$$

$$\frac{\partial^2 z}{\partial y^2} = 6$$

$$\frac{\partial^2 z}{\partial x \partial y} = -2$$

59. $z = \sqrt{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

61. $z = e^x \tan y$

$$\frac{\partial z}{\partial x} = e^x \tan y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \tan y$$

$$\frac{\partial^2 z}{\partial y \partial x} = e^x \sec^2 y$$

$$\frac{\partial z}{\partial y} = e^x \sec^2 y$$

$$\frac{\partial^2 z}{\partial y^2} = 2e^x \sec^2 y \tan y$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^x \sec^2 y$$

65. $z = x \sec y$

$$\frac{\partial z}{\partial x} = \sec y$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial^2 z}{\partial y \partial x} = \sec y \tan y$$

$$\frac{\partial z}{\partial y} = x \sec y \tan y$$

$$\frac{\partial^2 z}{\partial y^2} = x \sec y (\sec^2 y + \tan^2 y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \sec y \tan y$$

Therefore, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$.

There are no points for which $z_x = 0 = z_y$, because

$$\frac{\partial z}{\partial x} = \sec y \neq 0.$$

69. $f(x, y, z) = xyz$

$$f_x(x, y, z) = yz$$

$$f_y(x, y, z) = xz$$

$$f_{yy}(x, y, z) = 0$$

$$f_{xy}(x, y, z) = z$$

$$f_{yx}(x, y, z) = z$$

$$f_{yyx}(x, y, z) = 0$$

$$f_{xyy}(x, y, z) = 0$$

$$f_{yxy}(x, y, z) = 0$$

Therefore, $f_{xyy} = f_{yxy} = f_{yyx} = 0$.

63. $z = \arctan \frac{y}{x}$

$$\frac{\partial z}{\partial x} = \frac{1}{1 + (y^2/x^2)} \left(-\frac{y}{x^2} \right) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-(x^2 + y^2) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + (y^2/x^2)} \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

67. $z = \ln \left(\frac{x}{x^2 + y^2} \right) = \ln x - \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x} - \frac{2x}{x^2 + y^2} = \frac{y^2 - x^2}{x(x^2 + y^2)}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{x^4 - 4x^2y^2 - y^4}{x^2(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{4xy}{(x^2 + y^2)^2}$$

$$\frac{\partial z}{\partial y} = -\frac{2y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{4xy}{(x^2 + y^2)^2}$$

There are no points for which $z_x = z_y = 0$.

71. $f(x, y, z) = e^{-x} \sin yz$

$$f_x(x, y, z) = -e^{-x} \sin yz$$

$$f_y(x, y, z) = ze^{-x} \cos yz$$

$$f_{yy}(x, y, z) = -z^2 e^{-x} \sin yz$$

$$f_{xy}(x, y, z) = -ze^{-x} \cos yz$$

$$f_{yx}(x, y, z) = -ze^{-x} \cos yz$$

$$f_{yyx}(x, y, z) = z^2 e^{-x} \sin yz$$

$$f_{xyy}(x, y, z) = z^2 e^{-x} \sin yz$$

$$f_{yxy}(x, y, z) = z^2 e^{-x} \sin yz$$

Therefore, $f_{xyy} = f_{yxy} = f_{yyx}$.

73. $z = 5xy$

$$\frac{\partial z}{\partial x} = 5y$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial z}{\partial y} = 5x$$

$$\frac{\partial^2 z}{\partial y^2} = 0$$

Therefore, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 + 0 = 0$.

75. $z = e^x \sin y$

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \sin y$$

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -e^x \sin y$$

$$\text{Therefore, } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x \sin y - e^x \sin y = 0.$$

77. $z = \sin(x - ct)$

$$\frac{\partial z}{\partial t} = -c \cos(x - ct)$$

$$\frac{\partial^2 z}{\partial t^2} = -c^2 \sin(x - ct)$$

$$\frac{\partial z}{\partial x} = \cos(x - ct)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin(x - ct)$$

$$\text{Therefore, } \frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}.$$

79. $z = e^{-t} \cos \frac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t} \cos \frac{x}{c}$$

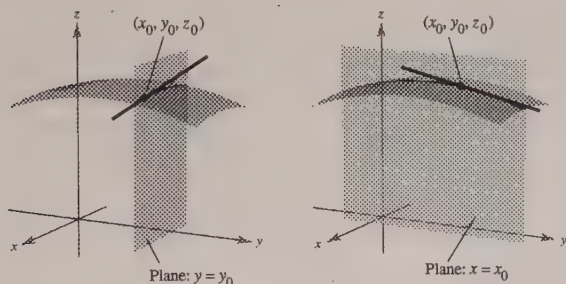
$$\frac{\partial z}{\partial x} = -\frac{1}{c} e^{-t} \sin \frac{x}{c}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2} e^{-t} \cos \frac{x}{c}$$

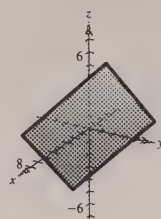
$$\text{Therefore, } \frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2}.$$

81. See the definition on page 859.

83.

 $\frac{\partial f}{\partial x}$ denotes the slope of the surface in the x -direction. $\frac{\partial f}{\partial y}$ denotes the slope of the surface in the y -direction.85. The plane $z = x + y = f(x, y)$ satisfies

$$\frac{\partial f}{\partial x} > 0 \text{ and } \frac{\partial f}{\partial y} > 0.$$



87. (a) $C = 32\sqrt{xy} + 175x + 205y + 1050$

$$\frac{\partial C}{\partial x} = 16\sqrt{\frac{y}{x}} + 175$$

$$\left. \frac{\partial C}{\partial x} \right|_{(80, 20)} = 16\sqrt{\frac{1}{4}} + 175 = 183$$

$$\frac{\partial C}{\partial y} = 16\sqrt{\frac{x}{y}} + 205$$

$$\left. \frac{\partial C}{\partial y} \right|_{(80, 20)} = 16\sqrt{4} + 205 = 237$$

(b) The fireplace-insert stove results in the cost increasing at a faster rate because

$$\frac{\partial C}{\partial y} > \frac{\partial C}{\partial x}.$$

89. An increase in either price will cause a decrease in demand.

$$91. T = 500 - 0.6x^2 - 1.5y^2$$

$$\frac{\partial T}{\partial x} = -1.2x, \frac{\partial T}{\partial x}(2, 3) = -2.4^\circ/\text{m}$$

$$\frac{\partial T}{\partial y} = -3y = \frac{\partial T}{\partial y}(2, 3) = -9^\circ/\text{m}$$

$$93. PV = mRT$$

$$T = \frac{PV}{mR} \Rightarrow \frac{\partial T}{\partial P} = \frac{V}{mR}$$

$$P = \frac{mRT}{V} \Rightarrow \frac{\partial P}{\partial V} = -\frac{mRT}{V^2}$$

$$V = \frac{mRT}{P} \Rightarrow \frac{\partial V}{\partial T} = \frac{mR}{P}$$

$$\begin{aligned} \frac{\partial T}{\partial P} \cdot \frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} &= \left(\frac{V}{mR} \right) \left(-\frac{mRT}{V^2} \right) \left(\frac{mR}{P} \right) \\ &= -\frac{mRT}{VP} = -\frac{mRT}{mRT} = -1 \end{aligned}$$

$$97. f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$(a) f_x(x, y) = \frac{(x^2 + y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2 + y^2)^2} = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{(x^2 + y^2)(x^3 - 3xy^2) - (x^3y - xy^3)(2y)}{(x^2 + y^2)^2} = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$$(b) f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0/[(\Delta x)^2] - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0/[(\Delta y)^2] - 0}{\Delta y} = 0$$

$$(c) f_{xy}(0, 0) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \bigg|_{(0, 0)} = \lim_{\Delta y \rightarrow 0} \frac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\Delta y(-(\Delta y)^4)}{((\Delta y)^2)^2(\Delta y)} = \lim_{\Delta y \rightarrow 0} (-1) = -1$$

$$f_{yx}(0, 0) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \bigg|_{(0, 0)} = \lim_{\Delta x \rightarrow 0} \frac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x((\Delta x)^4)}{((\Delta x)^2)^2(\Delta x)} = \lim_{\Delta x \rightarrow 0} 1 = 1$$

(d) f_{yx} or f_{xy} or both are not continuous at $(0, 0)$.

99. True

101. True

Section 12.4 Differentials

$$1. z = 3x^2y^3$$

$$dz = 6xy^3 dx + 9x^2y^2 dy$$

$$3. z = \frac{-1}{x^2 + y^2}$$

$$dz = \frac{2x}{(x^2 + y^2)^2} dx + \frac{2y}{(x^2 + y^2)^2} dy$$

$$= \frac{2}{(x^2 + y^2)^2} (x dx + y dy)$$

5. $z = x \cos y - y \cos x$

$$dz = (\cos y + y \sin x) dx + (-x \sin y - \cos x) dy = (\cos y + y \sin x) dx - (x \sin y + \cos x) dy$$

7. $z = e^x \sin y$

$$dz = (e^x \sin y) dx + (e^x \cos y) dy$$

9. $w = 2z^3 y \sin x$

$$dw = 2z^3 y \cos x dx + 2z^3 \sin x dy + 6z^2 y \sin x dz$$

11. (a) $f(1, 2) = 4$

$$f(1.05, 2.1) = 3.4875$$

$$\Delta z = f(1.05, 2.1) - f(1, 2) = -0.5125$$

(b) $dz = -2x dx - 2y dy$

$$= -2(0.05) - 4(0.1) = -0.5$$

13. (a) $f(1, 2) = \sin 2$

$$f(1.05, 2.1) = 1.05 \sin 2.1$$

$$\Delta z = f(1.05, 2.1) - f(1, 2) \approx -0.00293$$

(b) $dz = \sin y dx + x \cos y dy$

$$= (\sin 2)(0.05) + (\cos 2)(0.1) \approx 0.00385$$

15. (a) $f(1, 2) = -5$

$$f(1.05, 2.1) = -5.25$$

$$\Delta z = -0.25$$

(b) $dz = 3 dx - 4 dy$

$$= 3(0.05) - 4(0.1) = -0.25$$

17. Let $z = \sqrt{x^2 + y^2}$, $x = 5$, $y = 3$, $dx = 0.05$, $dy = 0.1$. Then: $dz = \frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy$

$$\sqrt{(5.05)^2 + (3.1)^2} - \sqrt{5^2 + 3^2} \approx \frac{5}{\sqrt{5^2 + 3^2}}(0.05) + \frac{3}{\sqrt{5^2 + 3^2}}(0.1) = \frac{0.55}{\sqrt{34}} \approx 0.094$$

19. Let $z = (1 - x^2)/y^2$, $x = 3$, $y = 6$, $dx = 0.05$, $dy = -0.05$. Then: $dz = -\frac{2x}{y^2} dx + \frac{-2(1 - x^2)}{y^3} dy$

$$\frac{1 - (3.05)^2}{(5.95)^2} - \frac{1 - 3^2}{6^2} \approx -\frac{2(3)}{6^2}(0.05) - \frac{2(1 - 3^2)}{6^3}(-0.05) \approx -0.012$$

21. See the definition on page 869.

23. The tangent plane to the surface $z = f(x, y)$ at the point P is a linear approximation of z .

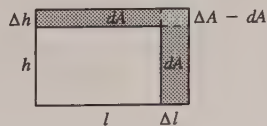
25. $A = lh$

$$dA = l dh + h dl$$

$$\Delta A = (l + dl)(h + dh) - lh$$

$$= h dl + l dh + dl dh$$

$$\Delta A - dA = dl dh$$



27. $V = \frac{\pi r^2 h}{3}$

$$r = 3$$

$$h = 6$$

$$dV = \frac{2\pi r h}{3} dr + \frac{\pi r^2}{3} dh = \frac{\pi r}{3}(2h dr + r dh)$$

$$\Delta V = \frac{\pi}{3}[(r + \Delta r)^2(h + \Delta h) - r^2 h]$$

$$= \frac{\pi}{3}[(3 + \Delta r)^2(6 + \Delta h) - 54]$$

Δr	Δh	dV	ΔV	$\Delta V - dV$
0.1	0.1	4.7124	4.8391	0.1267
0.1	-0.1	2.8274	2.8264	-0.0010
0.001	0.002	0.0565	0.0566	0.0001
-0.0001	0.0002	-0.0019	-0.0019	0.0000

29. (a) $dz = -1.83 dx - 1.09 dy$

$$\begin{aligned} \text{(b) } dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ &= -1.83(\pm 0.25) + (-1.09)(\pm 0.25) \\ &= \pm 0.73 \end{aligned}$$

Maximum propagated error: ± 0.73

$$\text{Relative error: } \frac{dz}{z} = \frac{\pm 0.73}{(-1.83)(7.2) - 1.09(8.5) + 28.7} = \frac{\pm 0.73}{6.259} \approx \pm 0.1166 = 11.67\%$$

31. $V = \pi r^2 h \Rightarrow dV = (2\pi r h) dr + (\pi r^2) dh$

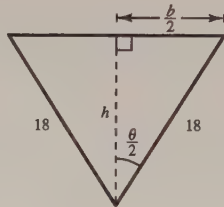
$$\begin{aligned} \frac{dV}{V} &= 2 \frac{dr}{r} + \frac{dh}{h} \\ &= 2(0.04) + (0.02) = 0.10 = 10\% \end{aligned}$$

33. $A = \frac{1}{2}ab \sin C$

$$\begin{aligned} dA &= \frac{1}{2}[(b \sin C) da + (a \sin C) db + (ab \cos C) dC] \\ &= \frac{1}{2}[4(\sin 45^\circ)(\pm \frac{1}{16}) + 3(\sin 45^\circ)(\pm \frac{1}{16}) + 12(\cos 45^\circ)(\pm 0.02)] \approx \pm 0.24 \text{ in.}^2 \end{aligned}$$

35. (a) $V = \frac{1}{2}bhl$

$$\begin{aligned} &= \left(18 \sin \frac{\theta}{2}\right) \left(18 \cos \frac{\theta}{2}\right) (16)(12) \\ &= 31,104 \sin \theta \text{ in.}^3 \\ &= 18 \sin \theta \text{ ft}^3 \end{aligned}$$



V is maximum when $\sin \theta = 1$ or $\theta = \pi/2$.

(b) $V = \frac{s^2}{2}(\sin \theta)l$

$$\begin{aligned} dV &= s(\sin \theta)l ds + \frac{s^2}{2}l(\cos \theta) d\theta + \frac{s^2}{2}(\sin \theta) dl \\ &= 18 \left(\sin \frac{\pi}{2} \right) (16)(12) \left(\frac{1}{2} \right) + \frac{18^2}{2} (16)(12) \left(\cos \frac{\pi}{2} \right) \left(\frac{\pi}{90} \right) + \frac{18^2}{2} \left(\sin \frac{\pi}{2} \right) \left(\frac{1}{2} \right) \\ &= 1809 \text{ in}^3 \approx 1.047 \text{ ft}^3 \end{aligned}$$

37. $P = \frac{E^2}{R}$

$$dP = \frac{2E}{R} dE - \frac{E^2}{R^2} dR$$

$$\frac{dP}{P} = \frac{\frac{2E}{R} dE - \frac{E^2}{R^2} dR}{\frac{E^2}{R}} = 2 \frac{dE}{E} - \frac{dR}{R} = 2(0.02) - (-0.03) = 0.07 = 7\%$$

$$39. L = 0.00021 \left(\ln \frac{2h}{r} - 0.75 \right)$$

$$dL = 0.00021 \left[\frac{dh}{h} - \frac{dr}{r} \right] = 0.00021 \left[\frac{(\pm 1/100)}{100} - \frac{(\pm 1/16)}{2} \right] \approx (\pm 6.6) \times 10^{-6}$$

$$L = 0.00021(\ln 100 - 0.75) \pm dL \approx 8.096 \times 10^{-4} \pm dL = 8.096 \times 10^{-4} \pm 6.6 \times 10^{-6} \text{ micro-henrys}$$

$$41. z = f(x, y) = x^2 - 2x + y$$

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \\ &= (x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + y + (\Delta y)) - (x^2 - 2x + y) \\ &= 2x(\Delta x) + (\Delta x)^2 - 2(\Delta x) + (\Delta y) \\ &= (2x - 2)\Delta x + \Delta y + \Delta x(\Delta x) + 0(\Delta y) \\ &= f_x(x, y)\Delta x + f_y(x, y)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \text{ where } \epsilon_1 = \Delta x \text{ and } \epsilon_2 = 0. \end{aligned}$$

$$\text{As } (\Delta x, \Delta y) \rightarrow (0, 0), \epsilon_1 \rightarrow 0 \text{ and } \epsilon_2 \rightarrow 0.$$

$$43. z = f(x, y) = x^2y$$

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \\ &= (x^2 + 2x(\Delta x) + (\Delta x)^2)(y + \Delta y) - x^2y \\ &= 2xy(\Delta x) + y(\Delta x)^2 + x^2\Delta y + 2x(\Delta x)(\Delta y) + (\Delta x)^2\Delta y \\ &= 2xy(\Delta x) + x^2\Delta y + (y\Delta x)\Delta x + [2x\Delta x + (\Delta x)^2]\Delta y \\ &= f_x(x, y)\Delta x + f_y(x, y)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \text{ where } \epsilon_1 = y(\Delta x) \text{ and } \epsilon_2 = 2x\Delta x + (\Delta x)^2. \end{aligned}$$

$$\text{As } (\Delta x, \Delta y) \rightarrow (0, 0), \epsilon_1 \rightarrow 0 \text{ and } \epsilon_2 \rightarrow 0.$$

$$45. f(x, y) = \begin{cases} \frac{3x^2y}{x^4 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$(a) f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{0}{(\Delta x)^4} - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\frac{0}{(\Delta y)^2} - 0}{\Delta y} = 0$$

Thus, the partial derivatives exist at $(0, 0)$.

$$(b) \text{ Along the line } y = x: \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{x \rightarrow 0} \frac{3x^3}{x^4 + x^2} = \lim_{x \rightarrow 0} \frac{3x}{x^2 + 1} = 0$$

$$\text{Along the curve } y = x^2: \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \frac{3x^4}{2x^4} = \frac{3}{2}$$

f is not continuous at $(0, 0)$. Therefore, f is not differentiable at $(0, 0)$. (See Theorem 12.5)

47. Essay. For example, we can use the equation $F = ma$:

$$dF = \frac{\partial F}{\partial m} dm + \frac{\partial F}{\partial a} da = a dm + m da.$$

Section 12.5 Chain Rules for Functions of Several Variables

1. $w = x^2 + y^2$

$x = e^t$

$y = e^{-t}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = 2xe^t + 2y(-e^{-t}) = 2(e^{2t} - e^{-2t})$$

3. $w = x \sec y$

$x = e^t$

$y = \pi - t$

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = (\sec y)(e^t) + (x \sec y \tan y)(-1) \\ &= e^t \sec(\pi - t)[1 - \tan(\pi - t)] \\ &= -e^t (\sec t + \sec t \tan t) \end{aligned}$$

5. $w = xy$, $x = 2 \sin t$, $y = \cos t$

$$\begin{aligned} \text{(a)} \quad \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = 2y \cos t + x(-\sin t) = 2y \cos t - x \sin t \\ &= 2(\cos^2 t - \sin^2 t) = 2 \cos 2t \end{aligned}$$

$$\text{(b)} \quad w = 2 \sin t \cos t = \sin 2t, \quad \frac{dw}{dt} = 2 \cos 2t$$

7. $w = x^2 + y^2 + z^2$

$x = e^t \cos t$

$y = e^t \sin t$

$z = e^t$

$$\text{(a)} \quad \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = 2x(-e^t \sin t + e^t \cos t) + 2y(e^t \cos t + e^t \sin t) + 2ze^t = 4e^{2t}$$

$$\text{(b)} \quad w = (e^t \cos t)^2 + (e^t \sin t)^2 + (e^t)^2 = 2e^{2t}, \quad \frac{dw}{dt} = 4e^{2t}$$

9. $w = xy + xz + yz$, $x = t - 1$, $y = t^2 - 1$, $z = t$

$$\begin{aligned} \text{(a)} \quad \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = (y + z) + (x + z)(2t) + (x + y) \\ &= (t^2 - 1 + t) + (t - 1 + t)(2t) + (t - 1 + t^2 - 1) = 3(2t^2 - 1) \end{aligned}$$

$$\text{(b)} \quad w = (t - 1)(t^2 - 1) + (t - 1)t + (t^2 - 1)t$$

$$\frac{dw}{dt} = 2t(t - 1) + (t^2 - 1) + 2t - 1 + 3t^2 - 1 = 3(2t^2 - 1)$$

11. Distance $= f(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(10 \cos 2t - 7 \cos t)^2 + (6 \sin 2t - 4 \sin t)^2}$

$$f'(t) = \frac{1}{2}[(10 \cos 2t - 7 \cos t)^2 + (6 \sin 2t - 4 \sin t)^2]^{-1/2}$$

$$[[2(10 \cos 2t - 7 \cos t)(-20 \sin 2t + 7 \sin t)] + [2(6 \sin 2t - 4 \sin t)(12 \cos 2t - 4 \cos t)]]$$

$$f'\left(\frac{\pi}{2}\right) = \frac{1}{2}[(-10)^2 + 4^2]^{-1/2}[[2(-10)(7)] + (2(-4)(-12))]$$

$$= \frac{1}{2}(116)^{-1/2}(-44) = \frac{-22}{2\sqrt{29}} = \frac{-11\sqrt{29}}{29} \approx -2.04$$

13. $w = \arctan(2xy)$, $x = \cos t$, $y = \sin t$, $t = 0$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

$$= \frac{2y}{1 + (4x^2y^2)}(-\sin t) + \frac{2x}{1 + (4x^2y^2)}(\cos t)$$

$$= \frac{2 \sin t}{1 + 4 \cos^2 t \sin^2 t}(-\sin t) + \frac{2 \cos t}{1 + 4 \cos^2 t \sin^2 t}(\cos t)$$

$$= \frac{2 \cos^2 t - 2 \sin^2 t}{1 + 4 \cos^2 t \sin^2 t}$$

$$\frac{d^2w}{dt^2} = \frac{(1 + 4 \cos^2 t \sin^2 t)(-8 \cos t \sin t) - (2 \cos^2 t - 2 \sin^2 t)(8 \cos^3 t \sin t - 8 \sin^3 t \cos t)}{(1 + 4 \cos^2 t \sin^2 t)^2}$$

$$= \frac{-8 \cos t \sin t(1 + 2 \sin^4 t + 2 \cos^4 t)}{(1 + 4 \cos^2 t \sin^2 t)^2}$$

At $t = 0$, $\frac{d^2w}{dt^2} = 0$.

15. $w = x^2 + y^2$

$$x = s + t$$

$$y = s - t$$

$$\frac{\partial w}{\partial s} = 2x + 2y = 2(x + y) = 4s$$

$$\frac{\partial w}{\partial t} = 2x + 2y(-1) = 2(x - y) = 4t$$

When $s = 2$ and $t = -1$,

$$\frac{\partial w}{\partial s} = 8 \text{ and } \frac{\partial w}{\partial t} = -4.$$

17. $w = x^2 - y^2$

$$x = s \cos t$$

$$y = s \sin t$$

$$\frac{\partial w}{\partial s} = 2x \cos t - 2y \sin t$$

$$= 2s \cos^2 t - 2s \sin^2 t = 2s \cos 2t$$

$$\frac{\partial w}{\partial t} = 2x(-s \sin t) - 2y(s \cos t) = -2s^2 \sin 2t$$

When $s = 3$ and $t = \frac{\pi}{4}$, $\frac{\partial w}{\partial s} = 0$ and $\frac{\partial w}{\partial t} = -18$.

19. $w = x^2 - 2xy + y^2$, $x = r + \theta$, $y = r - \theta$

(a) $\frac{\partial w}{\partial r} = (2x - 2y)(1) + (-2x + 2y)(1) = 0$

$$\frac{\partial w}{\partial \theta} = (2x - 2y)(1) + (-2x + 2y)(-1)$$

$$= 4x - 4y = 4(x - y)$$

$$= 4[(r + \theta) - (r - \theta)] = 8\theta$$

(b) $w = (r + \theta)^2 - 2(r + \theta)(r - \theta) + (r - \theta)^2$

$$= (r^2 + 2r\theta + \theta^2) - 2(r^2 - \theta^2) + (r^2 - 2r\theta + \theta^2)$$

$$= 4\theta^2$$

$$\frac{\partial w}{\partial r} = 0$$

$$\frac{\partial w}{\partial \theta} = 8\theta$$

21. $w = \arctan \frac{y}{x}$, $x = r \cos \theta$, $y = r \sin \theta$

(a) $\frac{\partial w}{\partial r} = \frac{-y}{x^2 + y^2} \cos \theta + \frac{x}{x^2 + y^2} \sin \theta = \frac{-r \sin \theta \cos \theta}{r^2} + \frac{r \cos \theta \sin \theta}{r^2} = 0$

$\frac{\partial w}{\partial \theta} = \frac{-y}{x^2 + y^2} (-r \sin \theta) + \frac{x}{x^2 + y^2} (r \cos \theta) = \frac{-(r \sin \theta)(-r \sin \theta)}{r^2} + \frac{(r \cos \theta)(r \cos \theta)}{r^2} = 1$

(b) $w = \arctan \frac{r \sin \theta}{r \cos \theta} = \arctan(\tan \theta) = \theta$

$\frac{\partial w}{\partial r} = 0$

$\frac{\partial w}{\partial \theta} = 1$

23. $w = xyz$, $x = s + t$, $y = s - t$, $z = st^2$

$\frac{\partial w}{\partial s} = yz(1) + xz(1) + xy(t^2)$

$= (s - t)st^2 + (s + t)st^2 + (s + t)(s - t)t^2$

$= 2s^2t^2 + s^2t^2 - t^4 = 3s^2t^2 - t^4 = t^2(3s^2 - t^2)$

$\frac{\partial w}{\partial t} = yz(1) + xz(-1) + xy(2st)$

$= (s - t)st^2 - (s + t)st^2 + (s + t)(s - t)(2st)$

$= -2st^3 + 2s^3t - 2st^3 = 2s^3t - 4st^3 = 2st(s^2 - 2t^2)$

25. $w = ze^{x/y}$, $x = s - t$, $y = s + t$, $z = st$

$\frac{\partial w}{\partial s} = \frac{z}{y} e^{x/y}(1) + \frac{-zx}{y^2} e^{x/y}(1) + e^{x/y}(t)$

$= e^{(s-t)/(s+t)} \left[\frac{st}{s+t} - \frac{(s-t)st}{(s+t)^2} + t \right]$

$= e^{(s-t)/(s+t)} \left[\frac{st(s+t) - s^2t + st^2 + t(s+t)^2}{(s+t)^2} \right]$

$= e^{(s-t)/(s+t)} \frac{t(s^2 + 4st + t^2)}{(s+t)^2}$

$\frac{\partial w}{\partial t} = \frac{z}{y} e^{x/y}(-1) + \frac{-zx}{y^2} e^{x/y}(1) + e^{x/y}(s)$

$= e^{(s-t)/(s+t)} \left[-\frac{st}{s+t} - \frac{st(s-t)}{(s+t)^2} + s \right]$

$= e^{(s-t)/(s+t)} \left[\frac{-st(s+t) - st(s-t) + s(s+t)^2}{(s+t)^2} \right]$

$= e^{(s-t)/(s+t)} \frac{s(s^2 + t^2)}{(s+t)^2}$

27. $x^2 - 3xy + y^2 - 2x + y - 5 = 0$

$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{2x - 3y - 2}{-3x + 2y + 1}$

$= \frac{3y - 2x + 2}{2y - 3x + 1}$

29. $\ln \sqrt{x^2 + y^2} + xy = 4$

$\frac{1}{2} \ln(x^2 + y^2) + xy - 4 = 0$

$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{\frac{x}{x^2 + y^2} + y}{\frac{y}{x^2 + y^2} + x} = -\frac{x + x^2y + y^3}{y + xy^2 + x^3}$

31. $F(x, y, z) = x^2 + y^2 + z^2 - 25$

$F_x = 2x$

$F_y = 2y$

$F_z = 2z$

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{x}{z}$

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y}{z}$

33. $F(x, y, z) = \tan(x + y) + \tan(y + z) - 1$

$F_x = \sec^2(x + y)$

$F_y = \sec^2(x + y) + \sec^2(y + z)$

$F_z = \sec^2(y + z)$

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\sec^2(x + y)}{\sec^2(y + z)}$

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\sec^2(x + y) + \sec^2(y + z)}{\sec^2(y + z)}$

$= -\left(\frac{\sec^2(x + y)}{\sec^2(y + z)} + 1 \right)$

$$35. F(x, y, z) = x^2 + 2yz + z^2 - 1$$

$$F_x = 2x$$

$$F_y = 2z$$

$$F_z = 2y + 2z$$

$$(i) \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{2y + 2z} = -\frac{x}{y + z}$$

$$(ii) \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2z}{2y + 2z} = -\frac{z}{y + z}$$

$$39. F(x, y, z, w) = xyz + xzw - yzw + w^2 - 5$$

$$F_x = yz + zw$$

$$F_y = xz - zw$$

$$F_z = xy + xw - yw$$

$$F_w = xz - yz + 2w$$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = -\frac{z(y + w)}{xz - yz + 2w}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = -\frac{z(x - w)}{xz - yz + 2w}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = -\frac{xy + xw - yw}{xz - yz + 2w}$$

$$43. f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$$

$$f(tx, ty) = \frac{(tx)(ty)}{\sqrt{(tx)^2 + (ty)^2}} = t \left(\frac{xy}{\sqrt{x^2 + y^2}} \right) = tf(x, y)$$

Degree: 1

$$\begin{aligned} xf_x(x, y) + yf_y(x, y) &= x \left(\frac{y^3}{(x^2 + y^2)^{3/2}} \right) + y \left(\frac{x^3}{(x^2 + y^2)^{3/2}} \right) \\ &= \frac{xy}{\sqrt{x^2 + y^2}} = 1f(x, y) \end{aligned}$$

$$45. f(x, y) = e^{x/y}$$

$$f(tx, ty) = e^{tx/ty} = e^{x/y} = f(x, y)$$

Degree: 0

$$xf_x(x, y) + yf_y(x, y) = x \left(\frac{1}{y^2} e^{x/y} \right) + y \left(-\frac{x}{y^2} e^{x/y} \right) = 0$$

49. $w = f(x, y)$ is the explicit form of a function of two variables, as in $z = x^2 + y^2$.
The implicit form is of the form $F(x, y, z) = 0$, as in $z - x^2 - y^2 = 0$.

$$51. A = \frac{1}{2}bh = \left(x \sin \frac{\theta}{2} \right) \left(x \cos \frac{\theta}{2} \right) = \frac{x^2}{2} \sin \theta$$

$$\begin{aligned} \frac{dA}{dt} &= \frac{\partial A}{\partial x} \frac{dx}{dt} + \frac{\partial A}{\partial \theta} \frac{d\theta}{dt} = x \sin \theta \frac{dx}{dt} + \frac{x^2}{2} \cos \theta \frac{d\theta}{dt} \\ &= 6 \left(\sin \frac{\pi}{4} \right) \left(\frac{1}{2} \right) + \frac{6^2}{2} \left(\cos \frac{\pi}{4} \right) \left(\frac{\pi}{90} \right) = \frac{3\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{10} \text{ m}^2/\text{hr} \\ &\approx 2.566 \text{ m}^2/\text{hr} \end{aligned}$$

$$37. F(x, y, z) = e^{xz} + xy$$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = -\frac{ze^{xz} + y}{xe^{xz}}$$

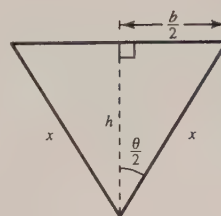
$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{-x}{xe^{xz}} = \frac{-1}{e^{xz}} = -e^{-xz}$$

$$41. F(x, y, z, w) = \cos xy + \sin yz + wz - 20$$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = \frac{y \sin xy}{z}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = \frac{x \sin xy - z \cos yz}{z}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = -\frac{y \cos yz + w}{z}$$



53. (a) $V = \frac{1}{3}\pi r^2 h$

$$\frac{dV}{dt} = \frac{1}{3}\pi \left(2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right) = \frac{1}{3}\pi [2(12)(36)(6) + (12)^2(-4)] = 1536\pi \text{ in.}^3/\text{min}$$

(b) $S = \pi r \sqrt{r^2 + h^2} + \pi r^2$ (Surface area includes base.)

$$\begin{aligned} \frac{dS}{dt} &= \pi \left[\left(\sqrt{r^2 + h^2} + \frac{r^2}{\sqrt{r^2 + h^2}} + 2r \right) \frac{dr}{dt} + \frac{rh}{\sqrt{r^2 + h^2}} \frac{dh}{dt} \right] \\ &= \pi \left[\left(\sqrt{12^2 + 36^2} + \frac{144}{\sqrt{12^2 + 36^2}} + 2(12) \right) (6) + \frac{36(12)}{\sqrt{12^2 + 36^2}} (-4) \right] \\ &= \pi \left[\left(12\sqrt{10} + \frac{12}{\sqrt{10}} \right) (6) + 144 + \frac{36}{\sqrt{10}} (-4) \right] \\ &= \frac{648\pi}{\sqrt{10}} + 144\pi \text{ in.}^2/\text{min} = \frac{36\pi}{5} (20 + 9\sqrt{10}) \text{ in.}^2/\text{min} \end{aligned}$$

55. $I = \frac{1}{2}m(r_1^2 + r_2^2)$

$$\frac{dI}{dt} = \frac{1}{2}m \left[2r_1 \frac{dr_1}{dt} + 2r_2 \frac{dr_2}{dt} \right] = m[(6)(2) + (8)(2)] = 28m \text{ cm}^2/\text{sec}$$

57. (a) $\tan \phi = \frac{2}{x}$

$$\tan(\theta + \phi) = \frac{4}{x}$$

$$\frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} = \frac{4}{x}$$

$$\frac{\tan \theta + (2/x)}{1 - (2/x)\tan \theta} = \frac{4}{x}$$

$$x \tan \theta + 2 = 4 - \frac{8}{x} \tan \theta$$

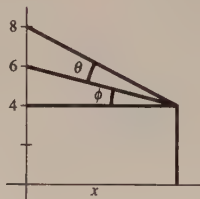
$$x^2 \tan \theta - 2x + 8 \tan \theta = 0$$

(b) $F(x, \theta) = (x^2 + 8)\tan \theta - 2x = 0$

$$\frac{d\theta}{dx} = -\frac{F_x}{F_\theta} = -\frac{2x \tan \theta - 2}{\sec^2 \theta (x^2 + 8)} = \frac{2 \cos^2 \theta - 2x \sin \theta \cos \theta}{x^2 + 8}$$

(c) $\frac{d\theta}{dx} = 0 \Rightarrow 2 \cos^2 \theta = 2x \sin \theta \cos \theta \Rightarrow \cos \theta = x \sin \theta \Rightarrow \tan \theta = \frac{1}{x}$

Thus, $x^2 \left(\frac{1}{x} \right) - 2x + 8 \left(\frac{1}{x} \right) = 0 \Rightarrow \frac{8}{x} = x \Rightarrow x = 2\sqrt{2} \text{ ft.}$



59. $w = f(x, y)$

$$x = u - v$$

$$y = v - u$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{dx}{du} + \frac{\partial w}{\partial y} \frac{dy}{du} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{dx}{dv} + \frac{\partial w}{\partial y} \frac{dy}{dv} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$$

61. $w = f(x, y)$, $x = r \cos \theta$, $y = r \sin \theta$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cos \theta + \frac{\partial w}{\partial y} \sin \theta$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} (-r \sin \theta) + \frac{\partial w}{\partial y} (r \cos \theta)$$

$$(a) \quad r \cos \theta \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} r \cos^2 \theta + \frac{\partial w}{\partial y} r \sin \theta \cos \theta$$

$$- \sin \theta \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} (r \sin^2 \theta) - \frac{\partial w}{\partial y} r \sin \theta \cos \theta$$

$$r \cos \theta \frac{\partial w}{\partial r} - \sin \theta \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} (r \cos^2 \theta + r \sin^2 \theta)$$

$$r \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} (r \cos \theta) - \frac{\partial w}{\partial \theta} \sin \theta$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \cos \theta - \frac{\partial w}{\partial \theta} \frac{\sin \theta}{r} \quad (\text{First Formula})$$

$$r \sin \theta \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} r \sin \theta \cos \theta + \frac{\partial w}{\partial y} r \sin^2 \theta$$

$$\cos \theta \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} (-r \sin \theta \cos \theta) + \frac{\partial w}{\partial y} (r \cos^2 \theta)$$

$$r \sin \theta \frac{\partial w}{\partial r} + \cos \theta \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial y} (r \sin^2 \theta + r \cos^2 \theta)$$

$$r \frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} r \sin \theta + \frac{\partial w}{\partial \theta} \cos \theta$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} \sin \theta + \frac{\partial w}{\partial \theta} \frac{\cos \theta}{r} \quad (\text{Second Formula})$$

$$(b) \quad \left(\frac{\partial w}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta} \right)^2 = \left(\frac{\partial w}{\partial x} \right)^2 \cos^2 \theta + 2 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \sin \theta \cos \theta + \left(\frac{\partial w}{\partial y} \right)^2 \sin^2 \theta + \left(\frac{\partial w}{\partial x} \right)^2 \sin^2 \theta -$$

$$2 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \sin \theta \cos \theta + \left(\frac{\partial w}{\partial y} \right)^2 \cos^2 \theta = \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2$$

63. Given $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, $x = r \cos \theta$ and $y = r \sin \theta$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta = \frac{\partial v}{\partial y} \cos \theta - \frac{\partial v}{\partial x} \sin \theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} (-r \sin \theta) + \frac{\partial v}{\partial y} (r \cos \theta) = r \left[\frac{\partial v}{\partial y} \cos \theta - \frac{\partial v}{\partial x} \sin \theta \right]$$

Therefore, $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$.

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x} \cos \theta + \frac{\partial v}{\partial y} \sin \theta = -\frac{\partial u}{\partial y} \cos \theta + \frac{\partial u}{\partial x} \sin \theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} (-r \sin \theta) + \frac{\partial u}{\partial y} (r \cos \theta) = -r \left[-\frac{\partial u}{\partial y} \cos \theta + \frac{\partial u}{\partial x} \sin \theta \right]$$

Therefore, $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$.

Section 12.6 Directional Derivatives and Gradients

1. $f(x, y) = 3x - 4xy + 5y$

$$\mathbf{v} = \frac{1}{2}(\mathbf{i} + \sqrt{3}\mathbf{j})$$

$$\nabla f(x, y) = (3 - 4y)\mathbf{i} + (-4x + 5)\mathbf{j}$$

$$\nabla f(1, 2) = -5\mathbf{i} + \mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = \frac{1}{2}(-5 + \sqrt{3})$$

5. $g(x, y) = \sqrt{x^2 + y^2}$

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$$

$$\nabla g = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$$

$$\nabla g(3, 4) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}g(3, 4) = \nabla g(3, 4) \cdot \mathbf{u} = -\frac{7}{25}$$

9. $f(x, y, z) = xy + yz + xz$

$$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$\nabla f(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$$

$$\nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{6}}{3}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} - \frac{\sqrt{6}}{6}\mathbf{k}$$

$$D_{\mathbf{u}}f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u} = \frac{2\sqrt{6}}{3}$$

13. $f(x, y) = x^2 + y^2$

$$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{2}{\sqrt{2}}x + \frac{2}{\sqrt{2}}y = \sqrt{2}(x + y)$$

3. $f(x, y) = xy$

$$\mathbf{v} = \mathbf{i} + \mathbf{j}$$

$$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$$

$$\nabla f(2, 3) = 3\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}f(2, 3) = \nabla f(2, 3) \cdot \mathbf{u} = \frac{5\sqrt{2}}{2}$$

7. $h(x, y) = e^x \sin y$

$$\mathbf{v} = -\mathbf{i}$$

$$\nabla h = e^x \sin y \mathbf{i} + e^x \cos y \mathbf{j}$$

$$\nabla h\left(1, \frac{\pi}{2}\right) = e\mathbf{i}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{i}$$

$$D_{\mathbf{u}}h\left(1, \frac{\pi}{2}\right) = \nabla h\left(1, \frac{\pi}{2}\right) \cdot \mathbf{u} = -e$$

11. $h(x, y, z) = x \arctan yz$

$$\mathbf{v} = \langle 1, 2, -1 \rangle$$

$$\nabla h(x, y, z) = \arctan yz \mathbf{i} + \frac{xz}{1 + (yz)^2} \mathbf{j} + \frac{xy}{1 + (yz)^2} \mathbf{k}$$

$$\nabla h(4, 1, 1) = \frac{\pi}{4}\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle$$

$$D_{\mathbf{u}}h(4, 1, 1) = \nabla h(4, 1, 1) \cdot \mathbf{u} = \frac{\pi + 8}{4\sqrt{6}} = \frac{(\pi + 8)\sqrt{6}}{24}$$

15. $f(x, y) = \sin(2x - y)$

$$\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\nabla f = 2 \cos(2x - y)\mathbf{i} - \cos(2x - y)\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \cos(2x - y) + \frac{\sqrt{3}}{2} \cos(2x - y)$$

$$= \left(\frac{2 + \sqrt{3}}{2} \right) \cos(2x - y)$$

17. $f(x, y) = x^2 + 4y^2$

$$\mathbf{v} = -2\mathbf{i} - 2\mathbf{j}$$

$$\nabla f = 2x\mathbf{i} + 8y\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$$

$$D_{\mathbf{u}}f = -\frac{2}{\sqrt{2}}x - \frac{8}{\sqrt{2}}y = -\sqrt{2}(x + 4y)$$

At $P = (3, 1)$, $D_{\mathbf{u}}f = -7\sqrt{2}$.

21. $f(x, y) = 3x - 5y^2 + 10$

$$\nabla f(x, y) = 3\mathbf{i} - 10y\mathbf{j}$$

$$\nabla f(2, 1) = 3\mathbf{i} - 10\mathbf{j}$$

25. $w = 3x^2y - 5yz + z^2$

$$\nabla w(x, y, z) = 6xy\mathbf{i} + (3x^2 - 5z)\mathbf{j} + (2z - 5y)\mathbf{k}$$

$$\nabla w(1, 1, -2) = 6\mathbf{i} + 13\mathbf{j} - 9\mathbf{k}$$

29. $\overrightarrow{PQ} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$

$$\nabla f(x, y) = -e^{-x}\cos y\mathbf{i} - e^{-x}\sin y\mathbf{j}$$

$$\nabla f(0, 0) = -\mathbf{i}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

33. $g(x, y) = \ln \sqrt{x^2 + y^2} = \frac{1}{2} \ln(x^2 + y^2)$

$$\nabla g(x, y) = \frac{1}{3} \left[\frac{2x}{x^2 + y^2} \mathbf{i} + \frac{2y}{x^2 + y^2} \mathbf{j} \right]$$

$$\nabla g(1, 2) = \frac{1}{3} \left(\frac{2}{5} \mathbf{i} + \frac{4}{5} \mathbf{j} \right) = \frac{2}{15} (\mathbf{i} + 2\mathbf{j})$$

$$\|\nabla g(1, 2)\| = \frac{2\sqrt{5}}{15}$$

37. $f(x, y, z) = xe^{yz}$

$$\nabla f(x, y, z) = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$$

$$\nabla f(2, 0, -4) = \mathbf{i} - 8\mathbf{j}$$

$$\|\nabla f(2, 0, -4)\| = \sqrt{65}$$

19. $h(x, y, z) = \ln(x + y + z)$

$$\mathbf{v} = 3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\nabla h = \frac{1}{x + y + z} (\mathbf{i} + \mathbf{j} + \mathbf{k})$$

At $(1, 0, 0)$, $\nabla h = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{19}} (3\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

$$D_{\mathbf{u}}h = \nabla h \cdot \mathbf{u} = \frac{7}{\sqrt{19}} = \frac{7\sqrt{19}}{19}$$

23. $z = \cos(x^2 + y^2)$

$$\nabla z(x, y) = -2x \sin(x^2 + y^2)\mathbf{i} - 2y \sin(x^2 + y^2)\mathbf{j}$$

$$\nabla z(3, -4) = -6 \sin 25\mathbf{i} + 8 \sin 25\mathbf{j} \approx 0.7941\mathbf{i} - 1.0588\mathbf{j}$$

27. $\overrightarrow{PQ} = 2\mathbf{i} + 4\mathbf{j}$, $\mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$

$$\nabla g(x, y) = 2x\mathbf{i} + 2y\mathbf{j}, \nabla g(1, 2) = 2\mathbf{i} + 4\mathbf{j}$$

$$D_{\mathbf{u}}g = \nabla g \cdot \mathbf{u} = \frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$$

31. $h(x, y) = x \tan y$

$$\nabla h(x, y) = \tan y\mathbf{i} + x \sec^2 y\mathbf{j}$$

$$\nabla h\left(2, \frac{\pi}{4}\right) = \mathbf{i} + 4\mathbf{j}$$

$$\left\| \nabla h\left(2, \frac{\pi}{4}\right) \right\| = \sqrt{17}$$

35. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

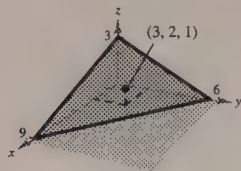
$$\nabla f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\nabla f(1, 4, 2) = \frac{1}{\sqrt{21}} (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$$

$$\|\nabla f(1, 4, 2)\| = 1$$

For Exercises 39–45, $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$ and $D_\theta f(x, y) = -\left(\frac{1}{3}\right) \cos \theta - \left(\frac{1}{2}\right) \sin \theta$.

39. $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$



41. (a) $D_{4\pi/3} f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)$
 $= \frac{2 + 3\sqrt{3}}{12}$

(b) $D_{-\pi/6} f(3, 2) = -\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)$
 $= \frac{3 - 2\sqrt{3}}{12}$

43. (a) $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$$

$$\mathbf{u} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{1}{5} - \frac{2}{5} = -\frac{1}{5}$$

(b) $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{10}$$

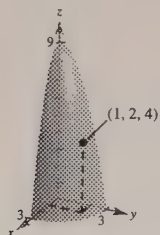
$$\mathbf{u} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{-11}{6\sqrt{10}} = -\frac{11\sqrt{10}}{60}$$

45. $\|\nabla f\| = \sqrt{\frac{1}{9} + \frac{1}{4}} = \frac{1}{6}\sqrt{13}$

For Exercises 47 and 49, $f(x, y) = 9 - x^2 - y^2$ and $D_\theta f(x, y) = -2x \cos \theta - 2y \sin \theta = -2(x \cos \theta + y \sin \theta)$.

47. $f(x, y) = 9 - x^2 - y^2$



49. $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$$\|\nabla f(1, 2)\| = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$$

51. (a) In the direction of the vector $-4\mathbf{i} + \mathbf{j}$.

(b) $\nabla f = \frac{1}{10}(2x - 3y)\mathbf{i} + \frac{1}{10}(-3x + 2y)\mathbf{j}$

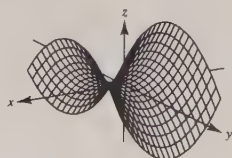
$$\nabla f(1, 2) = \frac{1}{10}(-4)\mathbf{i} + \frac{1}{10}(1)\mathbf{j} = -\frac{2}{5}\mathbf{i} + \frac{1}{10}\mathbf{j}$$

(Same direction as in part (a).)

(c) $-\nabla f = \frac{2}{5}\mathbf{i} - \frac{1}{10}\mathbf{j}$, the direction opposite that of the gradient.

53. $f(x, y) = x^2 - y^2$, $(4, -3, 7)$

(a)

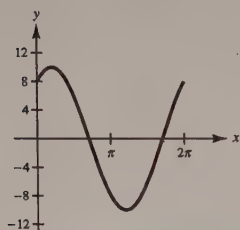


—CONTINUED—

53. —CONTINUED—

$$(b) D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = 2x \cos \theta - 2y \sin \theta$$

$$D_{\mathbf{u}} f(4, -3) = 8 \cos \theta + 6 \sin \theta$$



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$$(c) \text{Zeros: } \theta \approx 2.21, 5.36$$

 These are the angles θ for which $D_{\mathbf{u}} f(4, 3)$ equals zero.

$$(d) g(\theta) = D_{\mathbf{u}} f(4, -3) = 8 \cos \theta + 6 \sin \theta$$

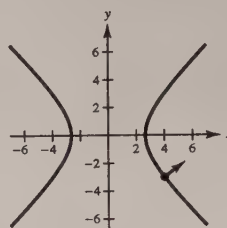
$$g'(\theta) = -8 \sin \theta + 6 \cos \theta$$

$$\text{Critical numbers: } \theta \approx 0.64, 3.79$$

 These are the angles for which $D_{\mathbf{u}} f(4, -3)$ is a maximum (0.64) and minimum (3.79).

$$(e) \|\nabla f(4, -3)\| = \|2(4)\mathbf{i} - 2(-3)\mathbf{j}\| = \sqrt{64 + 36} = 10, \text{ the maximum value of } D_{\mathbf{u}} f(4, -3), \text{ at } \theta \approx 0.64.$$

$$(f) f(x, y) = x^2 - y^2 = 7$$

 $\nabla f(4, -3) = 8\mathbf{i} + 6\mathbf{j}$ is perpendicular to the level curve at $(4, -3)$.


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$$55. f(x, y) = x^2 + y^2$$

$$c = 25, P = (3, 4)$$

$$\nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$$

$$x^2 + y^2 = 25$$

$$\nabla f(3, 4) = 6\mathbf{i} + 8\mathbf{j}$$

$$57. f(x, y) = \frac{x}{x^2 + y^2}$$

$$c = \frac{1}{2}, P = (1, 1)$$

$$\nabla f(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2} \mathbf{i} - \frac{2xy}{(x^2 + y^2)^2} \mathbf{j}$$

$$\frac{x}{x^2 + y^2} = \frac{1}{2}$$

$$x^2 + y^2 - 2x = 0$$

$$\nabla f(1, 1) = -\frac{1}{2} \mathbf{j}$$

$$59. 4x^2 - y = 6$$

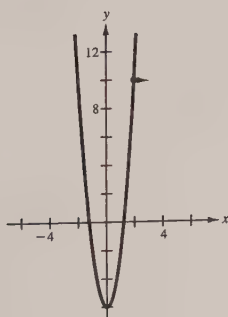
$$f(x, y) = 4x^2 - y$$

$$\nabla f(x, y) = 8x\mathbf{i} - \mathbf{j}$$

$$\nabla f(2, 10) = 16\mathbf{i} - \mathbf{j}$$

$$\frac{\nabla f(2, 10)}{\|\nabla f(2, 10)\|} = \frac{1}{\sqrt{257}} (16\mathbf{i} - \mathbf{j})$$

$$= \frac{\sqrt{257}}{257} (16\mathbf{i} - \mathbf{j})$$



$$61. 9x^2 + 4y^2 = 40$$

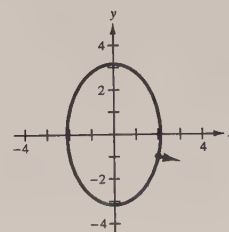
$$f(x, y) = 9x^2 + 4y^2$$

$$\nabla f(x, y) = 18x\mathbf{i} + 8y\mathbf{j}$$

$$\nabla f(2, -1) = 36\mathbf{i} - 8\mathbf{j}$$

$$\frac{\nabla f(2, -1)}{\|\nabla f(2, -1)\|} = \frac{1}{\sqrt{85}} (9\mathbf{i} - 2\mathbf{j})$$

$$= \frac{\sqrt{85}}{85} (9\mathbf{i} - 2\mathbf{j})$$



63. $T = \frac{x}{x^2 + y^2}$

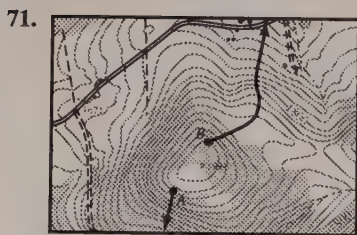
$$\nabla T = \frac{y^2 - x^2}{(x^2 + y^2)^2} \mathbf{i} - \frac{2xy}{(x^2 + y^2)^2} \mathbf{j}$$

$$\nabla T(3, 4) = \frac{7}{625} \mathbf{i} - \frac{24}{625} \mathbf{j} = \frac{1}{625} (7\mathbf{i} - 24\mathbf{j})$$

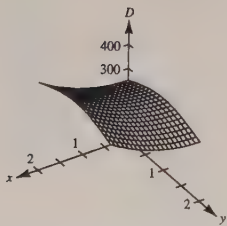
67. Let $f(x, y)$ be a function of two variables and $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ a unit vector.

(a) If $\theta = 0^\circ$, then $D_{\mathbf{u}} f = \frac{\partial f}{\partial x}$.

(b) If $\theta = 90^\circ$, then $D_{\mathbf{u}} f = \frac{\partial f}{\partial y}$.



75. (a)



(c) $D(1, 0.5) = 250 + 30(1) + 50 \sin \frac{\pi}{4} \approx 315.4$ ft

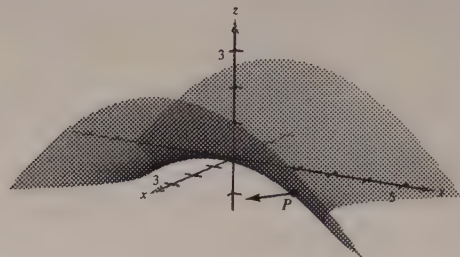
(e) $\frac{\partial D}{\partial y} = 25\pi \cos \frac{\pi y}{2}$ and $\frac{\partial D}{\partial y}(1, 0.5) = 25\pi \cos \frac{\pi}{4} \approx 55.5$

77. True

81. Let $f(x, y, z) = e^x \cos y + \frac{z^2}{2} + C$. Then $\nabla f(x, y, z) = e^x \cos y \mathbf{i} - e^x \sin y \mathbf{j} + z \mathbf{k}$.

65. See the definition, page 885.

69.



73. $T(x, y) = 400 - 2x^2 - y^2$,

$P = (10, 10)$

$$\frac{dx}{dt} = -4x$$

$$\frac{dy}{dt} = -2y$$

$$x(t) = C_1 e^{-4t}$$

$$y(t) = C_2 e^{-2t}$$

$$10 = x(0) = C_1$$

$$10 = y(0) = C_2$$

$$x(t) = 10e^{-4t}$$

$$y(t) = 10e^{-2t}$$

$$x = \frac{y^2}{10}$$

$$y^2(t) = 100e^{-4t}$$

$$y^2 = 10x$$

(b) The graph of $-D = -250 - 30x^2 - 50 \sin(\pi y/2)$ would model the ocean floor.

(d) $\frac{\partial D}{\partial x} = 60x$ and $\frac{\partial D}{\partial x}(1, 0.5) = 60$

(f) $\nabla D = 60x \mathbf{i} + 25\pi \cos\left(\frac{\pi y}{2}\right) \mathbf{j}$

$$\nabla D(1, 0.5) = 60\mathbf{i} + 55.5\mathbf{j}$$

79. True

Section 12.7 Tangent Planes and Normal Lines

1. $F(x, y, z) = 3x - 5y + 3z - 15 = 0$

$$3x - 5y + 3z = 15 \text{ Plane}$$

3. $F(x, y, z) = 4x^2 + 9y^2 - 4z^2 = 0$

$$4x^2 + 9y^2 = 4z^2 \text{ Elliptic cone}$$

5. $F(x, y, z) = x + y + z - 4$

$$\nabla F = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$= \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

7. $F(x, y, z) = \sqrt{x^2 + y^2} - z$

$$\nabla F(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$$

$$\nabla F(3, 4, 5) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{5}{5\sqrt{2}}\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}\right)$$

$$= \frac{1}{5\sqrt{2}}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$$

$$= \frac{\sqrt{2}}{10}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$$

9. $F(x, y, z) = x^2y^4 - z$

$$\nabla F(x, y, z) = 2xy^4\mathbf{i} + 4x^2y^3\mathbf{j} - \mathbf{k}$$

$$\nabla F(1, 2, 16) = 32\mathbf{i} + 32\mathbf{j} - \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{2049}}(32\mathbf{i} + 32\mathbf{j} - \mathbf{k})$$

$$= \frac{\sqrt{2049}}{2049}(32\mathbf{i} + 32\mathbf{j} - \mathbf{k})$$

11. $F(x, y, z) = \ln\left(\frac{x}{y-z}\right) = \ln x - \ln(y-z)$

$$\nabla F(x, y, z) = \frac{1}{x}\mathbf{i} - \frac{1}{y-z}\mathbf{j} + \frac{1}{y-z}\mathbf{k}$$

$$\nabla F(1, 4, 3) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$= \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

13. $F(x, y, z) = -x \sin y + z - 4$

$$\nabla F(x, y, z) = -\sin y\mathbf{i} - x \cos y\mathbf{j} + \mathbf{k}$$

$$\nabla F\left(6, \frac{\pi}{6}, 7\right) = -\frac{1}{2}\mathbf{i} - 3\sqrt{3}\mathbf{j} + \mathbf{k}$$

$$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{2}{\sqrt{113}}\left(-\frac{1}{2}\mathbf{i} - 3\sqrt{3}\mathbf{j} + \mathbf{k}\right)$$

$$= \frac{1}{\sqrt{113}}(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k})$$

$$= \frac{\sqrt{113}}{113}(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k})$$

15. $f(x, y) = 25 - x^2 - y^2, (3, 1, 15)$

$$F(x, y, z) = 25 - x^2 - y^2 - z$$

$$F_x(x, y, z) = -2x$$

$$F_y(x, y, z) = -2y$$

$$F_z(x, y, z) = -1$$

$$F_x(3, 1, 15) = -6$$

$$F_y(3, 1, 15) = -2$$

$$F_z(3, 1, 15) = -1$$

$$-6(x-3) - 2(y-1) - (z-15) = 0$$

$$0 = 6x + 2y + z - 35$$

$$6x + 2y + z = 35$$

17. $f(x, y) = \sqrt{x^2 + y^2}, (3, 4, 5)$

$$F(x, y, z) = \sqrt{x^2 + y^2} - z$$

$$F_x(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}} \quad F_y(x, y, z) = \frac{y}{\sqrt{x^2 + y^2}} \quad F_z(x, y, z) = -1$$

$$F_x(3, 4, 5) = \frac{3}{5} \quad F_y(3, 4, 5) = \frac{4}{5} \quad F_z(3, 4, 5) = -1$$

$$\frac{3}{5}(x - 3) + \frac{4}{5}(y - 4) - (z - 5) = 0$$

$$3(x - 3) + 4(y - 4) - 5(z - 5) = 0$$

$$3x + 4y - 5z = 0$$

19. $g(x, y) = x^2 - y^2, (5, 4, 9)$

$$G(x, y, z) = x^2 - y^2 - z$$

$$G_x(x, y, z) = 2x \quad G_y(x, y, z) = -2y \quad G_z(x, y, z) = -1$$

$$G_x(5, 4, 9) = 10 \quad G_y(5, 4, 9) = -8 \quad G_z(5, 4, 9) = -1$$

$$10(x - 5) - 8(y - 4) - (z - 9) = 0$$

$$10x - 8y - z = 9$$

21. $z = e^x(\sin y + 1), \left(0, \frac{\pi}{2}, 2\right)$

$$F(x, y, z) = e^x(\sin y + 1) - z$$

$$F_x(x, y, z) = e^x(\sin y + 1) \quad F_y(x, y, z) = e^x \cos y \quad F_z(x, y, z) = -1$$

$$F_x\left(0, \frac{\pi}{2}, 2\right) = 2 \quad F_y\left(0, \frac{\pi}{2}, 2\right) = 0 \quad F_z\left(0, \frac{\pi}{2}, 2\right) = -1$$

$$2x - z = -2$$

23. $h(x, y) = \ln \sqrt{x^2 + y^2}, (3, 4, \ln 5)$

$$H(x, y, z) = \ln \sqrt{x^2 + y^2} - z = \frac{1}{2} \ln(x^2 + y^2) - z$$

$$H_x(x, y, z) = \frac{x}{x^2 + y^2} \quad H_y(x, y, z) = \frac{y}{x^2 + y^2} \quad H_z(x, y, z) = -1$$

$$H_x(3, 4, \ln 5) = \frac{3}{25} \quad H_y(3, 4, \ln 5) = \frac{4}{25} \quad H_z(3, 4, \ln 5) = -1$$

$$\frac{3}{25}(x - 3) + \frac{4}{25}(y - 4) - (z - \ln 5) = 0$$

$$3(x - 3) + 4(y - 4) - 25(z - \ln 5) = 0$$

$$3x + 4y - 25z = 25(1 - \ln 5)$$

25. $x^2 + 4y^2 + z^2 = 36, (2, -2, 4)$

$$F(x, y, z) = x^2 + 4y^2 + z^2 - 36$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 8y \quad F_z(x, y, z) = 2z$$

$$F_x(2, -2, 4) = 4 \quad F_y(2, -2, 4) = -16 \quad F_z(2, -2, 4) = 8$$

$$4(x - 2) - 16(y + 2) + 8(z - 4) = 0$$

$$(x - 2) - 4(y + 2) + 2(z - 4) = 0$$

$$x - 4y + 2z = 18$$

27. $xy^2 + 3x - z^2 = 4$, $(2, 1, -2)$

$$F(x, y, z) = xy^2 + 3x - z^2 - 4$$

$$F_x(x, y, z) = y^2 + 3 \quad F_y(x, y, z) = 2xy \quad F_z(x, y, z) = -2z$$

$$F_x(2, 1, -2) = 4 \quad F_y(2, 1, -2) = 4 \quad F_z(2, 1, -2) = 4$$

$$4(x - 2) + 4(y - 1) + 4(z + 2) = 0$$

$$x + y + z = 1$$

29. $x^2 + y^2 + z = 9$, $(1, 2, 4)$

$$F(x, y, z) = x^2 + y^2 + z - 9$$

$$F_x(x, y, z) = 2x \quad F_y(x, y, z) = 2y \quad F_z(x, y, z) = 1$$

$$F_x(1, 2, 4) = 2 \quad F_y(1, 2, 4) = 4 \quad F_z(1, 2, 4) = 1$$

Direction numbers: 2, 4, 1

$$\text{Plane: } 2(x - 1) + 4(y - 2) + (z - 4) = 0, \quad 2x + 4y + z = 14$$

$$\text{Line: } \frac{x - 1}{2} = \frac{y - 2}{4} = \frac{z - 4}{1}$$

31. $xy - z = 0$, $(-2, -3, 6)$

$$F(x, y, z) = xy - z$$

$$F_x(x, y, z) = y \quad F_y(x, y, z) = x \quad F_z(x, y, z) = -1$$

$$F_x(-2, -3, 6) = -3 \quad F_y(-2, -3, 6) = -2 \quad F_z(-2, -3, 6) = -1$$

Direction numbers: 3, 2, 1

$$\text{Plane: } 3(x + 2) + 2(y + 3) + (z - 6) = 0, \quad 3x + 2y + z = -6$$

$$\text{Line: } \frac{x + 2}{3} = \frac{y + 3}{2} = \frac{z - 6}{1}$$

33. $z = \arctan \frac{y}{x}$, $\left(1, 1, \frac{\pi}{4}\right)$

$$F(x, y, z) = \arctan \frac{y}{x} - z$$

$$F_x(x, y, z) = \frac{-y}{x^2 + y^2} \quad F_y(x, y, z) = \frac{x}{x^2 + y^2} \quad F_z(x, y, z) = -1$$

$$F_x\left(1, 1, \frac{\pi}{4}\right) = -\frac{1}{2} \quad F_y\left(1, 1, \frac{\pi}{4}\right) = \frac{1}{2} \quad F_z\left(1, 1, \frac{\pi}{4}\right) = -1$$

Direction numbers: 1, -1, 2

$$\text{Plane: } (x - 1) - (y - 1) + 2\left(z - \frac{\pi}{4}\right) = 0, \quad x - y + 2z = \frac{\pi}{2}$$

$$\text{Line: } \frac{x - 1}{1} = \frac{y - 1}{-1} = \frac{z - (\pi/4)}{2}$$

$$35. z = f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}, \quad -2 \leq x \leq 2, \quad 0 \leq y \leq 3$$

$$(a) \text{ Let } F(x, y, z) = \frac{4xy}{(x^2 + 1)(y^2 + 1)} - z$$

$$\begin{aligned} \nabla F(x, y, z) &= \frac{4y}{y^2 + 1} \left(\frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} \right) \mathbf{i} + \frac{4x}{x^2 + 1} \left(\frac{y^2 + 1 - 2y^2}{(y^2 + 1)^2} \right) \mathbf{j} - \mathbf{k} \\ &= \frac{4y(1 - x^2)}{(y^2 + 1)(x^2 + 1)^2} \mathbf{i} + \frac{4x(1 - y^2)}{(x^2 + 1)(y^2 + 1)^2} \mathbf{j} - \mathbf{k} \end{aligned}$$

$$\nabla F(1, 1, 1) = -\mathbf{k}.$$

Direction numbers: 0, 0, -1.

Line: $x = 1, y = 1, z = 1 - t$

Tangent plane: $0(x - 1) + 0(y - 1) - 1(z - 1) = 0 \Rightarrow z = 1$

$$(b) \nabla F\left(-1, 2, -\frac{4}{5}\right) = 0\mathbf{i} + \frac{-4(-3)}{(2)(5)^2} \mathbf{j} - \mathbf{k} = \frac{6}{25} \mathbf{j} - \mathbf{k}$$

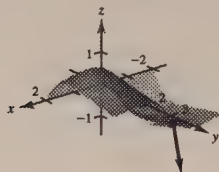
$$\text{Line: } x = -1, y = 2 + \frac{6}{25}t, z = -\frac{4}{5} - t$$

$$\text{Plane: } 0(x + 1) + \frac{6}{25}(y - 2) - 1\left(z + \frac{4}{5}\right) = 0$$

$$6y - 12 - 25z - 20 = 0$$

$$6y - 25z - 32 = 0$$

(c)



(d) At (1, 1, 1), the tangent plane is parallel to the xy -plane, implying that the surface is level there. At $(-1, 2, -\frac{4}{5})$, the function does not change in the x -direction.

$$37. F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

(Theorem 12.13)

$$39. F(x, y, z) = x^2 + y^2 - 5$$

$$G(x, y, z) = x - z$$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j}$$

$$\nabla G(x, y, z) = \mathbf{i} - \mathbf{k}$$

$$\nabla F(2, 1, 2) = 4\mathbf{i} + 2\mathbf{j}$$

$$\nabla G(2, 1, 2) = \mathbf{i} - \mathbf{k}$$

$$(a) \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} = -2(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

$$\text{Direction numbers: } 1, -2, 1. \quad \frac{x-2}{1} = \frac{y-1}{-2} = \frac{z-2}{1}$$

$$(b) \cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = \frac{4}{\sqrt{20}\sqrt{2}} = \frac{2}{\sqrt{10}} = \frac{\sqrt{10}}{5}; \text{ not orthogonal}$$

$$41. F(x, y, z) = x^2 + z^2 - 25$$

$$G(x, y, z) = y^2 + z^2 - 25$$

$$\nabla F = 2x\mathbf{i} + 2z\mathbf{k}$$

$$\nabla G = 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(3, 3, 4) = 6\mathbf{i} + 8\mathbf{k}$$

$$\nabla G(3, 3, 4) = 6\mathbf{j} + 8\mathbf{k}$$

—CONTINUED—

41. —CONTINUED—

$$(a) \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 8 \\ 0 & 6 & 8 \end{vmatrix} = -48\mathbf{i} - 48\mathbf{j} + 36\mathbf{k} = -12(4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$$

Direction numbers: 4, 4, -3. $\frac{x-3}{4} = \frac{y-3}{4} = \frac{z-4}{-3}$

$$(b) \cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = \frac{64}{(10)(10)} = \frac{16}{25}; \text{ not orthogonal}$$

$$43. F(x, y, z) = x^2 + y^2 + z^2 - 6 \quad G(x, y, z) = x - y - z$$

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \quad \nabla G(x, y, z) = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$\nabla F(2, 1, 1) = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \quad \nabla G(2, 1, 1) = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

$$(a) \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 2 \\ 1 & -1 & -1 \end{vmatrix} = 6\mathbf{j} - 6\mathbf{k} = 6(\mathbf{j} - \mathbf{k})$$

$$(b) \cos \theta = \frac{|\nabla F \cdot \nabla G|}{\|\nabla F\| \|\nabla G\|} = 0; \text{ orthogonal}$$

Direction numbers: 0, 1, -1. $x = 2, \frac{y-1}{1} = \frac{z-1}{-1}$

$$45. f(x, y) = 6 - x^2 - \frac{y^2}{4}, \quad g(x, y) = 2x + y$$

$$(a) F(x, y, z) = z + x^2 + \frac{y^2}{4} - 6 \quad G(x, y, z) = z - 2x - y$$

$$\nabla F(x, y, z) = 2x\mathbf{i} + \frac{1}{2}y\mathbf{j} + \mathbf{k} \quad \nabla G(x, y, z) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$$

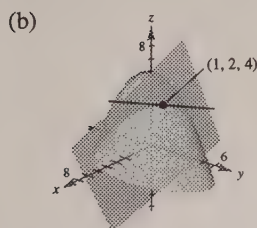
$$\nabla F(1, 2, 4) = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \quad \nabla G(1, 2, 4) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$$

The cross product of these gradients is parallel to the curve of intersection.

$$\nabla F(1, 2, 4) \times \nabla G(1, 2, 4) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j}$$

Using direction numbers 1, -2, 0, you get $x = 1 + t, y = 2 - 2t, z = 4$.

$$\cos \theta = \frac{\nabla F \cdot \nabla G}{\|\nabla F\| \|\nabla G\|} = \frac{-4 - 1 + 1}{\sqrt{6} \sqrt{6}} = \frac{-4}{6} \Rightarrow \theta \approx 48.2^\circ$$



$$47. F(x, y, z) = 3x^2 + 2y^2 - z - 15, \quad (2, 2, 5)$$

$$\nabla F(x, y, z) = 6x\mathbf{i} + 4y\mathbf{j} - \mathbf{k}$$

$$\nabla F(2, 2, 5) = 12\mathbf{i} + 8\mathbf{j} - \mathbf{k}$$

$$\cos \theta = \frac{|\nabla F(2, 2, 5) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 5)\|} = \frac{1}{\sqrt{209}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{209}}\right) \approx 86.03^\circ$$

$$49. F(x, y, z) = x^2 - y^2 + z, \quad (1, 2, 3)$$

$$\nabla F(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$$

$$\nabla F(1, 2, 3) = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

$$\cos \theta = \frac{|\nabla F(1, 2, 3) \cdot \mathbf{k}|}{\|\nabla F(1, 2, 3)\|} = \frac{1}{\sqrt{21}}$$

$$\theta = \arccos \frac{1}{\sqrt{21}} \approx 77.40^\circ$$

51. $F(x, y, z) = 3 - x^2 - y^2 + 6y - z$

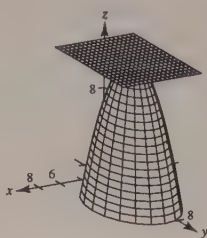
$$\nabla F(x, y, z) = -2x\mathbf{i} + (-2y + 6)\mathbf{j} - \mathbf{k}$$

$$-2x = 0, x = 0$$

$$-2y + 6 = 0, y = 3$$

$$z = 3 - 0^2 - 3^2 + 6(3) = 12$$

$$(0, 3, 12) \text{ (vertex of paraboloid)}$$



53. $T(x, y, z) = 400 - 2x^2 - y^2 - 4z^2, (4, 3, 10)$

$$\frac{dx}{dt} = -4kx$$

$$\frac{dy}{dt} = -2ky$$

$$\frac{dz}{dt} = -8kz$$

$$x(t) = C_1 e^{-4kt}$$

$$y(t) = C_2 e^{-2kt}$$

$$z(t) = C_3 e^{-8kt}$$

$$x(0) = C_1 = 4$$

$$y(0) = C_2 = 3$$

$$z(0) = C_3 = 10$$

$$x = 4e^{-4kt}$$

$$y = 3e^{-2kt}$$

$$z = 10e^{-8kt}$$

55. $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$$F_x(x, y, z) = \frac{2x}{a^2}$$

$$F_y(x, y, z) = \frac{2y}{b^2}$$

$$F_z(x, y, z) = \frac{2z}{c^2}$$

$$\text{Plane: } \frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + \frac{2z_0}{c^2}(z - z_0) = 0$$

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1$$

57. $F(x, y, z) = a^2 x^2 + b^2 y^2 - z^2$

$$F_x(x, y, z) = 2a^2 x$$

$$F_y(x, y, z) = 2b^2 y$$

$$F_z(x, y, z) = -2z$$

$$\text{Plane: } 2a^2 x_0(x - x_0) + 2b^2 y_0(y - y_0) - 2z_0(z - z_0) = 0$$

$$a^2 x_0 x + b^2 y_0 y - z_0 z = a^2 x_0^2 + b^2 y_0^2 - z_0^2 = 0$$

Hence, the plane passes through the origin.

59. $f(x, y) = e^{x-y}$

$$f_x(x, y) = e^{x-y}, \quad f_y(x, y) = -e^{x-y}$$

$$f_{xx}(x, y) = e^{x-y}, \quad f_{yy}(x, y) = e^{x-y}, \quad f_{xy}(x, y) = -e^{x-y}$$

$$(a) P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1 + x - y$$

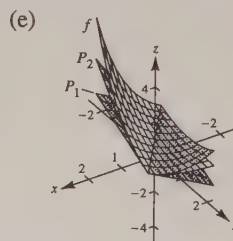
$$(b) P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2 \\ = 1 + x - y + \frac{1}{2}x^2 - xy + \frac{1}{2}y^2$$

$$(c) \text{ If } x = 0, P_2(0, y) = 1 - y + \frac{1}{2}y^2. \text{ This is the second-degree Taylor polynomial for } e^{-y}.$$

$$\text{If } y = 0, P_2(x, 0) = 1 + x + \frac{1}{2}x^2. \text{ This is the second-degree Taylor polynomial for } e^x.$$

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0.1	0.9048	0.9000	0.9050
0.2	0.1	1.1052	1.1000	1.1050
0.2	0.5	0.7408	0.7000	0.7450
1	0.5	1.6487	1.5000	1.6250



61. Given $w = F(x, y, z)$ where F is differentiable at

$$(x_0, y_0, z_0) \text{ and } \nabla F(x_0, y_0, z_0) \neq \mathbf{0},$$

the level surface of F at (x_0, y_0, z_0) is of the form $F(x, y, z) = C$ for some constant C . Let

$$G(x, y, z) = F(x, y, z) - C = 0.$$

Then $\nabla G(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0)$ where $\nabla G(x_0, y_0, z_0)$ is normal to $F(x_0, y_0, z_0) - C = 0$.

Therefore, $\nabla F(x_0, y_0, z_0)$ is normal to $F(x_0, y_0, z_0) = C$.

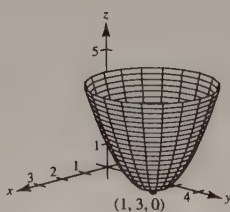
Section 12.8 Extrema of Functions of Two Variables

1. $g(x, y) = (x - 1)^2 + (y - 3)^2 \geq 0$

Relative minimum: $(1, 3, 0)$

$$g_x = 2(x - 1) = 0 \Rightarrow x = 1$$

$$g_y = 2(y - 3) = 0 \Rightarrow y = 3$$



3. $f(x, y) = \sqrt{x^2 + y^2 + 1} \geq 1$

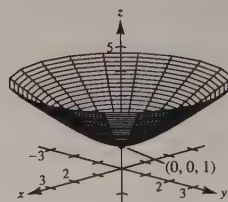
Relative minimum: $(0, 0, 1)$

Check: $f_x = \frac{x}{\sqrt{x^2 + y^2 + 1}} = 0 \Rightarrow x = 0$

$$f_y = \frac{y}{\sqrt{x^2 + y^2 + 1}} = 0 \Rightarrow y = 0$$

$$f_{xx} = \frac{y^2 + 1}{(x^2 + y^2 + 1)^{3/2}}, f_{yy} = \frac{x^2 + 1}{(x^2 + y^2 + 1)^{3/2}}, f_{xy} = \frac{-xy}{(x^2 + y^2 + 1)^{3/2}}$$

At the critical point $(0, 0)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(0, 0, 1)$ is a relative minimum.



5. $f(x, y) = x^2 + y^2 + 2x - 6y + 6 = (x + 1)^2 + (y - 3)^2 - 4 \geq -4$

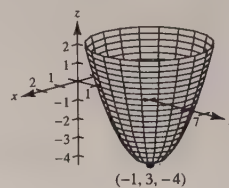
Relative minimum: $(-1, 3, -4)$

Check: $f_x = 2x + 2 = 0 \Rightarrow x = -1$

$$f_y = 2y - 6 = 0 \Rightarrow y = 3$$

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 0$$

At the critical point $(-1, 3)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-1, 3, -4)$ is a relative minimum.



7. $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$

$$\left. \begin{aligned} f_x &= 4x + 2y + 2 = 0 \\ f_y &= 2x + 2y = 0 \end{aligned} \right\} \text{Solving simultaneously yields } x = -1 \text{ and } y = 1.$$

$$f_{xx} = 4, f_{yy} = 2, f_{xy} = 2$$

At the critical point $(-1, 1)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-1, 1, -4)$ is a relative minimum.

9. $f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10$

$$\left. \begin{aligned} f_x &= -10x + 4y + 16 = 0 \\ f_y &= 4x - 2y = 0 \end{aligned} \right\} \text{Solving simultaneously yields } x = 8 \text{ and } y = 16.$$

$$f_{xx} = -10, f_{yy} = -2, f_{xy} = 4$$

At the critical point $(8, 16)$, $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(8, 16, 74)$ is a relative maximum.

11. $f(x, y) = 2x^2 + 3y^2 - 4x - 12y + 13$

$$f_x = 4x - 4 = 4(x - 1) = 0 \text{ when } x = 1.$$

$$f_y = 6y - 12 = 6(y - 2) = 0 \text{ when } y = 2.$$

$$f_{xx} = 4, f_{yy} = 6, f_{xy} = 0$$

At the critical point $(1, 2)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(1, 2, -1)$ is a relative minimum.

13. $f(x, y) = 2\sqrt{x^2 + y^2} + 3$

$$\left. \begin{aligned} f_x &= \frac{2x}{\sqrt{x^2 + y^2}} = 0 \\ f_y &= \frac{2y}{\sqrt{x^2 + y^2}} = 0 \end{aligned} \right\} x = 0, y = 0$$

Since $f(x, y) \geq 3$ for all (x, y) , $(0, 0, 3)$ is relative minimum.

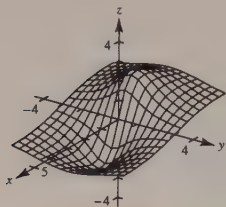
15. $g(x, y) = 4 - |x| - |y|$

$(0, 0)$ is the only critical point. Since $g(x, y) \leq 4$ for all (x, y) , $(0, 0, 4)$ is relative maximum.

17. $z = \frac{-4x}{x^2 + y^2 + 1}$

Relative minimum: $(1, 0, -2)$

Relative maximum: $(-1, 0, 2)$

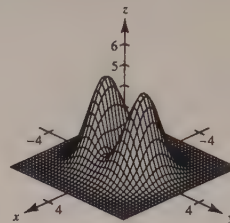


19. $z = (x^2 + 4y^2)e^{1-x^2-y^2}$

Relative minimum: $(0, 0, 0)$

Relative maxima: $(0, \pm 1, 4)$

Saddle points: $(\pm 1, 0, 1)$



21. $h(x, y) = x^2 - y^2 - 2x - 4y - 4$

$h_x = 2x - 2 = 2(x - 1) = 0$ when $x = 1$.

$h_y = -2y - 4 = -2(y + 2) = 0$ when $y = -2$.

$h_{xx} = 2$, $h_{yy} = -2$, $h_{xy} = 0$

At the critical point $(1, -2)$, $h_{xx} h_{yy} - (h_{xy})^2 < 0$. Therefore, $(1, -2, -1)$ is a saddle point.

23. $h(x, y) = x^2 - 3xy - y^2$

$\left. \begin{aligned} h_x &= 2x - 3y = 0 \\ h_y &= -3x - 2y = 0 \end{aligned} \right\}$ Solving simultaneously yields $x = 0$ and $y = 0$.

$h_{xx} = 2$, $h_{yy} = -2$, $h_{xy} = -3$

At the critical point $(0, 0)$, $h_{xx} h_{yy} - (h_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.

25. $f(x, y) = x^3 - 3xy + y^3$

$\left. \begin{aligned} f_x &= 3(x^2 - y) = 0 \\ f_y &= 3(-x + y^2) = 0 \end{aligned} \right\}$ Solving by substitution yields two critical points $(0, 0)$ and $(1, 1)$.

$f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3$

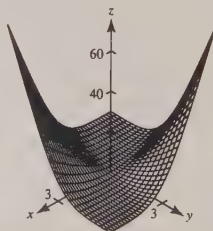
At the critical point $(0, 0)$, $f_{xx} f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point. At the critical point $(1, 1)$, $f_{xx} = 6 > 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(1, 1, -1)$ is a relative minimum.

27. $f(x, y) = e^{-x} \sin y$

$\left. \begin{aligned} f_x &= -e^{-x} \sin y = 0 \\ f_y &= e^{-x} \cos y = 0 \end{aligned} \right\}$ Since $e^{-x} > 0$ for all x and $\sin y$ and $\cos y$ are never both zero for a given value of y , there are no critical points.

29. $z = \frac{(x - y)^4}{x^2 + y^2} \geq 0$. $z = 0$ if $x = y \neq 0$.

Relative minimum at all points (x, x) , $x \neq 0$.



31. $f_{xx}f_{yy} - (f_{xy})^2 = (9)(4) - 6^2 = 0$

Insufficient information.

33. $f_{xx}f_{yy} - (f_{xy})^2 = (-9)(6) - 10^2 < 0$

 f has a saddle point at (x_0, y_0) .

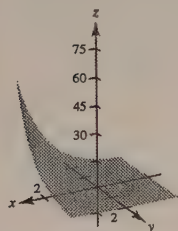
 35. (a) The function f defined on a region R containing (x_0, y_0) has a relative minimum at (x_0, y_0) if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) in R .

 (b) The function f defined on a region R containing (x_0, y_0) has a relative maximum at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in R .

(c) A saddle point is a critical point which is not a relative extremum.

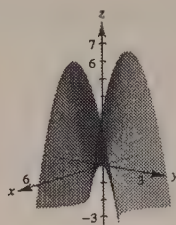
(d) See definition page 906.

37.



No extrema

39.



Saddle point

 41. In this case, the point A will be a saddle point. The function could be

$$f(x, y) = xy.$$

43. $d = f_{xx}f_{yy} - f_{xy}^2 = (2)(8) - f_{xy}^2 = 16 - f_{xy}^2 > 0$

$$\Rightarrow f_{xy}^2 < 16 \Rightarrow -4 < f_{xy} < 4$$

45. $f(x, y) = x^3 + y^3$

$$\left. \begin{aligned} f_x &= 3x^2 = 0 \\ f_y &= 3y^2 = 0 \end{aligned} \right\} \text{Solving yields } x = y = 0$$

$$f_{xx} = 6x, f_{yy} = 6y, f_{xy} = 0$$

 At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 = 0$ and the test fails. $(0, 0, 0)$ is a saddle point.

47. $f(x, y) = (x - 1)^2(y + 4)^2 \geq 0$

$$\left. \begin{aligned} f_x &= 2(x - 1)(y + 4)^2 = 0 \\ f_y &= 2(x - 1)^2(y + 4) = 0 \end{aligned} \right\} \text{Solving yields the critical points } (1, a) \text{ and } (b, -4).$$

$$f_{xx} = 2(y + 4)^2, f_{yy} = 2(x - 1)^2, f_{xy} = 4(x - 1)(y + 4)$$

 At both $(1, a)$ and $(b, -4)$, $f_{xx}f_{yy} - (f_{xy})^2 = 0$ and the test fails.

 Absolute minima: $(1, a, 0)$ and $(b, -4, 0)$

49. $f(x, y) = x^{2/3} + y^{2/3} \geq 0$

$$\left. \begin{aligned} f_x &= \frac{2}{3\sqrt[3]{x}} \\ f_y &= \frac{2}{3\sqrt[3]{y}} \end{aligned} \right\} f_x \text{ and } f_y \text{ are undefined at } x = 0, y = 0. \text{ The critical point is } (0, 0).$$

$$f_{xx} = -\frac{2}{9x\sqrt[3]{x}}, f_{yy} = -\frac{2}{9y\sqrt[3]{y}}, f_{xy} = 0$$

 At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2$ is undefined and the test fails.

 Absolute minimum: 0 at $(0, 0)$

51. $f(x, y, z) = x^2 + (y - 3)^2 + (z + 1)^2 \geq 0$

$$\left. \begin{aligned} f_x &= 2x = 0 \\ f_y &= 2(y - 3) = 0 \\ f_z &= 2(z + 1) = 0 \end{aligned} \right\} \text{Solving yields the critical point } (0, 3, -1).$$

 Absolute minimum: 0 at $(0, 3, -1)$

53. $f(x, y) = 12 - 3x - 2y$ has no critical points. On the line $y = x + 1$, $0 \leq x \leq 1$,

$$f(x, y) = f(x) = 12 - 3x - 2(x + 1) = -5x + 10$$

and the maximum is 10, the minimum is 5. On the line $y = -2x + 4$, $1 \leq x \leq 2$,

$$f(x, y) = f(x) = 12 - 3x - 2(-2x + 4) = x + 4$$

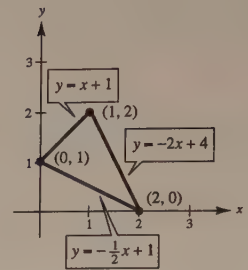
and the maximum is 6, the minimum is 5. On the line $y = -\frac{1}{2}x + 1$, $0 \leq x \leq 2$,

$$f(x, y) = f(x) = 12 - 3x - 2\left(-\frac{1}{2}x + 1\right) = -2x + 10$$

and the maximum is 10, the minimum is 6.

Absolute maximum: 10 at (0, 1)

Absolute minimum: 5 at (1, 2)



55. $f(x, y) = 3x^2 + 2y^2 - 4y$

$$\left. \begin{aligned} f_x = 6x = 0 &\Rightarrow x = 0 \\ f_y = 4y - 4 = 0 &\Rightarrow y = 1 \end{aligned} \right\} f(0, 1) = -2$$

On the line $y = 4$, $-2 \leq x \leq 2$,

$$f(x, y) = f(x) = 3x^2 + 32 - 16 = 3x^2 + 16$$

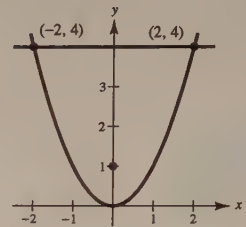
and the maximum is 28, the minimum is 16. On the curve $y = x^2$, $-2 \leq x \leq 2$,

$$f(x, y) = f(x) = 3x^2 + 2(x^2)^2 - 4x^2 = 2x^4 - x^2 = x^2(2x^2 - 1)$$

and the maximum is 28, the minimum is $-\frac{1}{8}$.

Absolute maximum: 28 at $(\pm 2, 4)$

Absolute minimum: -2 at $(0, 1)$



57. $f(x, y) = x^2 + xy$, $R = \{(x, y): |x| \leq 2, |y| \leq 1\}$

$$\left. \begin{aligned} f_x = 2x + y = 0 \\ f_y = x = 0 \end{aligned} \right\} x = y = 0$$

$$f(0, 0) = 0$$

Along $y = 1$, $-2 \leq x \leq 2$, $f = x^2 + x$, $f' = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$.

Thus, $f(-2, 1) = 2$, $f(-\frac{1}{2}, 1) = -\frac{1}{4}$ and $f(2, 1) = 6$.

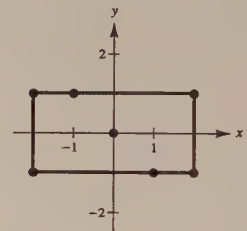
Along $y = -1$, $-2 \leq x \leq 2$, $f = x^2 - x$, $f' = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$.

Thus, $f(-2, -1) = 6$, $f(\frac{1}{2}, -1) = -\frac{1}{4}$, $f(2, -1) = 2$.

Along $x = 2$, $-1 \leq y \leq 1$, $f = 4 + 2y \Rightarrow f' = 2 \neq 0$.

Along $x = -2$, $-1 \leq y \leq 1$, $f = 4 - 2y \Rightarrow f' = -2 \neq 0$.

Thus, the maxima are $f(2, 1) = 6$ and $f(-2, -1) = 6$ and the minima are $f(-\frac{1}{2}, 1) = -\frac{1}{4}$ and $f(\frac{1}{2}, -1) = -\frac{1}{4}$.



59. $f(x, y) = x^2 + 2xy + y^2$, $R = \{(x, y): x^2 + y^2 \leq 8\}$

$$\left. \begin{aligned} f_x = 2x + 2y = 0 \\ f_y = 2x + 2y = 0 \end{aligned} \right\} y = -x$$

$$f(x, -x) = x^2 - 2x^2 + x^2 = 0$$

On the boundary $x^2 + y^2 = 8$, we have $y^2 = 8 - x^2$ and $y = \pm\sqrt{8 - x^2}$. Thus,

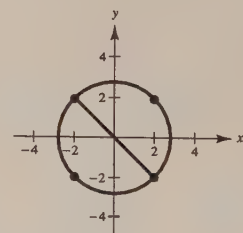
$$f = x^2 \pm 2x\sqrt{8 - x^2} + (8 - x^2) = 8 \pm 2x\sqrt{8 - x^2}$$

$$f' = \pm[(8 - x^2)^{-1/2}(-2x^2) + 2(8 - x^2)^{1/2}] = \pm \frac{16 - 4x^2}{\sqrt{8 - x^2}}$$

Then, $f' = 0$ implies $16 = 4x^2$ or $x = \pm 2$.

$$f(2, 2) = f(-2, -2) = 16 \quad \text{and} \quad f(2, -2) = f(-2, 2) = 0$$

Thus, the maxima are $f(2, 2) = 16$ and $f(-2, -2) = 16$, and the minima are $f(x, -x) = 0$, $|x| \leq 2$.



61. $f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}, R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$

$$f_x = \frac{4(1 - x^2)y}{(y^2 + 1)(x^2 + 1)^2} = 0 \Rightarrow x = 1 \text{ or } y = 0$$

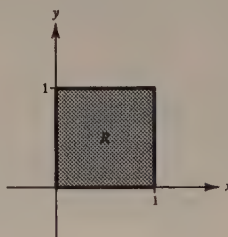
$$f_y = \frac{4(1 - y^2)x}{(x^2 + 1)(y^2 + 1)^2} \Rightarrow x = 0 \text{ or } y = 1$$

For $x = 0, y = 0$, also, and $f(0, 0) = 0$.

For $x = 1, y = 1, f(1, 1) = 1$.

The absolute maximum is $1 = f(1, 1)$.

The absolute minimum is $0 = f(0, 0)$. (In fact, $f(0, y) = f(x, 0) = 0$)



63. False

Let $f(x, y) = 1 - |x| - |y|$.

$(0, 0, 1)$ is a relative maximum, but $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

Section 12.9 Applications of Extrema of Functions of Two Variables

1. A point on the plane is given by $(x, y, 12 - 2x - 3y)$. The square of the distance from the origin to this point is

$$S = x^2 + y^2 + (12 - 2x - 3y)^2$$

$$S_x = 2x + 2(12 - 2x - 3y)(-2)$$

$$S_y = 2y + 2(12 - 2x - 3y)(-3)$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$5x + 6y = 24$$

$$3x + 5y = 18.$$

Solving simultaneously, we have $x = \frac{12}{7}, y = \frac{18}{7}$

$z = 12 - \frac{24}{7} - \frac{54}{7} = \frac{6}{7}$. Therefore, the distance from the origin to $(\frac{12}{7}, \frac{18}{7}, \frac{6}{7})$ is

$$\sqrt{\left(\frac{12}{7}\right)^2 + \left(\frac{18}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \frac{6\sqrt{14}}{7}.$$

5. Let x, y and z be the numbers. Since $x + y + z = 30, z = 30 - x - y$.

$$P = xyz = 30xy - x^2y - xy^2$$

$$P_x = 30y - 2xy - y^2 = y(30 - 2x - y) = 0 \quad 2x + y = 30$$

$$P_y = 30x - x^2 - 2xy = x(30 - x - 2y) = 0 \quad x + 2y = 30$$

Solving simultaneously yields $x = 10, y = 10$, and $z = 10$.

7. Let x, y , and z be the numbers and let $S = x^2 + y^2 + z^2$. Since $x + y + z = 30$, we have

$$S = x^2 + y^2 + (30 - x - y)^2$$

$$S_x = 2x + 2(30 - x - y)(-1) = 0 \quad 2x + y = 30$$

$$S_y = 2y + 2(30 - x - y)(-1) = 0 \quad x + 2y = 30.$$

Solving simultaneously yields $x = 10, y = 10$, and $z = 10$.

3. A point on the paraboloid is given by $(x, y, x^2 + y^2)$. The square of the distance from $(5, 5, 0)$ to a point on the paraboloid is given by

$$S = (x - 5)^2 + (y - 5)^2 + (x^2 + y^2)^2$$

$$S_x = 2(x - 5) + 4x(x^2 + y^2) = 0$$

$$S_y = 2(y - 5) + 4y(x^2 + y^2) = 0.$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$2x^3 + 2xy^2 + x - 5 = 0$$

$$2y^3 + 2x^2y + y - 5 = 0$$

Multiply the first equation by y and the second equation by x , and subtract to obtain $x = y$. Then, we have $x = 1, y = 1, z = 2$ and the distance is

$$\sqrt{(1 - 5)^2 + (1 - 5)^2 + (2 - 0)^2} = 6.$$

9. Let x , y , and z be the length, width, and height, respectively. Then the sum of the length and girth is given by $x + (2y + 2z) = 108$ or $x = 108 - 2y - 2z$. The volume is given by

$$V = xyz = 108zy - 2zy^2 - 2yz^2$$

$$V_y = 108z - 4yz - 2z^2 = z(108 - 4y - 2z) = 0$$

$$V_z = 108y - 2y^2 - 4yz = y(108 - 2y - 4z) = 0.$$

Solving the system $4y + 2z = 108$ and $2y + 4z = 108$, we obtain the solution $x = 36$ inches, $y = 18$ inches, and $z = 18$ inches.

11. Let $a + b + c = k$. Then

$$V = \frac{4\pi abc}{3} = \frac{4}{3}\pi ab(k - a - b)$$

$$= \frac{4}{3}\pi(kab - a^2b - ab^2)$$

$$V_a = \frac{4\pi}{3}(kb - 2ab - b^2) = 0 \left\{ \begin{array}{l} kb - 2ab - b^2 = 0 \\ ka - a^2 - 2ab = 0 \end{array} \right.$$

$$V_b = \frac{4\pi}{3}(ka - a^2 - 2ab) = 0 \left\{ \begin{array}{l} kb - 2ab - b^2 = 0 \\ ka - a^2 - 2ab = 0 \end{array} \right.$$

Solving this system simultaneously yields $a = b$ and substitution yields $b = k/3$. Therefore, the solution is $a = b = c = k/3$.

13. Let x , y , and z be the length, width, and height, respectively and let V_0 be the given volume.

Then $V_0 = xyz$ and $z = V_0/xy$. The surface area is

$$S = 2xy + 2yz + 2xz = 2\left(xy + \frac{V_0}{x} + \frac{V_0}{y}\right)$$

$$S_x = 2\left(y - \frac{V_0}{x^2}\right) = 0 \left\{ \begin{array}{l} x^2y - V_0 = 0 \\ xy^2 - V_0 = 0 \end{array} \right.$$

$$S_y = 2\left(x - \frac{V_0}{y^2}\right) = 0 \left\{ \begin{array}{l} x^2y - V_0 = 0 \\ xy^2 - V_0 = 0 \end{array} \right.$$

Solving simultaneously yields $x = \sqrt[3]{V_0}$, $y = \sqrt[3]{V_0}$, and $z = \sqrt[3]{V_0}$.

15. The distance from P to Q is $\sqrt{x^2 + 4}$. The distance from Q to R is $\sqrt{(y - x)^2 + 1}$. The distance from R to S is $10 - y$.

$$C = 3k\sqrt{x^2 + 4} + 2k\sqrt{(y - x)^2 + 1} + k(10 - y)$$

$$C_x = 3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(\frac{-(y - x)}{\sqrt{(y - x)^2 + 1}}\right) = 0$$

$$C_y = 2k\left(\frac{y - x}{\sqrt{(y - x)^2 + 1}}\right) - k = 0 \Rightarrow \frac{y - x}{\sqrt{(y - x)^2 + 1}} = \frac{1}{2}$$

$$3k\left(\frac{x}{\sqrt{x^2 + 4}}\right) + 2k\left(-\frac{1}{2}\right) = 0$$

$$\frac{x}{\sqrt{x^2 + 4}} = \frac{1}{3}$$

$$3x = \sqrt{x^2 + 4}$$

$$9x^2 = x^2 + 4$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{\sqrt{2}}{2}$$

$$2(y - x) = \sqrt{(y - x)^2 + 1}$$

$$4(y - x)^2 = (y - x)^2 + 1$$

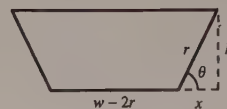
$$(y - x)^2 = \frac{1}{3}$$

$$y = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} = \frac{2\sqrt{3} + 3\sqrt{2}}{6}$$

Therefore, $x = \frac{\sqrt{2}}{2} \approx 0.707$ km and $y = \frac{2\sqrt{3} + 3\sqrt{2}}{6} \approx 1.284$ kms.

17. Let h be the height of the trough and r the length of the slanted sides. We observe that the area of a trapezoidal cross section is given by

$$A = h \left[\frac{(w - 2r) + [(w - 2r) + 2x]}{2} \right] = (w - 2r + x)h$$



where $x = r \cos \theta$ and $h = r \sin \theta$. Substituting these expressions for x and h , we have

$$A(r, \theta) = (w - 2r + r \cos \theta)(r \sin \theta) = wr \sin \theta - 2r^2 \sin \theta + r^2 \sin \theta \cos \theta$$

Now

$$A_r(r, \theta) = w \sin \theta - 4r \sin \theta + 2r \sin \theta \cos \theta = \sin \theta(w - 4r + 2r \cos \theta) = 0 \implies w = r(4 - 2 \cos \theta)$$

$$A_\theta(r, \theta) = wr \cos \theta - 2r^2 \cos \theta + r^2 \cos 2\theta = 0.$$

Substituting the expression for w from $A_r(r, \theta) = 0$ into the equation $A_\theta(r, \theta) = 0$, we have

$$r^2(4 - 2 \cos \theta) \cos \theta - 2r^2 \cos \theta + r^2(2 \cos^2 \theta - 1) = 0$$

$$r^2(2 \cos \theta - 1) = 0 \text{ or } \cos \theta = \frac{1}{2}.$$

Therefore, the first partial derivatives are zero when $\theta = \pi/3$ and $r = w/3$. (Ignore the solution $r = \theta = 0$.) Thus, the trapezoid of maximum area occurs when each edge of width $w/3$ is turned up 60° from the horizontal.

19. $R(x_1, x_2) = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$

$$R_{x_1} = -10x_1 - 2x_2 + 42 = 0, \quad 5x_1 + x_2 = 21$$

$$R_{x_2} = -16x_2 - 2x_1 + 102 = 0, \quad x_1 + 8x_2 = 51$$

Solving this system yields $x_1 = 3$ and $x_2 = 6$.

$$R_{x_1x_1} = -10$$

$$R_{x_1x_2} = -2$$

$$R_{x_2x_2} = -16$$

$$R_{x_1x_1} < 0 \text{ and } R_{x_1x_1}R_{x_2x_2} - (R_{x_1x_2})^2 > 0$$

Thus, revenue is maximized when $x_1 = 3$ and $x_2 = 6$.

21. $P(x_1, x_2) = 15(x_1 + x_2) - C_1 - C_2$

$$= 15x_1 + 15x_2 - (0.02x_1^2 + 4x_1 + 500) - (0.05x_2^2 + 4x_2 + 275)$$

$$= -0.02x_1^2 - 0.05x_2^2 + 11x_1 + 11x_2 - 775$$

$$P_{x_1} = -0.04x_1 + 11 = 0, \quad x_1 = 275$$

$$P_{x_2} = -0.10x_2 + 11 = 0, \quad x_2 = 110$$

$$P_{x_1x_1} = -0.04$$

$$P_{x_1x_2} = 0$$

$$P_{x_2x_2} = -0.10$$

$$P_{x_1x_1} < 0 \text{ and } P_{x_1x_1}P_{x_2x_2} - (P_{x_1x_2})^2 > 0$$

Therefore, profit is maximized when $x_1 = 275$ and $x_2 = 110$.

23. (a) $S(x, y) = d_1 + d_2 + d_3$

$$= \sqrt{(x-0)^2 + (y-0)^2} + \sqrt{(x+2)^2 + (y-2)^2} + \sqrt{(x-4)^2 + (y-2)^2}$$

$$= \sqrt{x^2 + y^2} + \sqrt{(x+2)^2 + (y-2)^2} + \sqrt{(x-4)^2 + (y-2)^2}$$

From the graph we see that the surface has a minimum.

(b) $S_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}} + \frac{x+2}{\sqrt{(x+2)^2 + (y-2)^2}} + \frac{x-4}{\sqrt{(x-4)^2 + (y-2)^2}}$

$$S_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}} + \frac{y-2}{\sqrt{(x+2)^2 + (y-2)^2}} + \frac{y-2}{\sqrt{(x-4)^2 + (y-2)^2}}$$

(c) $-\nabla S(1, 1) = -S_x(1, 1)\mathbf{i} - S_y(1, 1)\mathbf{j} = -\frac{1}{\sqrt{2}}\mathbf{i} - \left(\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{10}}\right)\mathbf{j}$

$$\tan \theta = \frac{(2/\sqrt{10}) - (1/\sqrt{2})}{-1/\sqrt{2}} = 1 - \frac{2}{\sqrt{5}} \Rightarrow \theta \approx 186.027^\circ$$

(d) $(x_2, y_2) = (x_1 - S_x(x_1, y_1)t, y_1 - S_y(x_1, y_1)t) = \left(1 - \frac{1}{\sqrt{2}}t, 1 + \left(\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right)t\right)$

$$S\left(1 - \frac{1}{\sqrt{2}}t, 1 + \left(\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right)t\right) = \sqrt{2 + \left(\frac{2\sqrt{10}}{5} - 2\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2}$$

$$+ \sqrt{10 - \left(\frac{2\sqrt{10}}{5} + 2\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2}$$

$$+ \sqrt{10 - \left(\frac{2\sqrt{10}}{5} - 4\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 1.344$. Thus, $(x_2, y_2) \approx (0.05, 0.90)$.

(e) $(x_3, y_3) = (x_2 - S_x(x_2, y_2)t, y_2 - S_y(x_2, y_2)t) \approx (0.05 + 0.03t, 0.90 - 0.26t)$

$$S(0.05 + 0.03t, 0.90 - 0.26t) = \sqrt{(0.05 + 0.03t)^2 + (0.90 - 0.26t)^2} + \sqrt{(2.05 + 0.03t)^2 + (-1.10 - 0.26t)^2}$$

$$+ \sqrt{(-3.95 + 0.03t)^2 + (-1.10 - 0.26t)^2}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 1.78$. Thus $(x_3, y_3) \approx (0.10, 0.44)$.

$$(x_4, y_4) = (x_3 - S_x(x_3, y_3)t, y_3 - S_y(x_3, y_3)t) \approx (0.10 - 0.09t, 0.44 - 0.01t)$$

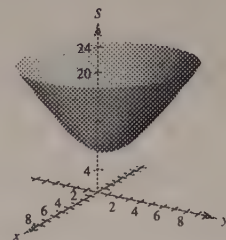
$$S(0.10 - 0.09t, 0.44 - 0.01t) = \sqrt{(0.10 - 0.09t)^2 + (0.44 - 0.01t)^2} + \sqrt{(2.10 - 0.09t)^2 + (-1.55 - 0.01t)^2}$$

$$+ \sqrt{(-3.90 - 0.09t)^2 + (-1.55 - 0.01t)^2}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 0.44$. Thus, $(x_4, y_4) \approx (0.06, 0.44)$.

Note: The minimum occurs at $(x, y) = (0.0555, 0.3992)$

(f) $-\nabla S(x, y)$ points in the direction that S decreases most rapidly. You would use $\nabla S(x, y)$ for maximization problems.



25. Write the equation to be maximized or minimized as a function of two variables. Set the partial derivatives equal to zero (or undefined) to obtain the critical points. Use the Second Partial Test to test for relative extrema using the critical points. Check the boundary points, too.

27. (a)

x	y	xy	x^2
-2	0	0	4
0	1	0	0
2	3	6	4
$\sum x_i = 0$	$\sum y_i = 4$	$\sum x_i y_i = 6$	$\sum x_i^2 = 8$

$$a = \frac{3(6) - 0(4)}{3(8) - 0^2} = \frac{3}{4}, \quad b = \frac{1}{3} \left[4 - \frac{3}{4}(0) \right] = \frac{4}{3},$$

$$y = \frac{3}{4}x + \frac{4}{3}$$

$$(b) S = \left(-\frac{3}{2} + \frac{4}{3} - 0 \right)^2 + \left(\frac{4}{3} - 1 \right)^2 + \left(\frac{3}{2} + \frac{4}{3} - 3 \right)^2 = \frac{1}{6}$$

31. (0, 0), (1, 1), (3, 4), (4, 2), (5, 5)

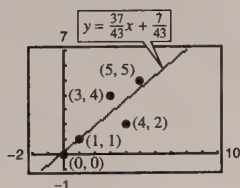
$$\sum x_i = 13, \quad \sum y_i = 12,$$

$$\sum x_i y_i = 46, \quad \sum x_i^2 = 51$$

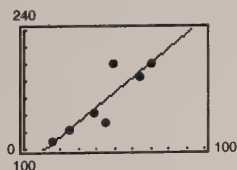
$$a = \frac{5(46) - 13(12)}{5(51) - (13)^2} = \frac{74}{86} = \frac{37}{43}$$

$$b = \frac{1}{5} \left[12 - \frac{37}{43}(13) \right] = \frac{7}{43}$$

$$y = \frac{37}{43}x + \frac{7}{43}$$


 35. (a) $y = 1.7236x + 79.7334$

(b)



(c) For each one-year increase in age, the pressure changes by 1.7236 (slope of line).

29. (a)

x	y	xy	x^2
0	4	0	0
1	3	3	1
1	1	1	1
2	0	0	4
$\sum x_i = 4$	$\sum y_i = 8$	$\sum x_i y_i = 4$	$\sum x_i^2 = 6$

$$a = \frac{4(4) - 4(8)}{4(6) - 4^2} = -2, \quad b = \frac{1}{4} [8 + 2(4)] = 4,$$

$$y = -2x + 4$$

$$(b) S = (4 - 4)^2 + (2 - 3)^2 + (2 - 1)^2 + (0 - 0)^2 = 2$$

33. (0, 6), (4, 3), (5, 0), (8, -4), (10, -5)

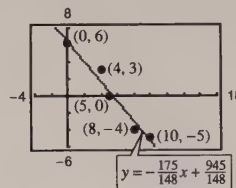
$$\sum x_i = 27, \quad \sum y_i = 0,$$

$$\sum x_i y_i = -70, \quad \sum x_i^2 = 205$$

$$a = \frac{5(-70) - (27)(0)}{5(205) - (27)^2} = \frac{-350}{296} = -\frac{175}{148}$$

$$b = \frac{1}{5} \left[0 - \left(-\frac{175}{148} \right)(27) \right] = \frac{945}{148}$$

$$y = -\frac{175}{148}x + \frac{945}{148}$$



37. (1.0, 32), (1.5, 41), (2.0, 48), (2.5, 53)

$$\sum x_i = 7, \quad \sum y_i = 174, \quad \sum x_i y_i = 322, \quad \sum x_i^2 = 13.5$$

$$a = 14, \quad b = 19, \quad y = 14x + 19$$

 When $x = 1.6$, $y = 41.4$ bushels per acre.

$$\begin{aligned}
 39. S(a, b, c) &= \sum_{i=1}^n (y_i - ax_i^2 - bx_i - c)^2 \\
 \frac{\partial S}{\partial a} &= \sum_{i=1}^n -2x_i^2(y_i - ax_i^2 - bx_i - c) = 0 \\
 \frac{\partial S}{\partial b} &= \sum_{i=1}^n -2x_i(y_i - ax_i^2 - bx_i - c) = 0 \\
 \frac{\partial S}{\partial c} &= -2 \sum_{i=1}^n (y_i - ax_i^2 - bx_i - c) = 0 \\
 a \sum_{i=1}^n x_i^4 + b \sum_{i=1}^n x_i^3 + c \sum_{i=1}^n x_i^2 &= \sum_{i=1}^n x_i^2 y_i \\
 a \sum_{i=1}^n x_i^3 + b \sum_{i=1}^n x_i^2 + c \sum_{i=1}^n x_i &= \sum_{i=1}^n x_i y_i \\
 a \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i + cn &= \sum_{i=1}^n y_i
 \end{aligned}$$

$$43. (0, 0), (2, 2), (3, 6), (4, 12)$$

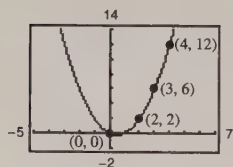
$$\begin{aligned}
 \sum x_i &= 9 \\
 \sum y_i &= 20 \\
 \sum x_i^2 &= 29 \\
 \sum x_i^3 &= 99 \\
 \sum x_i^4 &= 353 \\
 \sum x_i y_i &= 70 \\
 \sum x_i^2 y_i &= 254
 \end{aligned}$$

$$353a + 99b + 29c = 254$$

$$99a + 29b + 9c = 70$$

$$29a + 9b + 4c = 20$$

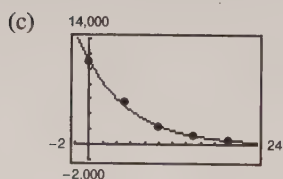
$$a = 1, b = -1, c = 0, y = x^2 - x$$



$$47. (a) \ln P = -0.1499h + 9.3018$$

$$(b) \ln P = -0.1499h + 9.3018$$

$$P = e^{-0.1499h + 9.3018} = 10,957.7e^{-0.1499h}$$



(d) Same answers.

$$41. (-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)$$

$$\sum x_i = 0$$

$$\sum y_i = 8$$

$$\sum x_i^2 = 10$$

$$\sum x_i^3 = 0$$

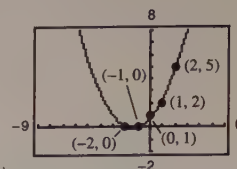
$$\sum x_i^4 = 34$$

$$\sum x_i y_i = 12$$

$$\sum x_i^2 y_i = 22$$

$$34a + 10c = 22, 10b = 12, 10a + 5c = 8$$

$$a = \frac{3}{7}, b = \frac{6}{5}, c = \frac{26}{35}, y = \frac{3}{7}x^2 + \frac{6}{5}x + \frac{26}{35}$$



$$45. (0, 0), (2, 15), (4, 30), (6, 50), (8, 65), (10, 70)$$

$$\sum x_i = 30,$$

$$\sum y_i = 230,$$

$$\sum x_i^2 = 220,$$

$$\sum x_i^3 = 1,800,$$

$$\sum x_i^4 = 15,664,$$

$$\sum x_i y_i = 1,670,$$

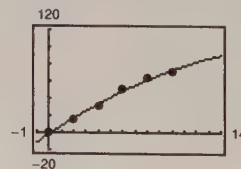
$$\sum x_i^2 y_i = 13,500$$

$$15,664a + 1,800b + 220c = 13,500$$

$$1,800a + 220b + 30c = 1,670$$

$$220a + 30b + 6c = 230$$

$$y = -\frac{25}{112}x^2 + \frac{541}{56}x - \frac{25}{14} \approx -0.22x^2 + 9.66x - 1.79$$



Section 12.10 Lagrange Multipliers

1. Maximize
- $f(x, y) = xy$
- .

Constraint: $x + y = 10$

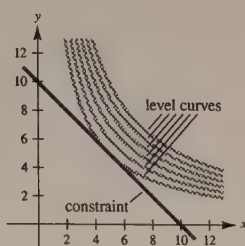
$\nabla f = \lambda \nabla g$

$y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$

$$\begin{cases} y = \lambda \\ x = \lambda \end{cases} \Rightarrow x = y$$

$x + y = 10 \Rightarrow x = y = 5$

$f(5, 5) = 25$



3. Minimize
- $f(x, y) = x^2 + y^2$
- .

Constraint: $x + y = 4$

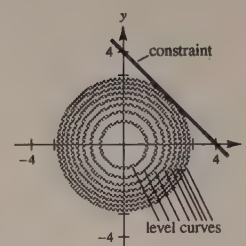
$\nabla f = \lambda \nabla g$

$2x\mathbf{i} + 2y\mathbf{j} = \lambda\mathbf{i} + \lambda\mathbf{j}$

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \end{cases} \Rightarrow x = y$$

$x + y = 4 \Rightarrow x = y = 2$

$f(2, 2) = 8$



5. Minimize
- $f(x, y) = x^2 - y^2$
- .

Constraint: $x - 2y = -6$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} - 2y\mathbf{j} = \lambda\mathbf{i} - 2\lambda\mathbf{j}$

$2x = \lambda \Rightarrow x = \frac{\lambda}{2}$

$-2y = -2\lambda \Rightarrow y = \lambda$

$x - 2y = -6 \Rightarrow -\frac{3}{2}\lambda = -6$

$\lambda = 4, x = 2, y = 4$

$f(2, 4) = -12$

7. Maximize
- $f(x, y) = 2x + 2xy + y$
- .

Constraint: $2x + y = 100$

$\nabla f = \lambda \nabla g$

$(2 + 2y)\mathbf{i} + (2x + 1)\mathbf{j} = 2\lambda\mathbf{i} + \lambda\mathbf{j}$

$$\begin{cases} 2 + 2y = 2\lambda \Rightarrow y = \lambda - 1 \\ 2x + 1 = \lambda \Rightarrow x = \frac{\lambda - 1}{2} \end{cases} \Rightarrow y = 2x$$

$2x + y = 100 \Rightarrow 4x = 100$

$x = 25, y = 50$

$f(25, 50) = 2600$

- 9.
- Note:**
- $f(x, y) = \sqrt{6 - x^2 - y^2}$
- is maximum when
- $g(x, y)$
- is maximum.

Maximize $g(x, y) = 6 - x^2 - y^2$.

Constraint: $x + y = 2$

$$\begin{cases} -2x = \lambda \\ -2y = \lambda \end{cases} \Rightarrow x = y$$

$x + y = 2 \Rightarrow x = y = 1$

$f(1, 1) = \sqrt{g(1, 1)} = 2$

11. Maximize
- $f(x, y) = e^{xy}$
- .

Constraint: $x^2 + y^2 = 8$

$$\begin{cases} ye^{xy} = 2x\lambda \\ xe^{xy} = 2y\lambda \end{cases} \Rightarrow x = y$$

$x^2 + y^2 = 8 \Rightarrow 2x^2 = 8$

$x = y = 2$

$f(2, 2) = e^4$

13. Maximize or minimize
- $f(x, y) = x^2 + 3xy + y^2$
- .

Constraint: $x^2 + y^2 \leq 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\begin{cases} 2x + 3y = 2x\lambda \\ 3x + 2y = 2y\lambda \end{cases} \Rightarrow x^2 = y^2$$

$x^2 + y^2 = 1 \Rightarrow x = \pm \frac{\sqrt{2}}{2}, y = \pm \frac{\sqrt{2}}{2}$

Maxima: $f\left(\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right) = \frac{5}{2}$

Minima: $f\left(\pm \frac{\sqrt{2}}{2}, \mp \frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$

Case 2: Inside the circle

$$\begin{cases} f_x = 2x + 3y = 0 \\ f_y = 3x + 2y = 0 \end{cases} \Rightarrow x = y = 0$$

$f_{xx} = 2, f_{yy} = 2, f_{xy} = 3, f_{xx}f_{yy} - (f_{xy})^2 \leq 0$

Saddle point: $f(0, 0) = 0$

By combining these two cases, we have a maximum of $\frac{5}{2}$ at

$\left(\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2}\right)$

and a minimum of $-\frac{1}{2}$ at

$\left(\pm \frac{\sqrt{2}}{2}, \mp \frac{\sqrt{2}}{2}\right).$

15. Minimize
- $f(x, y, z) = x^2 + y^2 + z^2$
- .

Constraint: $x + y + z = 6$

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \end{cases} \Rightarrow x = y = z$$

$x + y + z = 6 \Rightarrow x = y = z = 2$

$f(2, 2, 2) = 12$

19. Maximize
- $f(x, y, z) = xyz$
- .

Constraints: $x + y + z = 32$

$x - y + z = 0$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$$\begin{cases} yz = \lambda + \mu \\ xz = \lambda - \mu \\ xy = \lambda + \mu \end{cases} \Rightarrow yz = xy \Rightarrow x = z$$

$$\begin{cases} x + y + z = 32 \\ x - y + z = 0 \end{cases} \Rightarrow 2x + 2z = 32 \Rightarrow x = z = 8$$

$y = 16$

$f(8, 16, 8) = 1024$

17. Minimize
- $f(x, y, z) = x^2 + y^2 + z^2$
- .

Constraint: $x + y + z = 1$

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \end{cases} \Rightarrow x = y = z$$

$x + y + z = 1 \Rightarrow x = y = z = \frac{1}{3}$

$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$

21. Maximize
- $f(x, y, z) = xy + yz$
- .

Constraints: $x + 2y = 6$

$x - 3z = 0$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$y\mathbf{i} + (x + z)\mathbf{j} + y\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{j}) + \mu(\mathbf{i} - 3\mathbf{k})$

$$\begin{cases} y = \lambda + \mu \\ x + z = 2\lambda \\ y = -3\mu \end{cases} \Rightarrow y = \frac{3}{4}\lambda \Rightarrow x + z = \frac{8}{3}y$$

$x + 2y = 6 \Rightarrow y = 3 - \frac{x}{2}$

$x - 3z = 0 \Rightarrow z = \frac{x}{3}$

$x + \frac{x}{3} = \frac{8}{3}\left(3 - \frac{x}{2}\right)$

$x = 3, y = \frac{3}{2}, z = 1$

$f\left(3, \frac{3}{2}, 1\right) = 6$

23. Minimize the square of the distance
- $f(x, y) = x^2 + y^2$
- subject to the constraint
- $2x + 3y = -1$
- .

$$\begin{cases} 2x = 2\lambda \\ 2y = 3\lambda \end{cases} \Rightarrow y = \frac{3x}{2}$$

$2x + 3y = -1 \Rightarrow x = -\frac{2}{13}, y = -\frac{3}{13}$

The point on the line is $\left(-\frac{2}{13}, -\frac{3}{13}\right)$ and the desired distance is

$$d = \sqrt{\left(-\frac{2}{13}\right)^2 + \left(-\frac{3}{13}\right)^2} = \frac{\sqrt{13}}{13}$$

25. Minimize the square of the distance

$f(x, y, z) = (x - 2)^2 + (y - 1)^2 + (z - 1)^2$

subject to the constraint $x + y + z = 1$.

$$\begin{cases} 2(x - 2) = \lambda \\ 2(y - 1) = \lambda \\ 2(z - 1) = \lambda \end{cases} \Rightarrow y = z \text{ and } y = x - 1$$

$x + y + z = 1 \Rightarrow x + 2(x - 1) = 1$

$x = 1, y = z = 0$

The point on the plane is $(1, 0, 0)$ and the desired distance is

$$d = \sqrt{(1 - 2)^2 + (0 - 1)^2 + (0 - 1)^2} = \sqrt{3}$$

27. Maximize $f(x, y, z) = z$ subject to the constraints $x^2 + y^2 + z^2 = 36$ and $2x + y - z = 2$.

$$\begin{cases} 0 = 2x\lambda + 2\mu \\ 0 = 2y\lambda + \mu \\ 1 = 2z\lambda - \mu \end{cases} \Rightarrow x = 2y$$

$$x^2 + y^2 + z^2 = 36$$

$$2x + y - z = 2 \Rightarrow z = 2x + y - 2 = 5y - 2$$

$$(2y)^2 + y^2 + (5y - 2)^2 = 36$$

$$30y^2 - 20y - 32 = 0$$

$$15y^2 - 10y - 16 = 0$$

$$y = \frac{5 \pm \sqrt{265}}{15}$$

Choosing the positive value for y we have the point

$$\left(\frac{10 + 2\sqrt{265}}{15}, \frac{5 + \sqrt{265}}{15}, \frac{-1 + \sqrt{265}}{3} \right).$$

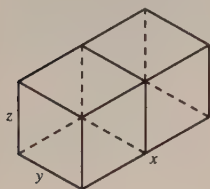
31. Maximize $V(x, y, z) = xyz$ subject to the constraint $x + 2y + 2z = 108$.

$$\begin{cases} yz = \lambda \\ xz = 2\lambda \\ xy = 2\lambda \end{cases} \Rightarrow y = z \text{ and } x = 2y$$

$$x + 2y + 2z = 108 \Rightarrow 6y = 108, y = 18$$

$$x = 36, y = z = 18$$

Volume is maximum when the dimensions are $36 \times 18 \times 18$ inches



29. Optimization problems that have restrictions or constraints on the values that can be used to produce the optimal solution are called constrained optimization problems.

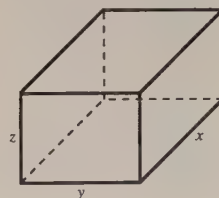
33. Minimize $C(x, y, z) = 5xy + 3(2xz + 2yz + xy)$ subject to the constraint $xyz = 480$.

$$\begin{cases} 8y + 6z = yz\lambda \\ 8x + 6z = xz\lambda \\ 6x + 6y = xy\lambda \end{cases} \Rightarrow x = y, 4y = 3z$$

$$xyz = 480 \Rightarrow \frac{4}{3}y^3 = 480$$

$$x = y = \sqrt[3]{360}, z = \frac{4}{3}\sqrt[3]{360}$$

Dimensions: $\sqrt[3]{360} \times \sqrt[3]{360} \times \frac{4}{3}\sqrt[3]{360}$ feet



35. Maximize $V(x, y, z) = (2x)(2y)(2z) = 8xyz$ subject to the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

$$\begin{cases} 8yz = \frac{2x}{a^2}\lambda \\ 8xz = \frac{2y}{b^2}\lambda \\ 8xy = \frac{2z}{c^2}\lambda \end{cases} \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{3x^2}{a^2} = 1, \frac{3y^2}{b^2} = 1, \frac{3z^2}{c^2} = 1$$

$$x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

Therefore, the dimensions of the box are $\frac{2\sqrt{3}a}{3} \times \frac{2\sqrt{3}b}{3} \times \frac{2\sqrt{3}c}{3}$.

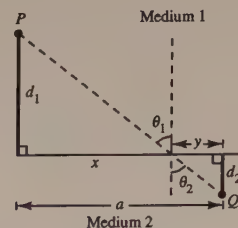
37. Using the formula $\text{Time} = \frac{\text{Distance}}{\text{Rate}}$, minimize $T(x, y) = \frac{\sqrt{d_1^2 + x^2}}{v_1} + \frac{\sqrt{d_2^2 + y^2}}{v_2}$ subject to the constraint $x + y = a$.

$$\left. \begin{aligned} \frac{x}{v_1 \sqrt{d_1^2 + x^2}} &= \lambda \\ \frac{y}{v_2 \sqrt{d_2^2 + y^2}} &= \lambda \end{aligned} \right\} \Rightarrow \frac{x}{v_1 \sqrt{d_1^2 + x^2}} = \frac{y}{v_2 \sqrt{d_2^2 + y^2}}$$

$$x + y = a$$

Since $\sin \theta_1 = \frac{x}{\sqrt{d_1^2 + x^2}}$ and $\sin \theta_2 = \frac{y}{\sqrt{d_2^2 + y^2}}$, we have

$$\frac{x/\sqrt{d_1^2 + x^2}}{v_1} = \frac{y/\sqrt{d_2^2 + y^2}}{v_2} \quad \text{or} \quad \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$



39. Maximize $P(p, q, r) = 2pq + 2pr + 2qr$.

Constraint: $p + q + r = 1$

$$\nabla P = \lambda \nabla g$$

$$\left. \begin{aligned} 2q + 2r &= \lambda \\ 2p + 2r &= \lambda \\ 2p + 2q &= \lambda \end{aligned} \right\} \Rightarrow 3\lambda = 4(p + q + r) = 4(1)$$

$$\Rightarrow \lambda = \frac{4}{3}$$

$$p + q + r = 1$$

$$\left. \begin{aligned} q + r &= \frac{2}{3} \\ p + q + r &= 1 \end{aligned} \right\} \Rightarrow p = \frac{1}{3}, q = \frac{1}{3}, r = \frac{1}{3}$$

$$P\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{3}.$$

41. Maximize $P(x, y) = 100x^{0.25}y^{0.75}$

subject to the constraint $48x + 36y = 100,000$.

$$25x^{-0.75}y^{0.75} = 48\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{48\lambda}{25}$$

$$75x^{0.25}y^{-0.25} = 36\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{36\lambda}{75}$$

$$\left(\frac{y}{x}\right)^{0.75} \left(\frac{y}{x}\right)^{0.25} = \left(\frac{48\lambda}{25}\right) \left(\frac{75}{36\lambda}\right)$$

$$\frac{y}{x} = 4$$

$$y = 4x$$

$$48x + 36y = 100,000 \Rightarrow 192x = 100,000$$

$$x = \frac{3125}{6}, y = \frac{6250}{3}$$

$$\text{Therefore, } P\left(\frac{3125}{6}, \frac{6250}{3}\right) \approx 147,314.$$

43. Minimize $C(x, y) = 48x + 36y$ subject to the constraint $100x^{0.25}y^{0.75} = 20,000$.

$$48 = 25x^{-0.75}y^{0.75}\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{48}{25\lambda}$$

$$36 = 75x^{0.25}y^{-0.25}\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{36}{75\lambda}$$

$$\left(\frac{y}{x}\right)^{0.75} \left(\frac{y}{x}\right)^{0.25} = \left(\frac{48}{25\lambda}\right) \left(\frac{75\lambda}{36}\right)$$

$$\frac{y}{x} = 4 \Rightarrow y = 4x$$

$$100x^{0.25}y^{0.75} = 20,000 \Rightarrow x^{0.25}(4x)^{0.75} = 200$$

$$x = \frac{200}{4^{0.75}} = \frac{200}{2\sqrt{2}} = 50\sqrt{2}$$

$$y = 4x = 200\sqrt{2}$$

$$\text{Therefore, } C(50\sqrt{2}, 200\sqrt{2}) \approx \$13,576.45.$$

45. (a) Maximize $g(\alpha, \beta, \gamma) = \cos \alpha \cos \beta \cos \gamma$ subject to the constraint $\alpha + \beta + \gamma = \pi$.

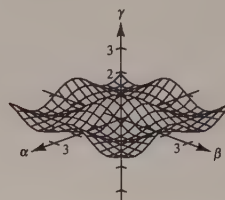
$$\left. \begin{aligned} -\sin \alpha \cos \beta \cos \gamma &= \lambda \\ -\cos \alpha \sin \beta \cos \gamma &= \lambda \\ -\cos \alpha \cos \beta \sin \gamma &= \lambda \end{aligned} \right\} \tan \alpha = \tan \beta = \tan \gamma \Rightarrow \alpha = \beta = \gamma$$

$$\alpha + \beta + \gamma = \pi \Rightarrow \alpha = \beta = \gamma = \frac{\pi}{3}$$

$$g\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{1}{8}$$

(b) $\alpha + \beta + \gamma = \pi \Rightarrow \gamma = \pi - (\alpha + \beta)$

$$\begin{aligned} g(\alpha + \beta) &= \cos \alpha \cos \beta \cos(\pi - (\alpha + \beta)) \\ &= \cos \alpha \cos \beta [\cos \pi \cos(\alpha + \beta) + \sin \pi \sin(\alpha + \beta)] \\ &= -\cos \alpha \cos \beta \cos(\alpha + \beta) \end{aligned}$$



Review Exercises for Chapter 12

1. No, it is not the graph of a function.

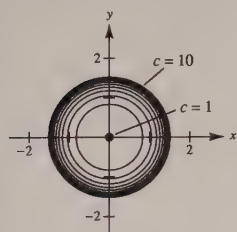
3. $f(x, y) = e^{x^2 + y^2}$

The level curves are of the form

$$c = e^{x^2 + y^2}$$

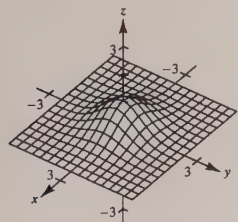
$$\ln c = x^2 + y^2.$$

The level curves are circles centered at the origin.



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7. $f(x, y) = e^{-(x^2 + y^2)}$



11. $\lim_{(x, y) \rightarrow (1, 1)} \frac{xy}{x^2 + y^2} = \frac{1}{2}$

Continuous except at $(0, 0)$.

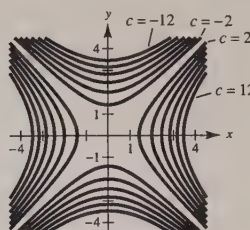
5. $f(x, y) = x^2 - y^2$

The level curves are of the form

$$c = x^2 - y^2$$

$$1 = \frac{x^2}{c} - \frac{y^2}{c}.$$

The level curves are hyperbolas.

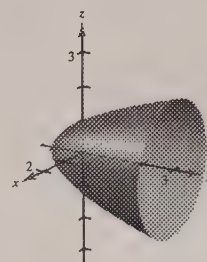


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9. $f(x, y, z) = x^2 - y + z^2 = 1$

$$y = x^2 + z^2 - 1$$

Elliptic paraboloid



13. $\lim_{(x, y) \rightarrow (0, 0)} \frac{-4x^2y}{x^4 + y^2}$

For $y = x^2$, $\frac{-4x^2y}{x^4 + y^2} = \frac{-4x^4}{x^4 + x^4} = -2$, for $x \neq 0$

For $y = 0$, $\frac{-4x^2y}{x^4 + y^2} = 0$, for $x \neq 0$

Thus, the limit does not exist. Continuous except at $(0, 0)$.

15. $f(x, y) = e^x \cos y$

$$f_x = e^x \cos y$$

$$f_y = -e^x \sin y$$

19. $g(x, y) = \frac{xy}{x^2 + y^2}$

$$g_x = \frac{y(x^2 + y^2) - xy(2x)}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$g_y = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$$

23. $u(x, t) = ce^{-n^2 t} \sin(nx)$

$$\frac{\partial u}{\partial x} = cne^{-n^2 t} \cos(nx)$$

$$\frac{\partial u}{\partial t} = -cn^2 e^{-n^2 t} \sin(nx)$$

27. $f(x, y) = 3x^2 - xy + 2y^3$

$$f_x = 6x - y$$

$$f_y = -x + 6y^2$$

$$f_{xx} = 6$$

$$f_{yy} = 12y$$

$$f_{xy} = -1$$

$$f_{yx} = -1$$

29. $h(x, y) = x \sin y + y \cos x$

$$h_x = \sin y - y \sin x$$

$$h_y = x \cos y + \cos x$$

$$h_{xx} = -y \cos x$$

$$h_{yy} = -x \sin y$$

$$h_{xy} = \cos y - \sin x$$

$$h_{yx} = \cos y - \sin x$$

31. $z = x^2 - y^2$

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial z}{\partial y} = -2y$$

$$\frac{\partial^2 z}{\partial y^2} = -2$$

$$\text{Therefore, } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

33. $z = \frac{y}{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = -2y \left[\frac{-4x^2}{(x^2 + y^2)^3} + \frac{1}{(x^2 + y^2)^2} \right] = 2y \frac{3x^2 - y^2}{(x^2 + y^2)^3}$$

$$\frac{\partial z}{\partial y} = \frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{(x^2 + y^2)^2(-2y) - 2(x^2 - y^2)(x^2 + y^2)(2y)}{(x^2 + y^2)^4}$$

$$= -2y \frac{3x^2 - y^2}{(x^2 + y^2)^3}$$

$$\text{Therefore, } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

17. $z = xe^y + ye^x$

$$\frac{\partial z}{\partial x} = e^y + ye^x$$

$$\frac{\partial z}{\partial y} = xe^y + e^x$$

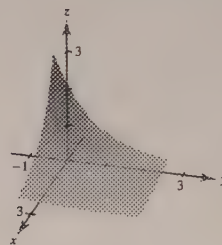
21. $f(x, y, z) = z \arctan \frac{y}{x}$

$$f_x = \frac{z}{1 + (y^2/x^2)} \left(-\frac{y}{x^2} \right) = \frac{-yz}{x^2 + y^2}$$

$$f_y = \frac{z}{1 + (y^2/x^2)} \left(\frac{1}{x} \right) = \frac{xz}{x^2 + y^2}$$

$$f_z = \arctan \frac{y}{x}$$

25.



35. $z = x \sin \frac{y}{x}$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \left(\sin \frac{y}{x} - \frac{y}{x} \cos \frac{y}{x} \right) dx + \left(\cos \frac{y}{x} \right) dy$$

37. $z^2 = x^2 + y^2$

$$2z \, dx = 2x \, dx + 2y \, dy$$

$$dz = \frac{x}{z} \, dx + \frac{y}{z} \, dy = \frac{5}{13} \left(\frac{1}{2} \right) + \frac{12}{13} \left(\frac{1}{2} \right) = \frac{17}{26} \approx 0.654 \text{ cm}$$

Percentage error: $\frac{dz}{z} = \frac{17/26}{13} \approx 0.0503 \approx 5\%$

39. $V = \frac{1}{3}\pi r^2 h$

$$\begin{aligned} dV &= \frac{2}{3}\pi r h \, dr + \frac{1}{3}\pi r^2 \, dh = \frac{2}{3}\pi(2)(5)\left(\pm\frac{1}{8}\right) + \frac{1}{3}\pi(2)^2\left(\pm\frac{1}{8}\right) \\ &= \pm\frac{5}{6}\pi \pm \frac{1}{6}\pi = \pm\pi \text{ in.}^3 \end{aligned}$$

41. $w = \ln(x^2 + y^2)$, $x = 2t + 3$, $y = 4 - t$

$$\begin{aligned} \text{Chain Rule: } \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= \frac{2x}{x^2 + y^2}(2) + \frac{2y}{x^2 + y^2}(-1) \\ &= \frac{2(2t + 3)2}{(2t + 3)^2 + (4 - t)^2} - \frac{2(4 - t)}{(2t + 3)^2 + (4 - t)^2} \\ &= \frac{10t + 4}{5t^2 + 4t + 25} \end{aligned}$$

Substitution: $w = \ln(x^2 + y^2) = \ln[(2t + 3)^2 + (4 - t)^2]$

$$\frac{dw}{dt} = \frac{2(2t + 3)(2) - 2(4 - t)}{(2t + 3)^2 + (4 - t)^2} = \frac{10t + 4}{5t^2 + 4t + 25}$$

43. $u = x^2 + y^2 + z^2$, $x = r \cos t$, $y = r \sin t$, $z = t$

$$\begin{aligned} \text{Chain Rule: } \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial r} \\ &= 2x \cos t + 2y \sin t + 2z(0) \\ &= 2(r \cos^2 t + r \sin^2 t) = 2r \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t} \\ &= 2x(-r \sin t) + 2y(r \cos t) + 2z \\ &= 2(-r^2 \sin t \cos t + r^2 \sin t \cos t) + 2t \\ &= 2t \end{aligned}$$

Substitution: $u(r, t) = r^2 \cos^2 t + r^2 \sin^2 t + t^2 = r^2 + t^2$

$$\frac{\partial u}{\partial r} = 2r$$

$$\frac{\partial u}{\partial t} = 2t$$

45. $x^2y - 2yz - xz - z^2 = 0$

$$\begin{aligned} 2xy - 2y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial x} - z - 2z \frac{\partial z}{\partial x} &= 0 \\ \frac{\partial z}{\partial x} &= \frac{-2xy + z}{-2y - x - 2z} = \frac{2xy - z}{x + 2y + 2z} \end{aligned}$$

$$\begin{aligned} x^2 - 2y \frac{\partial z}{\partial y} - 2z - x \frac{\partial z}{\partial y} - 2z \frac{\partial z}{\partial y} &= 0 \\ \frac{\partial z}{\partial y} &= \frac{-x^2 + 2z}{-2y - x - 2z} = \frac{x^2 - 2z}{x + 2y + 2z} \end{aligned}$$

47. $f(x, y) = x^2y$

$$\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$\nabla f(2, 1) = 4\mathbf{i} + 4\mathbf{j}$$

$$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}f(2, 1) = \nabla f(2, 1) \cdot \mathbf{u} = 2\sqrt{2} - 2\sqrt{2} = 0$$

51. $z = \frac{y}{x^2 + y^2}$

$$\nabla z = -\frac{2xy}{(x^2 + y^2)^2}\mathbf{i} + \frac{x^2 - y^2}{(x^2 + y^2)^2}\mathbf{j}$$

$$\nabla z(1, 1) = -\frac{1}{2}\mathbf{i} = \left\langle -\frac{1}{2}, 0 \right\rangle$$

$$\|\nabla z(1, 1)\| = \frac{1}{2}$$

55. $9x^2 - 4y^2 = 65$

$$f(x, y) = 9x^2 - 4y^2$$

$$\nabla f(x, y) = 18x\mathbf{i} - 8y\mathbf{j}$$

$$\nabla f(3, 2) = 54\mathbf{i} - 16\mathbf{j}$$

$$\text{Unit normal: } \frac{54\mathbf{i} - 16\mathbf{j}}{\|54\mathbf{i} - 16\mathbf{j}\|} = \frac{1}{\sqrt{793}}(27\mathbf{i} - 8\mathbf{j})$$

59. $F(x, y, z) = x^2 + y^2 - 4x + 6y + z + 9 = 0$

$$\nabla F = (2x - 4)\mathbf{i} + (2y + 6)\mathbf{j} + \mathbf{k}$$

$$\nabla F(2, -3, 4) = \mathbf{k}$$

Therefore, the equation of the tangent plane is

$$z - 4 = 0 \quad \text{or} \quad z = 4,$$

and the equation of the normal line is

$$x = 2, y = -3, z = 4 + t.$$

63. $f(x, y, z) = x^2 + y^2 + z^2 - 14$

$$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla f(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} \text{ Normal vector to plane.}$$

$$\cos \theta = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|} = \frac{6}{\sqrt{56}} = \frac{3\sqrt{14}}{14}$$

$$\theta = 36.7^\circ$$

49. $w = y^2 + xz$

$$\nabla w = z\mathbf{i} + 2y\mathbf{j} + x\mathbf{k}$$

$$\nabla w(1, 2, 2) = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

$$\mathbf{u} = \frac{1}{3}\mathbf{v} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$D_{\mathbf{u}}w(1, 2, 2) = \nabla w(1, 2, 2) \cdot \mathbf{u} = \frac{4}{3} - \frac{4}{3} + \frac{2}{3} = \frac{2}{3}$$

53. $z = e^{-x} \cos y$

$$\nabla z = -e^{-x} \cos y \mathbf{i} - e^{-x} \sin y \mathbf{j}$$

$$\nabla z\left(0, \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$

$$\left\| \nabla z\left(0, \frac{\pi}{4}\right) \right\| = 1$$

57. $F(x, y, z) = x^2y - z = 0$

$$\nabla F = 2xy\mathbf{i} + x^2\mathbf{j} - \mathbf{k}$$

$$\nabla F(2, 1, 4) = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$

Therefore, the equation of the tangent plane is

$$4(x - 2) + 4(y - 1) - (z - 4) = 0 \quad \text{or}$$

$$4x + 4y - z = 8,$$

and the equation of the normal line is

$$x = 4t + 2, y = 4t + 1, z = -t + 4.$$

61. $F(x, y, z) = x^2 - y^2 - z = 0$

$$G(x, y, z) = 3 - z = 0$$

$$\nabla F = 2x\mathbf{i} - 2y\mathbf{j} - \mathbf{k}$$

$$\nabla G = -\mathbf{k}$$

$$\nabla F(2, 1, 3) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 2(\mathbf{i} + 2\mathbf{j})$$

Therefore, the equation of the tangent line is

$$\frac{x - 2}{1} = \frac{y - 1}{2}, z = 3.$$

65. $f(x, y) = x^3 - 3xy + y^2$

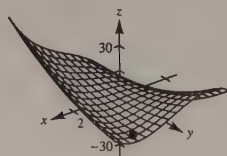
$$f_x = 3x^2 - 3y = 3(x^2 - y) = 0$$

$$f_y = -3x + 2y = 0$$

$$f_{xx} = 6x$$

$$f_{yy} = 2$$

$$f_{xy} = -3$$



From $f_x = 0$, we have $y = x^2$. Substituting this into $f_y = 0$, we have $-3x + 2x^2 = x(2x - 3) = 0$. Thus, $x = 0$ or $\frac{3}{2}$.

At the critical point $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.

At the critical point $(\frac{3}{2}, \frac{9}{4})$, $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} > 0$. Therefore, $(\frac{3}{2}, \frac{9}{4}, -\frac{27}{16})$ is a relative minimum.

67. $f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$

$$f_x = y - \frac{1}{x^2} = 0, \quad x^2y = 1$$

$$f_y = x - \frac{1}{y^2} = 0, \quad xy^2 = 1$$

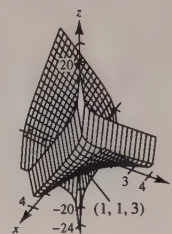
Thus, $x^2y = xy^2$ or $x = y$ and substitution yields the critical point $(1, 1)$.

$$f_{xx} = \frac{2}{x^3}$$

$$f_{xy} = 1$$

$$f_{yy} = \frac{2}{y^3}$$

At the critical point $(1, 1)$, $f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0$. Thus, $(1, 1, 3)$ is a relative minimum.



69. The level curves are hyperbolas. There is a critical point at $(0, 0)$, but there are no relative extrema. The gradient is normal to the level curve at any given point at (x_0, y_0) .

71. $P(x_1, x_2) = R - C_1 - C_2$

$$= [225 - 0.4(x_1 + x_2)](x_1 + x_2) - (0.05x_1^2 + 15x_1 + 5400) - (0.03x_2^2 + 15x_2 + 6100)$$

$$= -0.45x_1^2 - 0.43x_2^2 - 0.8x_1x_2 + 210x_1 + 210x_2 - 11,500$$

$$P_{x_1} = -0.9x_1 - 0.8x_2 + 210 = 0$$

$$0.9x_1 + 0.8x_2 = 210$$

$$P_{x_2} = -0.86x_2 - 0.8x_1 + 210 = 0$$

$$0.8x_1 + 0.86x_2 = 210$$

Solving this system yields $x_1 \approx 94$ and $x_2 \approx 157$.

$$P_{x_1x_1} = -0.9$$

$$P_{x_1x_2} = -0.8$$

$$P_{x_2x_2} = -0.86$$

$$P_{x_1x_1} < 0$$

$$P_{x_1x_1}P_{x_2x_2} - (P_{x_1x_2})^2 > 0$$

Therefore, profit is maximum when $x_1 \approx 94$ and $x_2 \approx 157$.

73. Maximize $f(x, y) = 4x + xy + 2y$ subject to the constraint $20x + 4y = 2000$.

$$\begin{cases} 4 + y = 20\lambda \\ x + 2 = 4\lambda \end{cases} \Rightarrow 5x - y = -6$$

$$20x + 4y = 2000 \Rightarrow 5x + y = 500$$

$$\frac{5x - y = -6}{5x + y = 500}$$

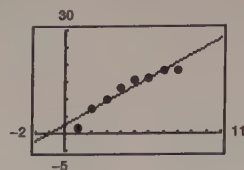
$$10x = 494$$

$$x = 49.4$$

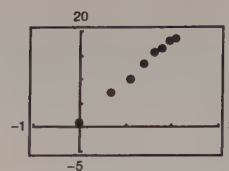
$$y = 253$$

$$f(49.4, 253) = 13,201.8$$

75. (a) $y = 2.29t + 2.34$



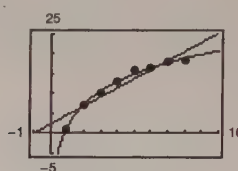
(b)



Yes, the data appears more linear.

- (c) $y = 8.37 \ln t + 1.54$

(d)



The logarithmic model is a better fit.

77. Optimize $f(x, y, z) = xy + yz + xz$ subject to the constraint $x + y + z = 1$.

$$\begin{cases} y + z = \lambda \\ x + z = \lambda \\ x + y = \lambda \end{cases} \Rightarrow x = y = z$$

$$x + y + z = 1 \Rightarrow x = y = z = \frac{1}{3}$$

$$\text{Maximum: } f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$$

79. $PQ = \sqrt{x^2 + 4}$, $QR = \sqrt{y^2 + 1}$, $RS = z$; $x + y + z = 10$

$$C = 3\sqrt{x^2 + 4} + 2\sqrt{y^2 + 1} + z$$

$$\text{Constraint: } x + y + z = 10$$

$$\nabla C = \lambda \nabla g$$

$$\frac{3x}{\sqrt{x^2 + 4}}\mathbf{i} + \frac{2y}{\sqrt{y^2 + 1}}\mathbf{j} + \mathbf{k} = \lambda[\mathbf{i} + \mathbf{j} + \mathbf{k}]$$

$$3x = \lambda\sqrt{x^2 + 4}$$

$$2y = \lambda\sqrt{y^2 + 1}$$

$$1 = \lambda$$

$$9x^2 = x^2 + 4 \Rightarrow x^2 = \frac{4}{8} = \frac{1}{2}$$

$$4y^2 = y^2 + 1 \Rightarrow y^2 = \frac{1}{3}$$

$$\text{Hence, } x = \frac{\sqrt{2}}{2}, y = \frac{\sqrt{3}}{3}, z = 10 - \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3} \approx 8.716 \text{ m.}$$

Problem Solving for Chapter 12

1. (a) The three sides have lengths 5, 6, and 5.

$$\text{Thus, } s = \frac{16}{2} = 8 \text{ and } A = \sqrt{8(3)(2)(3)} = 12$$

- (b) Let $f(a, b, c) = (\text{area})^2 = s(s-a)(s-b)(s-c)$, subject to the constraint $a + b + c = \text{constant}$ (perimeter).

Using Lagrange multipliers,

$$-s(s-b)(s-c) = \lambda$$

$$-s(s-a)(s-c) = \lambda$$

$$-s(s-a)(s-b) = \lambda$$

From the first 2 equations $s-b = s-a \Rightarrow a = b$.

Similarly, $b = c$ and hence $a = b = c$ which is an equilateral triangle.

- (c) Let $f(a, b, c) = a + b + c$, subject to $(\text{Area})^2 = s(s-a)(s-b)(s-c)$ constant.

Using Lagrange multipliers,

$$1 = -\lambda s(s-b)(s-c)$$

$$1 = -\lambda s(s-a)(s-c)$$

$$1 = -\lambda s(s-a)(s-b)$$

Hence, $s-a = s-b \Rightarrow a = b$ and $a = b = c$.

3. (a) $F(x, y, z) = xyz - 1 = 0$

$$F_x = yz, F_y = xz, F_z = xy$$

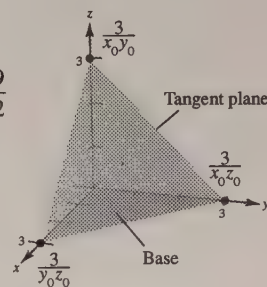
Tangent plane:

$$y_0 z_0 (x - x_0) + x_0 z_0 (y - y_0) + x_0 y_0 (z - z_0) = 0$$

$$y_0 z_0 x + x_0 z_0 y + x_0 y_0 z = 3x_0 y_0 z_0 = 3$$

$$(b) V = \frac{1}{3}(\text{base})(\text{height})$$

$$= \frac{1}{3} \left(\frac{1}{2} \frac{3}{y_0 z_0} \frac{3}{x_0 z_0} \right) \left(\frac{3}{x_0 y_0} \right) = \frac{9}{2}$$



5. We cannot use Theorem 12.9 since f is not a differentiable function of x and y . Hence, we use the definition of directional derivatives.

$$D_u f(x, y) = \lim_{t \rightarrow 0} \frac{f(x + t \cos \theta, y + t \sin \theta) - f(x, y)}{t}$$

$$D_u f(0, 0) = \lim_{t \rightarrow 0} \frac{f\left[0 + \left(\frac{t}{\sqrt{2}}\right), 0 + \left(\frac{t}{\sqrt{2}}\right)\right] - f(0, 0)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{1}{t} \left[\frac{4 \left(\frac{t}{\sqrt{2}}\right) \left(\frac{t}{\sqrt{2}}\right)}{\left(\frac{t^2}{2}\right) + \left(\frac{t^2}{2}\right)} \right] = \lim_{t \rightarrow 0} \frac{1}{t} \left[\frac{2t^2}{t^2} \right] = \lim_{t \rightarrow 0} \frac{2}{t} \text{ which does not exist.}$$

If $f(0, 0) = 2$, then

$$D_u f(0, 0) = \lim_{t \rightarrow 0} \frac{f\left(0 + \frac{t}{\sqrt{2}}, 0 + \frac{t}{\sqrt{2}}\right) - 2}{t} = \lim_{t \rightarrow 0} \frac{1}{t} \left[\frac{2t^2}{t^2} - 2 \right] = 0$$

which implies that the directional derivative exists.

7. $H = k(5xy + 6xz + 6yz)$

$$z = \frac{1000}{xy} \Rightarrow H = k\left(5xy + \frac{6000}{y} + \frac{6000}{x}\right).$$

$$H_x = 5y - \frac{6000}{x^2} = 0 \Rightarrow 5yx^2 = 6000$$

By symmetry, $x = y \Rightarrow x^3 = y^3 = 1200$.

Thus, $x = y = 2\sqrt[3]{150}$ and $z = \frac{5}{3}\sqrt[3]{150}$.

9. (a) $\frac{\partial f}{\partial x} = Cax^{a-1}y^{1-a}, \frac{\partial f}{\partial y} = C(1-a)x^ay^{-a}$

$$\begin{aligned} x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} &= Cax^ay^{1-a} + C(1-a)x^ay^{1-a} \\ &= [Ca + C(1-a)]x^ay^{1-a} \\ &= Cx^ay^{1-a} = f \end{aligned}$$

(b) $f(tx, ty) = C(tx)^a(ty)^{1-a} = Ct^ax^at^{1-a}y^{1-a}$
 $= Cx^ay^{1-a}(t) = tf(x, y)$

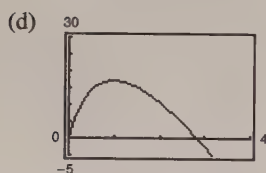
11. (a) $x = 64(\cos 45^\circ)t = 32\sqrt{2}t$

$$y = 64(\sin 45^\circ)t - 16t^2 = 32\sqrt{2}t - 16t^2$$

(b) $\tan \alpha = \frac{y}{x + 50}$

$$\alpha = \arctan\left(\frac{y}{x + 50}\right) = \arctan\left(\frac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)$$

(c) $\frac{d\alpha}{dt} = \frac{1}{1 + \left(\frac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)^2} \cdot \frac{-64(8\sqrt{2}t^2 + 25t - 25\sqrt{2})}{(32\sqrt{2}t + 50)^2} = \frac{-16(8\sqrt{2}t^2 + 25t - 25\sqrt{2})}{64t^4 - 256\sqrt{2}t^3 + 1024t^2 + 800\sqrt{2}t + 625}$



No. The rate of change of α is greatest when the projectile is closest to the camera.

(e) $\frac{d\alpha}{dt} = 0$ when

$$8\sqrt{2}t^2 + 25t - 25\sqrt{2} = 0$$

$$t = \frac{-25 + \sqrt{25^2 - 4(8\sqrt{2})(-25\sqrt{2})}}{2(8\sqrt{2})} \approx 0.98 \text{ second.}$$

No, the projectile is at its maximum height when $dy/dt = 32\sqrt{2} - 32t = 0$ or $t = \sqrt{2} \approx 1.41$ seconds.

13. (a) There is a minimum at $(0, 0, 0)$, maxima at $(0, \pm 1, 2/e)$ and saddle point at $(\pm 1, 0, 1/e)$:

$$f_x = (x^2 + 2y^2)e^{-(x^2+y^2)}(-2x) + (2x)e^{-(x^2+y^2)}$$

$$= e^{-(x^2+y^2)}[(x^2 + 2y^2)(-2x) + 2x]$$

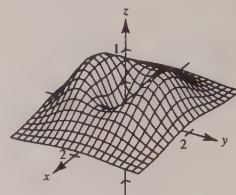
$$= e^{-(x^2+y^2)}[-2x^3 + 4xy^2 + 2x] = 0 \Rightarrow x^3 + 2xy^2 - x = 0$$

$$f_y = (x^2 + 2y^2)e^{-(x^2+y^2)}(-2y) + (4y)e^{-(x^2+y^2)}$$

$$= e^{-(x^2+y^2)}[(x^2 + 2y^2)(-2y) + 4y]$$

$$= e^{-(x^2+y^2)}[-4y^3 - 2x^2y + 4y] = 0 \Rightarrow 2y^3 + x^2y - 2y = 0$$

Solving the two equations $x^3 + 2xy^2 - x = 0$ and $2y^3 + x^2y - 2y = 0$, you obtain the following critical points: $(0, \pm 1)$, $(\pm 1, 0)$, $(0, 0)$. Using the second derivative test, you obtain the results above.



—CONTINUED—

13. —CONTINUED—

(b) As in part (a), you obtain

$$f_x = e^{-(x^2+y^2)}[2x(x^2 - 1 - 2y^2)]$$

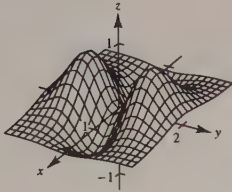
$$f_y = e^{-(x^2+y^2)}[2y(2 + x^2 - 2y^2)]$$

 The critical numbers are $(0, 0)$, $(0, \pm 1)$, $(\pm 1, 0)$.

These yield

 $(\pm 1, 0, -1/e)$ minima

 $(0, \pm 1, 2/e)$ maxima

 $(0, 0, 0)$ saddle

 (c) In general, for $\alpha > 0$ you obtain

 $(0, 0, 0)$ minimum

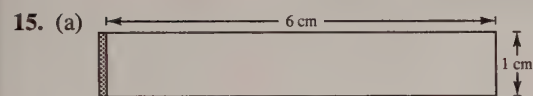
 $(0, \pm 1, \beta/e)$ maxima

 $(\pm 1, 0, \alpha/e)$ saddle

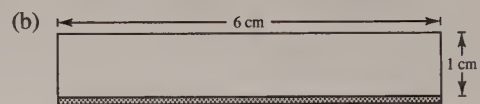
 For $\alpha < 0$, you obtain

 $(\pm 1, 0, \alpha/e)$ minima

 $(0, \pm 1, \beta/e)$ maxima

 $(0, 0, 0)$ saddle


(c) The height has more effect since the shaded region in (b) is larger than the shaded region in (a).



$$(d) A = hl \Rightarrow dA = l dh + h dl$$

$$\text{If } dl = 0.01 \text{ and } dh = 0, \text{ then } dA = 1(0.01) = 0.01.$$

$$\text{If } dh = 0.01 \text{ and } dl = 0, \text{ then } dA = 6(0.01) = 0.06.$$

17. Essay

$$19. u(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)]$$

$$\text{Let } r = x - ct \text{ and } s = x + ct. \text{ Then } u(r, s) = \frac{1}{2}[f(r) + f(s)].$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial t} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial t} = \frac{1}{2} \frac{df}{dr}(-c) + \frac{1}{2} \frac{df}{ds}(c)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2} \frac{d^2 f}{dr^2}(-c)^2 + \frac{1}{2} \frac{d^2 f}{ds^2}(c)^2 = \frac{c^2}{2} \left[\frac{d^2 f}{dr^2} + \frac{d^2 f}{ds^2} \right]$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} = \frac{1}{2} \frac{df}{dr}(1) + \frac{1}{2} \frac{df}{ds}(1)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2} \frac{d^2 f}{dr^2}(1)^2 + \frac{1}{2} \frac{d^2 f}{ds^2}(1)^2 = \frac{1}{2} \left[\frac{d^2 f}{dr^2} + \frac{d^2 f}{ds^2} \right]$$

$$\text{Thus, } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

CHAPTER 13

Multiple Integration

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CHAPTER 13

Multiple Integration

Section 13.1 Iterated Integrals and Area in the Plane

Solutions to Odd-Numbered Exercises

$$1. \int_0^x (2x - y) dy = \left[2xy - \frac{1}{2}y^2 \right]_0^x = \frac{3}{2}x^2$$

$$3. \int_1^{2y} \frac{y}{x} dx = \left[y \ln x \right]_1^{2y} = y \ln 2y - 0 = y \ln 2y, (y > 0)$$

$$5. \int_0^{\sqrt{4-x^2}} x^2 y dy = \left[\frac{1}{2} x^2 y^2 \right]_0^{\sqrt{4-x^2}} = \frac{4x^2 - x^4}{2}$$

$$7. \int_e^y \frac{y \ln x}{x} dx = \left[\frac{1}{2} y \ln^2 x \right]_e^y = \frac{1}{2} y [\ln^2 y - \ln^2 e] = \frac{y}{2} [(\ln y)^2 - y^2], (y > 0)$$

$$9. \int_0^{x^3} y e^{-y/x} dy = \left[-x y e^{-y/x} \right]_0^{x^3} + x \int_0^{x^3} e^{-y/x} dy = -x^4 e^{-x^2} - \left[x^2 e^{-y/x} \right]_0^{x^3} = x^2 (1 - e^{-x^2} - x^2 e^{-x^2})$$

$$u = y, du = dy, dv = e^{-y/x} dy, v = -x e^{-y/x}$$

$$11. \int_0^1 \int_0^2 (x + y) dy dx = \int_0^1 \left[xy + \frac{1}{2} y^2 \right]_0^2 dx = \int_0^1 (2x + 2) dx = \left[x^2 + 2x \right]_0^1 = 3$$

$$13. \int_0^1 \int_0^x \sqrt{1-x^2} dy dx = \int_0^1 \left[y \sqrt{1-x^2} \right]_0^x dx = \int_0^1 x \sqrt{1-x^2} dx = \left[-\frac{1}{2} \left(\frac{2}{3} \right) (1-x^2)^{3/2} \right]_0^1 = \frac{1}{3}$$

$$15. \int_1^2 \int_0^4 (x^2 - 2y^2 + 1) dx dy = \int_1^2 \left[\frac{1}{3} x^3 - 2xy^2 + x \right]_0^4 dy \\ = \int_1^2 \left(\frac{64}{3} - 8y^2 + 4 \right) dy = \frac{4}{3} \int_1^2 (19 - 6y^2) dy = \left[\frac{4}{3} (19y - 2y^3) \right]_1^2 = \frac{20}{3}$$

$$17. \int_0^1 \int_0^{\sqrt{1-y^2}} (x + y) dx dy = \int_0^1 \left[\frac{1}{2} x^2 + xy \right]_0^{\sqrt{1-y^2}} dy \\ = \int_0^1 \left[\frac{1}{2} (1-y^2) + y \sqrt{1-y^2} \right] dy = \left[\frac{1}{2} y - \frac{1}{6} y^3 - \frac{1}{2} \left(\frac{2}{3} \right) (1-y^2)^{3/2} \right]_0^1 = \frac{2}{3}$$

$$19. \int_0^2 \int_0^{\sqrt{4-y^2}} \frac{2}{\sqrt{4-y^2}} dx dy = \int_0^2 \left[\frac{2x}{\sqrt{4-y^2}} \right]_0^{\sqrt{4-y^2}} dy = \int_0^2 2 dy = \left[2y \right]_0^2 = 4$$

$$21. \int_0^{\pi/2} \int_0^{\sin \theta} \theta r dr d\theta = \int_0^{\pi/2} \left[\theta \frac{r^2}{2} \right]_0^{\sin \theta} d\theta = \int_0^{\pi/2} \frac{1}{2} \theta \sin^2 \theta d\theta \\ = \frac{1}{4} \int_0^{\pi/2} (\theta - \theta \cos 2\theta) d\theta = \frac{1}{4} \left[\frac{\theta^2}{2} - \left(\frac{1}{4} \cos 2\theta + \frac{\theta}{2} \sin 2\theta \right) \right]_0^{\pi/2} = \frac{\pi^2}{32} + \frac{1}{8}$$

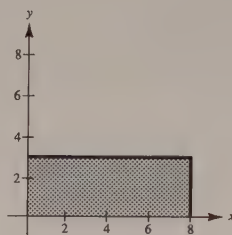
$$23. \int_1^\infty \int_0^{1/x} y \, dy \, dx = \int_1^\infty \left[\frac{y^2}{2} \right]_0^{1/x} dx = \frac{1}{2} \int_1^\infty \frac{1}{x^2} dx = \left[-\frac{1}{2x} \right]_1^\infty = 0 + \frac{1}{2} = \frac{1}{2}$$

$$25. \int_1^\infty \int_1^\infty \frac{1}{xy} \, dx \, dy = \int_1^\infty \left[\frac{1}{y} \ln x \right]_1^\infty dy = \int_1^\infty \left[\frac{1}{y}(\infty) - \frac{1}{y}(0) \right] dy$$

Diverges

$$27. A = \int_0^8 \int_0^3 dy \, dx = \int_0^8 \left[y \right]_0^3 dx = \int_0^8 3 \, dx = \left[3x \right]_0^8 = 24$$

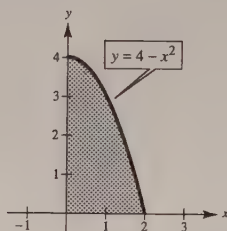
$$A = \int_0^3 \int_0^8 dx \, dy = \int_0^3 \left[x \right]_0^8 dy = \int_0^3 8 \, dy = \left[8y \right]_0^3 = 24$$



$$29. A = \int_0^2 \int_0^{4-x^2} dy \, dx = \int_0^2 \left[y \right]_0^{4-x^2} dx$$

$$= \int_0^2 (4 - x^2) \, dx$$

$$= \left[4x - \frac{x^3}{3} \right]_0^2 = \frac{16}{3}$$



$$A = \int_0^4 \int_0^{\sqrt{4-y}} dx \, dy$$

$$= \int_0^4 \left[x \right]_0^{\sqrt{4-y}} dy = \int_0^4 \sqrt{4-y} \, dy = -\int_0^4 (4-y)^{1/2} (-1) \, dy = \left[-\frac{2}{3}(4-y)^{3/2} \right]_0^4 = \frac{2}{3}(8) = \frac{16}{3}$$

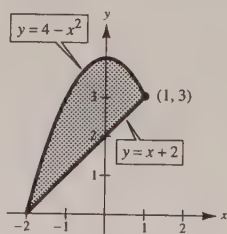
$$31. A = \int_{-2}^1 \int_{x+2}^{4-x^2} dy \, dx$$

$$= \int_{-2}^1 \left[y \right]_{x+2}^{4-x^2} dx$$

$$= \int_{-2}^1 (4 - x^2 - x - 2) \, dx$$

$$= \int_{-2}^1 (2 - x - x^2) \, dx$$

$$= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-2}^1 = \frac{9}{2}$$



$$A = \int_0^3 \int_{-\sqrt{4-y}}^{y-2} dx \, dy + 2 \int_3^4 \int_0^{\sqrt{4-y}} dx \, dy$$

$$= \int_0^3 \left[x \right]_{-\sqrt{4-y}}^{y-2} dy + 2 \int_3^4 \left[x \right]_0^{\sqrt{4-y}} dy$$

$$= \int_0^3 (y - 2 + \sqrt{4-y}) \, dy + 2 \int_3^4 \sqrt{4-y} \, dy$$

$$= \left[\frac{1}{2}y^2 - 2y - \frac{2}{3}(4-y)^{3/2} \right]_0^3 - \left[\frac{4}{3}(4-y)^{3/2} \right]_3^4 = \frac{9}{2}$$

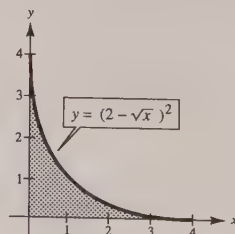
$$33. \int_0^4 \int_0^{(2-\sqrt{x})^2} dy \, dx = \int_0^4 \left[y \right]_0^{(2-\sqrt{x})^2} dx$$

$$= \int_0^4 (4 - 4\sqrt{x} + x) \, dx$$

$$= \left[4x - \frac{8}{3}x\sqrt{x} + \frac{x^2}{2} \right]_0^4 = \frac{8}{3}$$

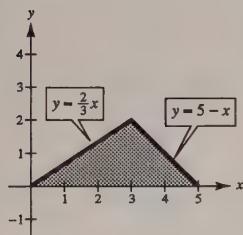
$$\int_0^4 \int_0^{(2-\sqrt{y})^2} dx \, dy = \frac{8}{3}$$

Integration steps are similar to those above.

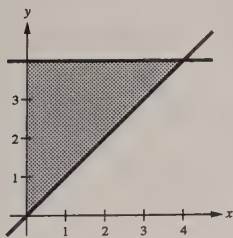


$$\begin{aligned}
 35. A &= \int_0^3 \int_0^{2x/3} dy \, dx + \int_3^5 \int_0^{5-x} dy \, dx \\
 &= \int_0^3 \left[y \right]_0^{2x/3} dx + \int_3^5 \left[y \right]_0^{5-x} dx \\
 &= \int_0^3 \frac{2x}{3} dx + \int_3^5 (5-x) dx \\
 &= \left[\frac{1}{3}x^2 \right]_0^3 + \left[5x - \frac{1}{2}x^2 \right]_3^5 = 5
 \end{aligned}$$

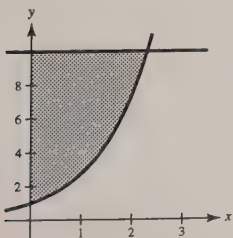
$$\begin{aligned}
 A &= \int_0^2 \int_{3y/2}^{5-y} dx \, dy \\
 &= \int_0^2 \left[x \right]_{3y/2}^{5-y} dy \\
 &= \int_0^2 \left(5 - y - \frac{3y}{2} \right) dy \\
 &= \int_0^2 \left(5 - \frac{5y}{2} \right) dy = \left[5y - \frac{5}{4}y^2 \right]_0^2 = 5
 \end{aligned}$$



$$\begin{aligned}
 39. \int_0^4 \int_0^y f(x, y) \, dx \, dy, \quad 0 \leq x \leq y, \quad 0 \leq y \leq 4 \\
 = \int_0^4 \int_x^4 f(x, y) \, dy \, dx
 \end{aligned}$$



$$\begin{aligned}
 43. \int_1^{10} \int_0^{\ln y} f(x, y) \, dx \, dy, \quad 0 \leq x \leq \ln y, \quad 1 \leq y \leq 10 \\
 = \int_0^{\ln 10} \int_{e^x}^{10} f(x, y) \, dy \, dx
 \end{aligned}$$

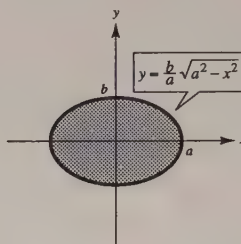


$$\begin{aligned}
 37. \frac{A}{4} &= \int_0^a \int_0^{(b/a)\sqrt{a^2-x^2}} dy \, dx = \int_0^a \left[y \right]_0^{(b/a)\sqrt{a^2-x^2}} dx \\
 &= \frac{b}{a} \int_0^a \sqrt{a^2-x^2} \, dx = ab \int_0^{\pi/2} \cos^2 \theta \, d\theta \\
 (x &= a \sin \theta, \, dx = a \cos \theta \, d\theta) \\
 &= \frac{ab}{2} \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta = \left[\frac{ab}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) \right]_0^{\pi/2} \\
 &= \frac{\pi ab}{4}
 \end{aligned}$$

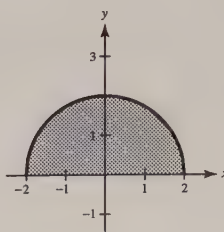
Therefore, $A = \pi ab$.

$$\frac{A}{4} = \int_0^b \int_0^{(a/b)\sqrt{b^2-y^2}} dx \, dy = \frac{\pi ab}{4}$$

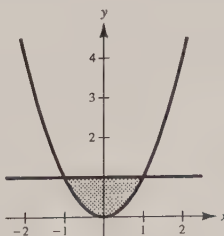
Therefore, $A = \pi ab$. Integration steps are similar to those above.



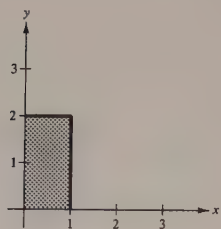
$$\begin{aligned}
 41. \int_{-2}^2 \int_0^{\sqrt{4-x^2}} f(x, y) \, dy \, dx, \quad 0 \leq y \leq \sqrt{4-x^2}, \quad -2 \leq x \leq 2 \\
 = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx \, dy
 \end{aligned}$$



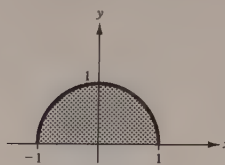
$$\begin{aligned}
 45. \int_{-1}^1 \int_{x^2}^1 f(x, y) \, dy \, dx, \quad x^2 \leq y \leq 1, \quad -1 \leq x \leq 1 \\
 = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) \, dx \, dy
 \end{aligned}$$



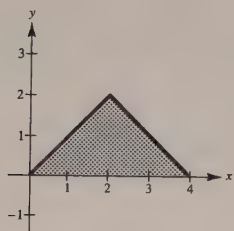
$$47. \int_0^1 \int_0^2 dy \, dx = \int_0^2 \int_0^1 dx \, dy = 2$$



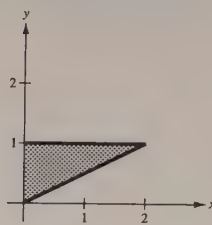
$$49. \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx \, dy = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy \, dx = \frac{\pi}{2}$$



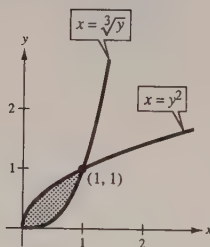
$$51. \int_0^2 \int_0^x dy \, dx + \int_2^4 \int_0^{4-x} dy \, dx = \int_0^2 \int_y^{4-y} dx \, dy = 4$$



$$53. \int_0^2 \int_{x/2}^1 dy \, dx = \int_0^1 \int_0^{2y} dx \, dy = 1$$



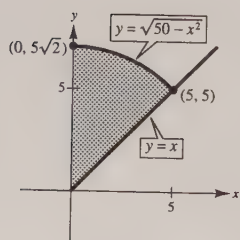
$$55. \int_0^1 \int_{y^2}^{\sqrt[3]{y}} dx \, dy = \int_0^1 \int_{x^3}^{\sqrt{x}} dy \, dx = \frac{5}{12}$$



57. The first integral arises using vertical representative rectangles. The second two integrals arise using horizontal representative rectangles.

$$\begin{aligned} \int_0^5 \int_x^{\sqrt{50-x^2}} x^2 y^2 \, dy \, dx &= \int_0^5 \left[\frac{1}{3} x^2 (50-x^2)^{3/2} - \frac{1}{3} x^5 \right] dx \\ &= \frac{15625}{24} \pi \end{aligned}$$

$$\begin{aligned} \int_0^5 \int_0^y x^2 y^2 \, dx \, dy + \int_5^{5\sqrt{2}} \int_0^{\sqrt{50-y^2}} x^2 y^2 \, dx \, dy &= \int_0^5 \frac{1}{3} y^5 \, dy + \int_5^{5\sqrt{2}} \frac{1}{3} (50-y^2)^{3/2} y^2 \, dy \\ &= \frac{15625}{24} \pi \end{aligned}$$



$$\begin{aligned}
 59. \int_0^2 \int_x^2 x \sqrt{1+y^3} dy dx &= \int_0^2 \int_0^y x \sqrt{1+y^3} dx dy = \int_0^2 \left[\sqrt{1+y^3} \cdot \frac{x^2}{2} \right]_0^y dy \\
 &= \frac{1}{2} \int_0^2 \sqrt{1+y^3} y^2 dy = \left[\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{3} (1+y^3)^{3/2} \right]_0^2 = \frac{1}{9} (27) - \frac{1}{9} (1) = \frac{26}{9}
 \end{aligned}$$

$$\begin{aligned}
 61. \int_0^1 \int_y^1 \sin(x^2) dx dy &= \int_0^1 \int_0^x \sin(x^2) dy dx = \int_0^1 \left[y \sin(x^2) \right]_0^x dx \\
 &= \int_0^1 x \sin(x^2) dx = \left[-\frac{1}{2} \cos(x^2) \right]_0^1 = -\frac{1}{2} \cos 1 + \frac{1}{2} (1) = \frac{1}{2} (1 - \cos 1) \approx 0.2298
 \end{aligned}$$

$$63. \int_0^2 \int_x^{2x} (x^3 + 3y^2) dy dx = \frac{1664}{105} \approx 15.848$$

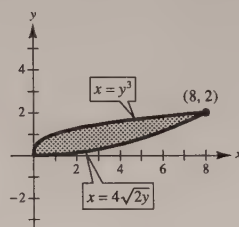
$$65. \int_0^4 \int_0^y \frac{2}{(x+1)(y+1)} dx dy = (\ln 5)^2 \approx 2.590$$

$$67. (a) x = y^3 \Leftrightarrow y = x^{1/3}$$

$$x = 4\sqrt{2}y \Leftrightarrow x^2 = 32y \Leftrightarrow y = \frac{x^2}{32}$$

$$(b) \int_0^8 \int_{x^2/32}^{x^{1/3}} (x^2y - xy^2) dy dx$$

$$(c) \text{ Both integrals equal } 67520/693 \approx 97.43$$



$$69. \int_0^2 \int_0^{4-x^2} e^{xy} dy dx \approx 20.5648$$

$$71. \int_0^{2\pi} \int_0^{1+\cos\theta} 6r^2 \cos\theta dr d\theta = \frac{15\pi}{2}$$

73. An iterated integral is a double integral of a function of two variables. First integrate with respect to one variable while holding the other variable constant. Then integrate with respect to the second variable.

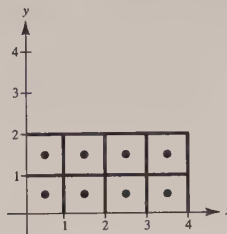
75. The region is a rectangle.

77. True

Section 13.2 Double Integrals and Volume

For Exercise 1–3, $\Delta x_i = \Delta y_i = 1$ and the midpoints of the squares are

$$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{5}{2}, \frac{1}{2}\right), \left(\frac{7}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{5}{2}, \frac{3}{2}\right), \left(\frac{7}{2}, \frac{3}{2}\right).$$



$$1. f(x, y) = x + y$$

$$\sum_{i=1}^8 f(x_i, y_i) \Delta x_i \Delta y_i = 1 + 2 + 3 + 4 + 2 + 3 + 4 + 5 = 24$$

$$\int_0^4 \int_0^2 (x + y) dy dx = \int_0^4 \left[xy + \frac{y^2}{2} \right]_0^2 dx = \int_0^4 (2x + 2) dx = \left[x^2 + 2x \right]_0^4 = 24$$

3. $f(x, y) = x^2 + y^2$

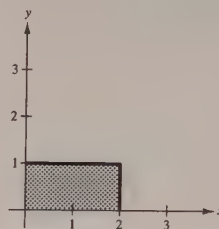
$$\sum_{i=1}^8 f(x_i, y_i) \Delta x_i \Delta y_i = \frac{2}{4} + \frac{10}{4} + \frac{26}{4} + \frac{50}{4} + \frac{10}{4} + \frac{18}{4} + \frac{34}{4} + \frac{58}{4} = 52$$

$$\int_0^4 \int_0^2 (x^2 + y^2) dy dx = \int_0^4 \left[x^2 y + \frac{y^3}{3} \right]_0^2 dx = \int_0^4 \left(2x^2 + \frac{8}{3} \right) dx = \left[\frac{2x^3}{3} + \frac{8x}{3} \right]_0^4 = \frac{160}{3}$$

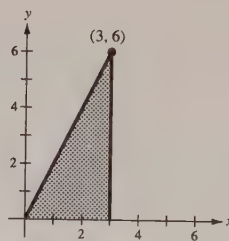
5. $\int_0^4 \int_0^4 f(x, y) dy dx \approx (32 + 31 + 28 + 23) + (31 + 30 + 27 + 22) + (28 + 27 + 24 + 19) + (23 + 22 + 19 + 14)$
 $= 400$

Using the corner of the i th square furthest from the origin, you obtain 272.

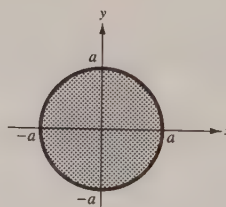
7. $\int_0^2 \int_0^1 (1 + 2x + 2y) dy dx = \int_0^2 \left[y + 2xy + y^2 \right]_0^1 dx$
 $= \int_0^2 (2 + 2x) dx$
 $= \left[2x + x^2 \right]_0^2$
 $= 8$



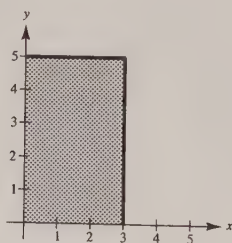
9. $\int_0^6 \int_{y/2}^3 (x + y) dx dy = \int_0^6 \left[\frac{1}{2}x^2 + xy \right]_{y/2}^3 dy$
 $= \int_0^6 \left(\frac{9}{2} + 3y - \frac{5}{8}y^2 \right) dy$
 $= \left[\frac{9}{2}y + \frac{3}{2}y^2 - \frac{5}{24}y^3 \right]_0^6$
 $= 36$



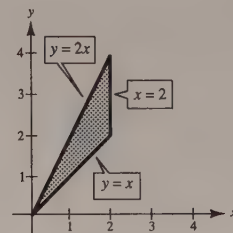
11. $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x + y) dy dx = \int_{-a}^a \left[xy + \frac{1}{2}y^2 \right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$
 $= \int_{-a}^a 2x\sqrt{a^2-x^2} dx$
 $= \left[-\frac{2}{3}(a^2-x^2)^{3/2} \right]_{-a}^a = 0$



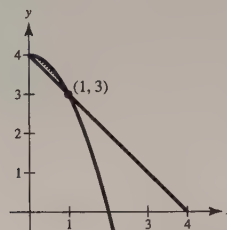
13. $\int_0^5 \int_0^3 xy dx dy = \int_0^5 \int_0^3 xy dy dx$
 $= \int_0^5 \left[\frac{1}{2}xy^2 \right]_0^3 dx$
 $= \frac{25}{2} \int_0^3 x dx$
 $= \left[\frac{25}{4}x^2 \right]_0^3 = \frac{225}{4}$



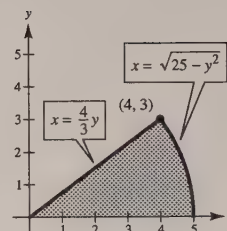
$$\begin{aligned}
 15. \int_0^2 \int_{y/2}^y \frac{y}{x^2 + y^2} dx dy + \int_2^4 \int_{y/2}^2 \frac{y}{x^2 + y^2} dx dy &= \int_0^2 \int_x^{2x} \frac{y}{x^2 + y^2} dy dx \\
 &= \frac{1}{2} \int_0^2 \left[\ln(x^2 + y^2) \right]_x^{2x} dx \\
 &= \frac{1}{2} \int_0^2 (\ln 5x^2 - \ln 2x^2) dx \\
 &= \frac{1}{2} \ln \frac{5}{2} \int_0^2 dx \\
 &= \left[\frac{1}{2} \left(\ln \frac{5}{2} \right) x \right]_0^2 = \ln \frac{5}{2}
 \end{aligned}$$



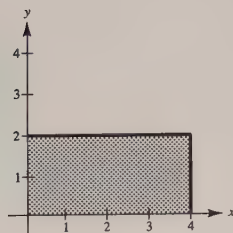
$$\begin{aligned}
 17. \int_3^4 \int_{4-y}^{\sqrt{4-y}} -2y \ln x dx dy &= \int_0^1 \int_{4-x}^{4-x^2} -2y \ln x dy dx \\
 &= - \int_0^1 \left[\ln x \cdot y^2 \right]_{4-x}^{4-x^2} dx \\
 &= - \int_0^1 [\ln x (4 - x^2)^2 - (4 - x)^2] dx \\
 &= \frac{26}{25}
 \end{aligned}$$



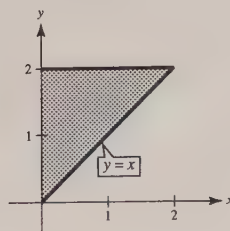
$$\begin{aligned}
 19. \int_0^4 \int_0^{3x/4} x dy dx + \int_4^5 \int_0^{\sqrt{25-x^2}} x dy dx &= \int_0^3 \int_{4y/3}^{\sqrt{25-y^2}} x dx dy \\
 &= \int_0^3 \left[\frac{1}{2} x^2 \right]_{4y/3}^{\sqrt{25-y^2}} dy \\
 &= \frac{25}{18} \int_0^3 (9 - y^2) dy \\
 &= \left[\frac{25}{18} \left(9y - \frac{1}{3} y^3 \right) \right]_0^3 = 25
 \end{aligned}$$



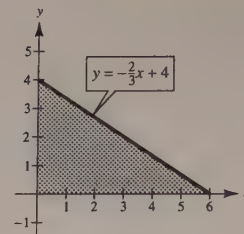
$$\begin{aligned}
 21. \int_0^4 \int_0^2 \frac{y}{2} dy dx &= \int_0^4 \left[\frac{y^2}{4} \right]_0^2 dx \\
 &= \int_0^4 dx = 4
 \end{aligned}$$



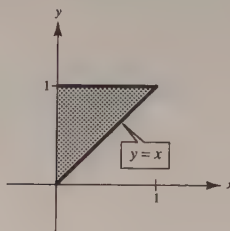
$$\begin{aligned}
 23. \int_0^2 \int_0^y (4 - x - y) dx dy &= \int_0^2 \left[4x - \frac{x^2}{2} - xy \right]_0^y dy \\
 &= \int_0^2 \left(4y - \frac{y^2}{2} - y^2 \right) dy \\
 &= \left[2y^2 - \frac{y^3}{6} - \frac{y^3}{3} \right]_0^2 \\
 &= 8 - \frac{8}{6} - \frac{8}{3} = 4
 \end{aligned}$$



$$\begin{aligned}
 25. \int_0^6 \int_0^{(-2/3)x+4} \left(\frac{12-2x-3y}{4} \right) dy dx &= \int_0^6 \left[\frac{1}{4} \left(12y - 2xy - \frac{3}{2}y^2 \right) \right]_0^{(-2/3)x+4} dx \\
 &= \int_0^6 \left(\frac{1}{6}x^2 - 2x + 6 \right) dx \\
 &= \left[\frac{1}{18}x^3 - x^2 + 6x \right]_0^6 \\
 &= 12
 \end{aligned}$$



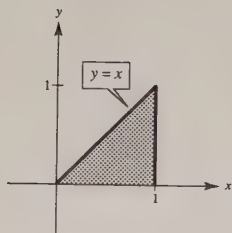
$$\begin{aligned}
 27. \int_0^1 \int_0^y (1-xy) dx dy &= \int_0^1 \left[x - \frac{x^2y}{2} \right]_0^y dy \\
 &= \int_0^1 \left(y - \frac{y^3}{2} \right) dy \\
 &= \left[\frac{y^2}{2} - \frac{y^4}{8} \right]_0^1 \\
 &= \frac{3}{8}
 \end{aligned}$$



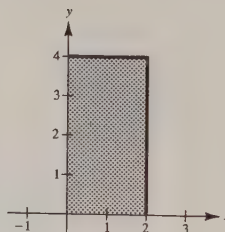
$$29. \int_0^\infty \int_0^\infty \frac{1}{(x+1)^2(y+1)^2} dy dx = \int_0^\infty \left[-\frac{1}{(x+1)^2(y+1)} \right]_0^\infty dx = \int_0^\infty \frac{1}{(x+1)^2} dx = \left[-\frac{1}{x+1} \right]_0^\infty = 1$$

$$31. 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (4-x^2-y^2) dy dx = 8\pi$$

$$\begin{aligned}
 33. V &= \int_0^1 \int_0^x xy dy dx \\
 &= \int_0^1 \left[\frac{1}{2}xy^2 \right]_0^x dx = \frac{1}{2} \int_0^1 x^3 dx \\
 &= \left[\frac{1}{8}x^4 \right]_0^1 = \frac{1}{8}
 \end{aligned}$$

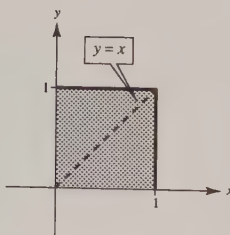


$$\begin{aligned}
 35. V &= \int_0^2 \int_0^4 x^2 dy dx \\
 &= \int_0^2 \left[x^2y \right]_0^4 dx = \int_0^2 4x^2 dx \\
 &= \left[\frac{4x^3}{3} \right]_0^2 = \frac{32}{3}
 \end{aligned}$$

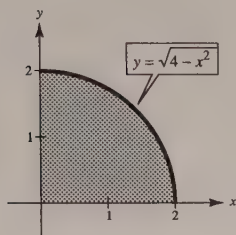


37. Divide the solid into two equal parts.

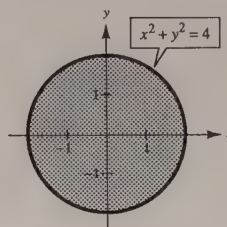
$$\begin{aligned}
 V &= 2 \int_0^1 \int_0^x \sqrt{1-x^2} dy dx \\
 &= 2 \int_0^1 \left[y\sqrt{1-x^2} \right]_0^x dx \\
 &= 2 \int_0^1 x\sqrt{1-x^2} dx \\
 &= \left[-\frac{2}{3}(1-x^2)^{3/2} \right]_0^1 = \frac{2}{3}
 \end{aligned}$$



$$\begin{aligned}
 39. V &= \int_0^2 \int_0^{\sqrt{4-x^2}} (x+y) dy dx \\
 &= \int_0^2 \left[xy + \frac{1}{2}y^2 \right]_0^{\sqrt{4-x^2}} dx \\
 &= \int_0^2 \left(x\sqrt{4-x^2} + 2 - \frac{1}{2}x^2 \right) dx \\
 &= \left[-\frac{1}{3}(4-x^2)^{3/2} + 2x - \frac{1}{6}x^3 \right]_0^2 = \frac{16}{3}
 \end{aligned}$$



$$\begin{aligned}
 41. V &= 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) dy dx \\
 &= 4 \int_0^2 \left[x^2\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2} \right] dx, \quad x = 2 \sin \theta \\
 &= 4 \int_0^{\pi/2} \left(16 \cos^2 \theta - \frac{32}{3} \cos^4 \theta \right) d\theta \\
 &= 4 \left[16 \left(\frac{\pi}{4} \right) - \frac{32}{3} \left(\frac{3\pi}{16} \right) \right] \\
 &= 8\pi
 \end{aligned}$$



$$43. V = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2) dy dx = 8\pi$$

$$45. V = \int_0^2 \int_0^{-0.5x+1} \frac{2}{1+x^2+y^2} dy dx \approx 1.2315$$

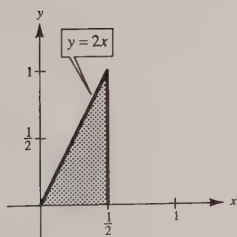
47. f is a continuous function such that $0 \leq f(x, y) \leq 1$ over a region R of area 1. Let $f(m, n)$ = the minimum value of f over R and $f(M, N)$ = the maximum value of f over R . Then

$$f(m, n) \iint_R dA \leq \iint_R f(x, y) dA \leq f(M, N) \iint_R dA.$$

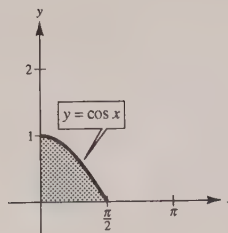
Since $\iint_R dA = 1$ and $0 \leq f(m, n) \leq f(M, N) \leq 1$, we have $0 \leq f(m, n)(1) \leq \iint_R f(x, y) dA \leq f(M, N)(1) \leq 1$.

Therefore, $0 \leq \iint_R f(x, y) dA \leq 1$.

$$\begin{aligned}
 49. \int_0^1 \int_{y/2}^{1/2} e^{-x^2} dx dy &= \int_0^{1/2} \int_0^{2x} e^{-x^2} dy dx \\
 &= \int_0^{1/2} 2xe^{-x^2} dx \\
 &= \left[-e^{-x^2} \right]_0^{1/2} \\
 &= -e^{-1/4} + 1 \\
 &= 1 - e^{-1/4} \approx 0.221
 \end{aligned}$$



$$\begin{aligned}
 51. \int_0^1 \int_0^{\arccos y} \sin x \sqrt{1 + \sin^2 x} dx dy \\
 &= \int_0^{\pi/2} \int_0^{\cos x} \sin x \sqrt{1 + \sin^2 x} dy dx \\
 &= \int_0^{\pi/2} (1 + \sin^2 x)^{1/2} \sin x \cos x dx \\
 &= \left[\frac{1}{2} \cdot \frac{2}{3} (1 + \sin^2 x)^{3/2} \right]_0^{\pi/2} = \frac{1}{3} [2\sqrt{2} - 1]
 \end{aligned}$$



$$53. \text{Average} = \frac{1}{8} \int_0^4 \int_0^2 x \, dy \, dx = \frac{1}{8} \int_0^4 2x \, dx = \left[\frac{x^2}{8} \right]_0^4 = 2$$

$$\begin{aligned} 55. \text{Average} &= \frac{1}{4} \int_0^2 \int_0^2 (x^2 + y^2) \, dx \, dy \\ &= \frac{1}{4} \int_0^2 \left[\frac{x^3}{3} + xy^2 \right]_0^2 dy = \frac{1}{4} \int_0^2 \left(\frac{8}{3} + 2y^2 \right) dy \\ &= \left[\frac{1}{4} \left(\frac{8}{3}y + \frac{2}{3}y^3 \right) \right]_0^2 = \frac{8}{3} \end{aligned}$$

57. See the definition on page 946.

59. The value of $\iint_R f(x, y) \, dA$ would be kB .

$$\begin{aligned} 61. \text{Average} &= \frac{1}{1250} \int_{300}^{325} \int_{200}^{250} 100x^{0.6}y^{0.4} \, dx \, dy \\ &= \frac{1}{1250} \int_{300}^{325} \left[(100y^{0.4}) \frac{x^{1.6}}{1.6} \right]_{200}^{250} dy = \frac{128,844.1}{1250} \int_{300}^{325} y^{0.4} \, dy = 103.0753 \left[\frac{y^{1.4}}{1.4} \right]_{300}^{325} \approx 25,645.24 \end{aligned}$$

63. $f(x, y) \geq 0$ for all (x, y) and

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dA &= \int_0^5 \int_0^2 \frac{1}{10} \, dy \, dx = \int_0^5 \frac{1}{5} \, dx = 1 \\ P(0 \leq x \leq 2, 1 \leq y \leq 2) &= \int_0^2 \int_1^2 \frac{1}{10} \, dy \, dx = \int_0^2 \frac{1}{10} \, dx = \frac{1}{5}. \end{aligned}$$

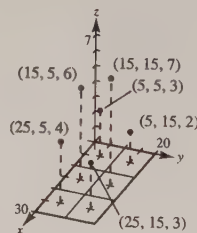
65. $f(x, y) \geq 0$ for all (x, y) and

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dA &= \int_0^3 \int_3^6 \frac{1}{27} (9 - x - y) \, dy \, dx \\ &= \int_0^3 \frac{1}{27} \left[9y - xy - \frac{y^2}{2} \right]_3^6 dx = \int_0^3 \left(\frac{1}{2} - \frac{1}{9}x \right) dx = \left[\frac{x}{2} - \frac{x^2}{18} \right]_0^3 = 1 \\ P(0 \leq x \leq 1, 4 \leq y \leq 6) &= \int_0^1 \int_4^6 \frac{1}{27} (9 - x - y) \, dy \, dx = \int_0^1 \frac{2}{27} (4 - x) \, dx = \frac{7}{27}. \end{aligned}$$

67. Divide the base into six squares, and assume the height at the center of each square is the height of the entire square.

Thus,

$$V \approx (4 + 3 + 6 + 7 + 3 + 2)(100) = 2500m^3.$$



$$69. \int_0^1 \int_0^2 \sin \sqrt{x+y} \, dy \, dx \quad m=4, n=8$$

(a) 1.78435

(b) 1.7879

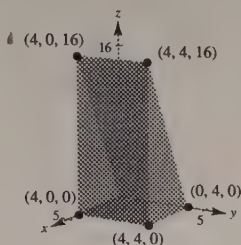
$$71. \int_4^6 \int_0^2 y \cos \sqrt{x} \, dx \, dy \quad m=4, n=8$$

(a) 11.0571

(b) 11.0414

73. $V \approx 125$

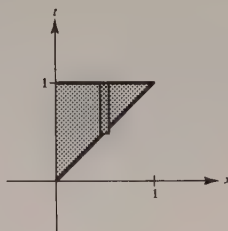
Matches d.



75. False

$$V = 8 \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2} \, dx \, dy$$

$$\begin{aligned} 77. \text{ Average} &= \int_0^1 f(x) \, dx = \int_0^1 \int_1^x e^{t^2} \, dt \, dx = - \int_0^1 \int_x^1 e^{t^2} \, dt \, dx \\ &= - \int_0^1 \int_0^t e^{t^2} \, dx \, dt = - \int_0^1 t e^{t^2} \, dt \\ &= \left[-\frac{1}{2} e^{t^2} \right]_0^1 = -\frac{1}{2} (e - 1) = \frac{1}{2} (1 - e) \end{aligned}$$



Section 13.3 Change of Variables: Polar Coordinates

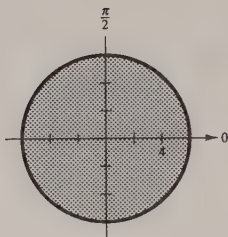
1. Rectangular coordinates

3. Polar coordinates

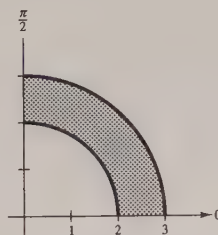
$$5. R = \{(r, \theta): 0 \leq r \leq 8, 0 \leq \theta \leq \pi\}$$

$$7. R = \{(r, \theta): 0 \leq r \leq 3 + 3 \sin \theta, 0 \leq \theta \leq 2\pi\} \text{ Cardioid}$$

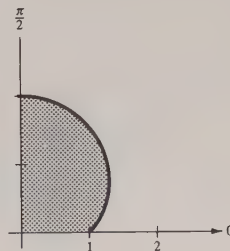
$$\begin{aligned} 9. \int_0^{2\pi} \int_0^6 3r^2 \sin \theta \, dr \, d\theta &= \int_0^{2\pi} \left[r^3 \sin \theta \right]_0^6 d\theta \\ &= \int_0^{2\pi} 216 \sin \theta \, d\theta \\ &= \left[-216 \cos \theta \right]_0^{2\pi} = 0 \end{aligned}$$



$$\begin{aligned} 11. \int_0^{\pi/2} \int_2^3 \sqrt{9-r^2} \, r \, dr \, d\theta &= \int_0^{\pi/2} \left[-\frac{1}{3} (9-r^2)^{3/2} \right]_2^3 d\theta \\ &= \left[\frac{5\sqrt{5}}{3} \theta \right]_0^{\pi/2} \\ &= \frac{5\sqrt{5}\pi}{6} \end{aligned}$$



$$\begin{aligned} 13. \int_0^{\pi/2} \int_0^{1+\sin \theta} \theta r \, dr \, d\theta &= \int_0^{\pi/2} \left[\frac{\theta r^2}{2} \right]_0^{1+\sin \theta} d\theta \\ &= \int_0^{\pi/2} \frac{1}{2} \theta (1 + \sin \theta)^2 d\theta \\ &= \left[\frac{1}{8} \theta^2 + \sin \theta - \theta \cos \theta + \frac{1}{2} \theta \left(-\frac{1}{2} \cos \theta \cdot \sin \theta + \frac{1}{2} \theta \right) + \frac{1}{8} \sin^2 \theta \right]_0^{\pi/2} \\ &= \frac{3}{32} \pi^2 + \frac{9}{8} \end{aligned}$$

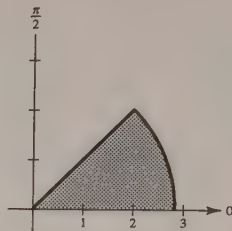


$$15. \int_0^a \int_0^{\sqrt{a^2-y^2}} y \, dx \, dy = \int_0^{\pi/2} \int_0^a r^2 \sin \theta \, dr \, d\theta = \frac{a^3}{3} \int_0^{\pi/2} \sin \theta \, d\theta = \left[\frac{a^3}{3} (-\cos \theta) \right]_0^{\pi/2} = \frac{a^3}{3}$$

$$17. \int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{3/2} \, dy \, dx = \int_0^{\pi/2} \int_0^3 r^4 \, dr \, d\theta = \frac{243}{5} \int_0^{\pi/2} d\theta = \frac{243\pi}{10}$$

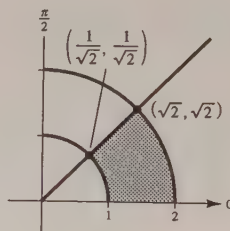
$$19. \int_0^2 \int_0^{\sqrt{2x-x^2}} xy \, dy \, dx = \int_0^{\pi/2} \int_0^{2 \cos \theta} r^3 \cos \theta \sin \theta \, dr \, d\theta = 4 \int_0^{\pi/2} \cos^5 \theta \sin \theta \, d\theta = \left[-\frac{4 \cos^6 \theta}{6} \right]_0^{\pi/2} = \frac{2}{3}$$

$$\begin{aligned} 21. \int_0^2 \int_0^x \sqrt{x^2 + y^2} \, dy \, dx + \int_2^{\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{x^2 + y^2} \, dy \, dx &= \int_0^{\pi/4} \int_0^{2\sqrt{2}} r^2 \, dr \, d\theta \\ &= \int_0^{\pi/4} \frac{16\sqrt{2}}{3} \, d\theta \\ &= \frac{4\sqrt{2}\pi}{3} \end{aligned}$$



$$\begin{aligned} 23. \int_0^2 \int_0^{\sqrt{4-x^2}} (x+y) \, dy \, dx &= \int_0^{\pi/2} \int_0^2 (r \cos \theta + r \sin \theta) r \, dr \, d\theta = \int_0^{\pi/2} \int_0^2 (\cos \theta + \sin \theta) r^2 \, dr \, d\theta \\ &= \frac{8}{3} \int_0^{\pi/2} (\cos \theta + \sin \theta) \, d\theta = \left[\frac{8}{3} (\sin \theta - \cos \theta) \right]_0^{\pi/2} = \frac{16}{3} \end{aligned}$$

$$\begin{aligned} 25. \int_0^{1/\sqrt{2}} \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} \arctan \frac{y}{x} \, dx \, dy + \int_{1/\sqrt{2}}^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \arctan \frac{y}{x} \, dx \, dy \\ &= \int_0^{\pi/4} \int_1^2 \theta r \, dr \, d\theta \\ &= \int_0^{\pi/4} \frac{3}{2} \theta \, d\theta = \left[\frac{3\theta^2}{4} \right]_0^{\pi/4} = \frac{3\pi^2}{64} \end{aligned}$$



$$\begin{aligned} 27. V &= \int_0^{\pi/2} \int_0^1 (r \cos \theta)(r \sin \theta) r \, dr \, d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} \int_0^1 r^3 \sin 2\theta \, dr \, d\theta = \frac{1}{8} \int_0^{\pi/2} \sin 2\theta \, d\theta = \left[-\frac{1}{16} \cos 2\theta \right]_0^{\pi/2} = \frac{1}{8} \end{aligned}$$

$$29. V = \int_0^{2\pi} \int_0^5 r^2 \, dr \, d\theta = \int_0^{2\pi} \frac{125}{3} \, d\theta = \frac{250\pi}{3}$$

$$\begin{aligned} 31. V &= 2 \int_0^{\pi/2} \int_0^{4 \cos \theta} \sqrt{16-r^2} \, r \, dr \, d\theta = 2 \int_0^{\pi/2} \left[-\frac{1}{3} (\sqrt{16-r^2})^3 \right]_0^{4 \cos \theta} d\theta = -\frac{2}{3} \int_0^{\pi/2} (64 \sin^3 \theta - 64) \, d\theta \\ &= \frac{128}{3} \int_0^{\pi/2} [1 - \sin \theta (1 - \cos^2 \theta)] \, d\theta = \frac{128}{3} \left[\theta + \cos \theta - \frac{\cos^3 \theta}{3} \right]_0^{\pi/2} = \frac{64}{9} (3\pi - 4) \end{aligned}$$

$$33. V = \int_0^{2\pi} \int_a^4 \sqrt{16 - r^2} r dr d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(\sqrt{16 - r^2})^3 \right]_a^4 d\theta = \frac{1}{3}(\sqrt{16 - a^2})^3(2\pi)$$

One-half the volume of the hemisphere is $(64\pi)/3$.

$$\frac{2\pi}{3}(16 - a^2)^{3/2} = \frac{64\pi}{3}$$

$$(16 - a^2)^{3/2} = 32$$

$$16 - a^2 = 32^{2/3}$$

$$a^2 = 16 - 32^{2/3} = 16 - 8\sqrt[3]{2}$$

$$a = \sqrt{4(4 - 2\sqrt[3]{2})} = 2\sqrt{4 - 2\sqrt[3]{2}} \approx 2.4332$$

$$\begin{aligned} 35. \text{ Total Volume} = V &= \int_0^{2\pi} \int_0^4 25e^{-r^2/4} r dr d\theta \\ &= \int_0^{2\pi} \left[-50e^{-r^2/4} \right]_0^4 d\theta \\ &= \int_0^{2\pi} -50(e^{-4} - 1) d\theta \\ &= (1 - e^{-4}) 100\pi \approx 308.40524 \end{aligned}$$

Let c be the radius of the hole that is removed.

$$\begin{aligned} \frac{1}{10} V &= \int_0^{2\pi} \int_0^c 25e^{-r^2/4} r dr d\theta = \int_0^{2\pi} \left[-50e^{-r^2/4} \right]_0^c d\theta \\ &= \int_0^{2\pi} -50(e^{-c^2/4} - 1) d\theta \Rightarrow 30.84052 = 100\pi(1 - e^{-c^2/4}) \end{aligned}$$

$$\Rightarrow e^{-c^2/4} = 0.90183$$

$$-\frac{c^2}{4} = -0.10333$$

$$c^2 = 0.41331$$

$$c = 0.6429$$

$$\Rightarrow \text{diameter} = 2c = 1.2858$$

$$37. A = \int_0^\pi \int_0^{6\cos\theta} r dr d\theta = \int_0^\pi 18 \cos^2 \theta d\theta = 9 \int_0^\pi (1 + \cos 2\theta) d\theta = \left[9\left(\theta + \frac{1}{2} \sin 2\theta\right) \right]_0^\pi = 9\pi$$

$$\begin{aligned} 39. \int_0^{2\pi} \int_0^{1+\cos\theta} r dr d\theta &= \frac{1}{2} \int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2} \right) d\theta = \frac{1}{2} \left[\theta + 2\sin\theta + \frac{1}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) \right]_0^{2\pi} = \frac{3\pi}{2} \end{aligned}$$

$$41. 3 \int_0^{\pi/3} \int_0^{2\sin 3\theta} r dr d\theta = \frac{3}{2} \int_0^{\pi/3} 4 \sin^2 3\theta d\theta = 3 \int_0^{\pi/3} (1 - \cos 6\theta) d\theta = 3 \left[\theta - \frac{1}{6} \sin 6\theta \right]_0^{\pi/3} = \pi$$

43. Let R be a region bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$, and the lines $\theta = a$ and $\theta = b$.

When using polar coordinates to evaluate a double integral over R , R can be partitioned into small polar sectors.

45. r -simple regions have fixed bounds for θ .

θ -simple regions have fixed bounds for r .

47. You would need to insert a factor of r because of the $r dr d\theta$ nature of polar coordinate integrals. The plane regions would be sectors of circles.

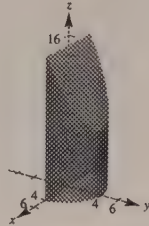
$$49. \int_{\pi/4}^{\pi/2} \int_0^5 r \sqrt{1+r^3} \sin \sqrt{\theta} dr d\theta \approx 56.051$$

[Note: This integral equals $\left(\int_{\pi/4}^{\pi/2} \sin \sqrt{\theta} d\theta \right) \left(\int_0^5 r \sqrt{1+r^3} dr \right)$]

51. Volume = base \times height

$$\approx 8\pi \times 12 \approx 300$$

Answer (c)



53. False

Let $f(r, \theta) = r - 1$ where R is the circular sector $0 \leq r \leq 6$ and $0 \leq \theta \leq \pi$. Then,

$$\int_R \int (r - 1) dA > 0 \quad \text{but} \quad r - 1 \neq 0 \text{ for all } r.$$

$$55. (a) I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dA = 4 \int_0^{\pi/2} \int_0^{\infty} e^{-r^2/2} r dr d\theta = 4 \int_0^{\pi/2} \left[-e^{-r^2/2} \right]_0^{\infty} d\theta = 4 \int_0^{\pi/2} d\theta = 2\pi$$

(b) Therefore, $I = \sqrt{2\pi}$.

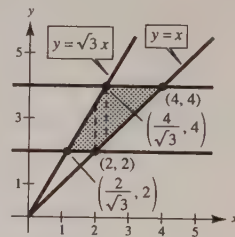
$$57. \int_{-7}^7 \int_{-\sqrt{49-x^2}}^{\sqrt{49-x^2}} 4000e^{-0.01(x^2+y^2)} dy dx = \int_0^{2\pi} \int_0^7 4000e^{-0.01r^2} r dr d\theta = \int_0^{2\pi} \left[-200,000e^{-0.01r^2} \right]_0^7 d\theta$$

$$= 2\pi(-200,000)(e^{-0.49} - 1) = 400,000\pi(1 - e^{-0.49}) \approx 486,788$$

$$59. (a) \int_2^4 \int_{y/\sqrt{3}}^y f dx dy$$

$$(b) \int_{2/\sqrt{3}}^2 \int_2^{\sqrt{3}x} f dy dx + \int_2^{4/\sqrt{3}} \int_x^{\sqrt{3}x} f dy dx + \int_{4/\sqrt{3}}^4 \int_x^4 f dy dx$$

$$(c) \int_{\pi/4}^{\pi/3} \int_{2 \csc \theta}^{4 \csc \theta} f r dr d\theta$$



$$61. A = \frac{\Delta \theta r_2^2}{2} - \frac{\Delta \theta r_1^2}{2} = \Delta \theta \left(\frac{r_1 + r_2}{2} \right) (r_2 - r_1) = r \Delta r \Delta \theta$$

Section 13.4 Center of Mass and Moments of Inertia

$$1. m = \int_0^4 \int_0^3 xy dy dx = \int_0^4 \left[\frac{xy^2}{2} \right]_0^3 dx = \int_0^4 \frac{9}{2} x dx = \left[\frac{9x^2}{4} \right]_0^4 = 36$$

$$3. m = \int_0^{\pi/2} \int_0^2 (r \cos \theta)(r \sin \theta) r dr d\theta = \int_0^{\pi/2} \int_0^2 \cos \theta \sin \theta \cdot r^3 dr d\theta$$

$$= \int_0^{\pi/2} 4 \cos \theta \sin \theta d\theta$$

$$= \left[4 \frac{\sin^2 \theta}{2} \right]_0^{\pi/2} = 2$$

$$5. (a) \quad m = \int_0^a \int_0^b k \, dy \, dx = kab$$

$$M_x = \int_0^a \int_0^b ky \, dy \, dx = \frac{kab^2}{2}$$

$$M_y = \int_0^a \int_0^b kx \, dy \, dx = \frac{ka^2b}{2}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^2b/2}{kab} = \frac{a}{2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kab^2/2}{kab} = \frac{b}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{b}{2} \right)$$

$$(b) \quad m = \int_0^a \int_0^b ky \, dy \, dx = \frac{kab^2}{2}$$

$$M_x = \int_0^a \int_0^b ky^2 \, dy \, dx = \frac{kab^3}{3}$$

$$M_y = \int_0^a \int_0^b kxy \, dy \, dx = \frac{ka^2b^2}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^2b^2/4}{kab^2/2} = \frac{a}{2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kab^3/3}{kab^2/2} = \frac{2}{3}b$$

$$(\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{2}{3}b \right)$$

$$(c) \quad m = \int_0^a \int_0^b kx \, dy \, dx = k \int_0^a xb \, dx = \frac{1}{2}ka^2b$$

$$M_x = \int_0^a \int_0^b kxy \, dy \, dx = \frac{ka^2b^2}{4}$$

$$M_y = \int_0^a \int_0^b kx^2 \, dy \, dx = \frac{ka^3b}{3}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^3b/3}{ka^2b/2} = \frac{2}{3}a$$

$$\bar{y} = \frac{M_x}{m} = \frac{ka^2b^2/4}{ka^2b/2} = \frac{b}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{2}{3}a, \frac{b}{2} \right)$$

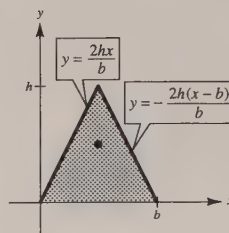
$$7. (a) \quad m = \frac{k}{2}bh$$

$$\bar{x} = \frac{b}{2} \text{ by symmetry}$$

$$\begin{aligned} M_x &= \int_0^{b/2} \int_0^{2hx/b} ky \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} ky \, dy \, dx \\ &= \frac{kbh^2}{12} + \frac{kbh^2}{12} = \frac{kbh^2}{6} \end{aligned}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kbh^2/6}{kbh/2} = \frac{h}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{2}, \frac{h}{3} \right)$$



—CONTINUED—

7. —CONTINUED—

$$(b) \quad m = \int_0^{b/2} \int_0^{2hx/b} ky \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} ky \, dy \, dx = \frac{kbh^2}{6}$$

$$M_x = \int_0^{b/2} \int_0^{2hx/b} ky^2 \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} ky^2 \, dy \, dx = \frac{kbh^3}{12}$$

$$M_y = \int_0^{b/2} \int_0^{2hx/b} kxy \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} kxy \, dy \, dx = \frac{kb^2h^2}{12}$$

$$\bar{x} = \frac{M_y}{m} = \frac{kb^2h^2/12}{kbh^2/6} = \frac{b}{2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kbh^3/12}{kbh^2/6} = \frac{h}{2}$$

$$(c) \quad m = \int_0^{b/2} \int_0^{2hx/b} kx \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} kx \, dy \, dx$$

$$= \frac{1}{12}kb^2h + \frac{1}{6}kb^2h = \frac{1}{4}kb^2h$$

$$M_x = \int_0^{b/2} \int_0^{2hx/b} kxy \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} kxy \, dy \, dx$$

$$= \frac{1}{32}kh^2b^2 + \frac{5}{96}kh^2b^2 = \frac{1}{12}kh^2b^2$$

$$M_y = \int_0^{b/2} \int_0^{2hx/b} kx^2 \, dy \, dx + \int_{b/2}^b \int_0^{-2h(x-b)/b} kx^2 \, dy \, dx$$

$$= \frac{1}{32}kb^3h + \frac{11}{96}kb^3h = \frac{7}{48}kb^3h$$

$$\bar{x} = \frac{M_y}{m} = \frac{7kb^3h/48}{kb^2h/4} = \frac{7}{12}b$$

$$\bar{y} = \frac{M_x}{m} = \frac{kh^2b^2/12}{kb^2h/4} = \frac{h}{3}$$

$$9. (a) \quad \text{The } x\text{-coordinate changes by 5: } (\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{b}{2}\right)$$

$$(b) \quad \text{The } x\text{-coordinate changes by 5: } (\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{2b}{3}\right)$$

$$(c) \quad m = \int_5^{a+5} \int_0^b kx \, dy \, dx = \frac{1}{2}k(a+5)^2b - \frac{25}{2}kb$$

$$M_x = \int_5^{a+5} \int_0^b kxy \, dy \, dx = \frac{1}{4}k(a+5)^2b^2 - \frac{25}{4}kb^2$$

$$M_y = \int_5^{a+5} \int_0^b kx^2 \, dy \, dx = \frac{1}{3}k(a+5)^3b - \frac{125}{3}kb$$

$$\bar{x} = \frac{M_y}{m} = \frac{2(a^2 + 15a + 75)}{3(a+10)}$$

$$\bar{y} = \frac{M_x}{m} = \frac{b}{2}$$

$$11. (a) \quad \bar{x} = 0 \text{ by symmetry}$$

$$m = \frac{\pi a^2 k}{2}$$

$$M_x = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} yk \, dy \, dx = \frac{2a^3k}{3}$$

$$\bar{y} = \frac{M_x}{m} = \frac{2a^3k}{3} \cdot \frac{2}{\pi a^2k} = \frac{4a}{3\pi}$$

$$(b) \quad m = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} k(a-y)y \, dy \, dx = \frac{a^4k}{24}(16-3\pi)$$

$$M_x = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} k(a-y)y^2 \, dy \, dx = \frac{a^5k}{120}(15\pi-32)$$

$$M_y = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} kx(a-y)y \, dy \, dx = 0$$

$$\bar{x} = \frac{M_y}{m} = 0$$

$$\bar{y} = \frac{M_x}{m} = \frac{a}{5} \left[\frac{15\pi-32}{16-3\pi} \right]$$

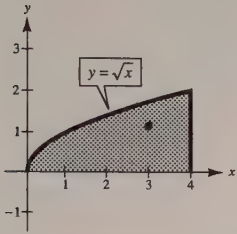
$$13. \quad m = \int_0^4 \int_0^{\sqrt{x}} kxy \, dy \, dx = \frac{32k}{3}$$

$$M_x = \int_0^4 \int_0^{\sqrt{x}} kxy^2 \, dy \, dx = \frac{256k}{21}$$

$$M_y = \int_0^4 \int_0^{\sqrt{x}} kx^2y \, dy \, dx = 32k$$

$$\bar{x} = \frac{M_y}{m} = \frac{32k}{\frac{32k}{3}} = 3$$

$$\bar{y} = \frac{M_x}{m} = \frac{256k}{\frac{32k}{3}} = \frac{8}{7}$$

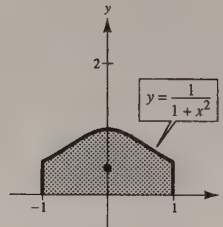


$$15. \quad \bar{x} = 0 \text{ by symmetry}$$

$$m = \int_{-1}^1 \int_0^{1/(1+x^2)} k \, dy \, dx = \frac{k\pi}{2}$$

$$M_x = \int_{-1}^1 \int_0^{1/(1+x^2)} ky \, dy \, dx = \frac{k}{8}(2 + \pi)$$

$$\bar{y} = \frac{M_x}{m} = \frac{k}{8}(2 + \pi) \cdot \frac{2}{k\pi} = \frac{2 + \pi}{4\pi}$$

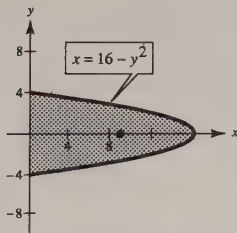


$$17. \quad \bar{y} = 0 \text{ by symmetry}$$

$$m = \int_{-4}^4 \int_0^{16-y^2} kx \, dx \, dy = \frac{8192k}{15}$$

$$M_y = \int_{-4}^4 \int_0^{16-y^2} kx^2 \, dx \, dy = \frac{524,288k}{105}$$

$$\bar{x} = \frac{M_y}{m} = \frac{524,288k}{\frac{8192k}{15}} = \frac{64}{7}$$

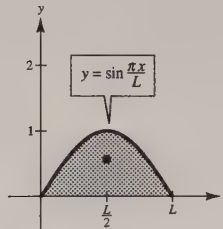


$$19. \quad \bar{x} = \frac{L}{2} \text{ by symmetry}$$

$$m = \int_0^L \int_0^{\sin \pi x/L} ky \, dy \, dx = \frac{kL}{4}$$

$$M_x = \int_0^L \int_0^{\sin \pi x/L} ky^2 \, dy \, dx = \frac{4kL}{9\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{4kL}{9\pi} \cdot \frac{4}{kL} = \frac{16}{9\pi}$$



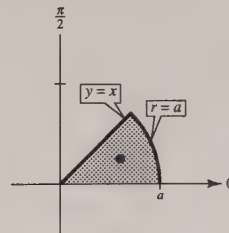
$$21. \quad m = \frac{\pi a^2 k}{8}$$

$$M_x = \int_R \int k y \, dA = \int_0^{\pi/4} \int_0^a kr^2 \sin \theta \, dr \, d\theta = \frac{ka^3(2 - \sqrt{2})}{6}$$

$$M_y = \int_R \int k x \, dA = \int_0^{\pi/4} \int_0^a kr^2 \cos \theta \, dr \, d\theta = \frac{ka^3\sqrt{2}}{6}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^3\sqrt{2}}{\frac{\pi a^2 k}{8}} = \frac{4a\sqrt{2}}{3\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{ka^3(2 - \sqrt{2})}{\frac{\pi a^2 k}{8}} = \frac{4a(2 - \sqrt{2})}{3\pi}$$



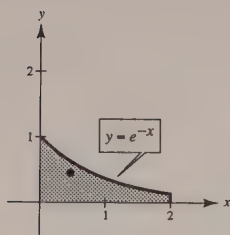
$$23. \quad m = \int_0^2 \int_0^{e^{-x}} ky \, dy \, dx = \frac{k}{4}(1 - e^{-4})$$

$$M_x = \int_0^2 \int_0^{e^{-x}} ky^2 \, dy \, dx = \frac{k}{9}(1 - e^{-6})$$

$$M_y = \int_0^2 \int_0^{e^{-x}} kxy \, dy \, dx = \frac{k(1 - 5e^{-4})}{8}$$

$$\bar{x} = \frac{M_y}{m} = \frac{k(e^4 - 5)}{8e^4} \cdot \frac{4e^4}{k(e^4 - 1)} = \frac{e^4 - 5}{2(e^4 - 1)} \approx 0.46$$

$$\bar{y} = \frac{M_x}{m} = \frac{k(e^6 - 1)}{9e^6} \cdot \frac{4e^4}{k(e^4 - 1)} = \frac{4}{9} \left[\frac{e^6 - 1}{e^6 - e^2} \right] \approx 0.45$$



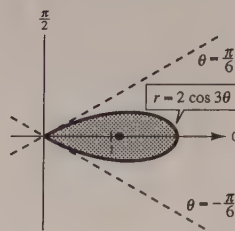
$$25. \quad \bar{y} = 0 \text{ by symmetry}$$

$$m = \iint_R k \, dA = \int_{-\pi/6}^{\pi/6} \int_0^{2 \cos 3\theta} kr \, dr \, d\theta = \frac{k\pi}{3}$$

$$M_y = \iint_R kx \, dA$$

$$= \int_{-\pi/6}^{\pi/6} \int_0^{2 \cos 3\theta} kr^2 \cos \theta \, dr \, d\theta = \frac{27\sqrt{3}}{40}k \approx 1.17k$$

$$\bar{x} = \frac{M_y}{m} = \frac{81\sqrt{3}}{40\pi} \approx 1.12$$



$$27. \quad m = bh$$

$$I_x = \int_0^b \int_0^h y^2 \, dy \, dx = \frac{bh^3}{3}$$

$$I_y = \int_0^b \int_0^h x^2 \, dy \, dx = \frac{b^3h}{3}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{b^3h}{3} \cdot \frac{1}{bh}} = \sqrt{\frac{b^2}{3}} = \frac{b}{\sqrt{3}} = \frac{\sqrt{3}}{3}b$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{bh^3}{3} \cdot \frac{1}{bh}} = \sqrt{\frac{h^2}{3}} = \frac{h}{\sqrt{3}} = \frac{\sqrt{3}}{3}h$$

$$29. \quad m = \pi a^2$$

$$I_x = \iint_R y^2 \, dA = \int_0^{2\pi} \int_0^a r^3 \sin^2 \theta \, dr \, d\theta = \frac{a^4\pi}{4}$$

$$I_y = \iint_R x^2 \, dA = \int_0^{2\pi} \int_0^a r^3 \cos^2 \theta \, dr \, d\theta = \frac{a^4\pi}{4}$$

$$I_0 = I_x + I_y = \frac{a^4\pi}{4} + \frac{a^4\pi}{4} = \frac{a^4\pi}{2}$$

$$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{a^4\pi}{4} \cdot \frac{1}{\pi a^2}} = \frac{a}{2}$$

$$31. \quad m = \frac{\pi a^2}{4}$$

$$I_x = \iint_R y^2 \, dA = \int_0^{\pi/2} \int_0^a r^3 \sin^2 \theta \, dr \, d\theta = \frac{\pi a^4}{16}$$

$$I_y = \iint_R x^2 \, dA = \int_0^{\pi/2} \int_0^a r^3 \cos^2 \theta \, dr \, d\theta = \frac{\pi a^4}{16}$$

$$I_0 = I_x + I_y = \frac{\pi a^4}{16} + \frac{\pi a^4}{16} = \frac{\pi a^4}{8}$$

$$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{\pi a^4}{16} \cdot \frac{4}{\pi a^2}} = \frac{a}{2}$$

$$33. \quad \rho = ky$$

$$m = k \int_0^a \int_0^b y \, dy \, dx = \frac{kab^2}{2}$$

$$I_x = k \int_0^a \int_0^b y^3 \, dy \, dx = \frac{kab^4}{4}$$

$$I_y = k \int_0^a \int_0^b x^2 y \, dy \, dx = \frac{ka^3b^2}{6}$$

$$I_0 = I_x + I_y = \frac{3kab^4 + 2kb^2a^3}{12}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{ka^3b^2/6}{kab^2/2}} = \sqrt{\frac{a^2}{3}} = \frac{a}{\sqrt{3}} = \frac{\sqrt{3}}{3}a$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{kab^4/4}{kab^2/2}} = \sqrt{\frac{b^2}{2}} = \frac{b}{\sqrt{2}} = \frac{\sqrt{2}}{2}b$$

35. $\rho = kx$

$$m = k \int_0^2 \int_0^{4-x^2} x \, dy \, dx = 4k$$

$$I_x = k \int_0^2 \int_0^{4-x^2} xy^2 \, dy \, dx = \frac{32k}{3}$$

$$I_y = k \int_0^2 \int_0^{4-x^2} x^3 \, dy \, dx = \frac{16k}{3}$$

$$I_0 = I_x + I_y = 16k$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{16k/3}{4k}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32k/3}{4k}} = \sqrt{\frac{8}{3}} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

39. $\rho = kx$

$$m = \int_0^1 \int_{x^2}^{\sqrt{x}} kx \, dy \, dx = \frac{3k}{20}$$

$$I_x = \int_0^1 \int_{x^2}^{\sqrt{x}} kxy^2 \, dy \, dx = \frac{3k}{56}$$

$$I_y = \int_0^1 \int_{x^2}^{\sqrt{x}} kx^3 \, dy \, dx = \frac{k}{18}$$

$$I_0 = I_x + I_y = \frac{55k}{504}$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{k/18}{3k/20}} = \frac{\sqrt{30}}{9}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{3k/56}{3k/20}} = \frac{\sqrt{70}}{14}$$

41. $I = 2k \int_{-b}^b \int_0^{\sqrt{b^2-x^2}} (x-a)^2 \, dy \, dx = 2k \int_{-b}^b (x-a)^2 \sqrt{b^2-x^2} \, dx$

$$= 2k \left[\int_{-b}^b x^2 \sqrt{b^2-x^2} \, dx - 2a \int_{-b}^b x \sqrt{b^2-x^2} \, dx + a^2 \int_{-b}^b \sqrt{b^2-x^2} \, dx \right]$$

$$= 2k \left[\frac{\pi b^4}{8} + 0 + \frac{\pi a^2 b^2}{2} \right] = \frac{k\pi b^2}{4} (b^2 + 4a^2)$$

37. $\rho = kxy$

$$m = \int_0^4 \int_0^{\sqrt{x}} kxy \, dy \, dx = \frac{32k}{3}$$

$$I_x = \int_0^4 \int_0^{\sqrt{x}} kxy^3 \, dy \, dx = 16k$$

$$I_y = \int_0^4 \int_0^{\sqrt{x}} kx^3 y \, dy \, dx = \frac{512k}{5}$$

$$I_0 = I_x + I_y = \frac{592k}{5}$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k/5}{32k}} = \sqrt{\frac{48}{5}} = \frac{4\sqrt{15}}{5}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16k}{32k}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$$

43. $I = \int_0^4 \int_0^{\sqrt{x}} kx(x-6)^2 \, dy \, dx = \int_0^4 kx\sqrt{x}(x^2 - 12x + 36) \, dx = k \left[\frac{2}{9}x^{9/2} - \frac{24}{7}x^{7/2} + \frac{72}{5}x^{5/2} \right]_0^4 = \frac{42,752k}{315}$

$$\begin{aligned}
45. \quad I &= \int_0^a \int_0^{\sqrt{a^2-x^2}} k(a-y)(y-a)^2 dy dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(a-y)^3 dy dx = \int_0^a \left[-\frac{k}{4}(a-y)^4 \right]_0^{\sqrt{a^2-x^2}} dx \\
&= -\frac{k}{4} \int_0^a \left[a^4 - 4a^3y + 6a^2y^2 - 4ay^3 + y^4 \right]_0^{\sqrt{a^2-x^2}} dx \\
&= -\frac{k}{4} \int_0^a \left[a^4 - 4a^3\sqrt{a^2-x^2} + 6a^2(a^2-x^2) - 4a(a^2-x^2)\sqrt{a^2-x^2} + (a^4 - 2a^2x^2 + x^4) - a^4 \right] dx \\
&= -\frac{k}{4} \int_0^a \left[7a^4 - 8a^2x^2 + x^4 - 8a^3\sqrt{a^2-x^2} + 4ax^2\sqrt{a^2-x^2} \right] dx \\
&= -\frac{k}{4} \left[7a^4x - \frac{8a^2}{3}x^3 + \frac{x^5}{5} - 4a^3 \left(x\sqrt{a^2-x^2} + a^2 \arcsin \frac{x}{a} \right) + \frac{a}{2} \left(x(2x^2 - a^2)\sqrt{a^2-x^2} + a^4 \arcsin \frac{x}{a} \right) \right]_0^a \\
&= -\frac{k}{4} \left(7a^5 - \frac{8}{3}a^5 + \frac{1}{5}a^5 - 2a^5\pi + \frac{1}{4}a^5\pi \right) = a^5k \left(\frac{7\pi}{16} - \frac{17}{15} \right)
\end{aligned}$$

47. $\rho(x, y) = ky$. \bar{y} will increase

49. $\rho(x, y) = kxy$.

Both \bar{x} and \bar{y} will increase

51. Let $\rho(x, y)$ be a continuous density function on the planar lamina R .

The movements of mass with respect to the x - and y -axes are

$$M_x = \iint_R y \rho(x, y) dA \text{ and } M_y = \iint_R x \rho(x, y) dA.$$

If m is the mass of the lamina, then the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right).$$

53. See the definition on page 968

55. $\bar{y} = \frac{L}{2}$, $A = bL$, $h = \frac{L}{2}$

$$\begin{aligned}
I_{\bar{y}} &= \int_0^b \int_0^L \left(y - \frac{L}{2} \right)^2 dy dx \\
&= \int_0^b \left[\frac{[y - (L/2)]^3}{3} \right]_0^L dx = \frac{L^3b}{12} \\
y_a &= \bar{y} - \frac{I_{\bar{y}}}{hA} = \frac{L}{2} - \frac{L^3b/12}{(L/2)(bL)} = \frac{L}{3}
\end{aligned}$$

57. $\bar{y} = \frac{2L}{3}$, $A = \frac{bL}{2}$, $h = \frac{L}{3}$

$$\begin{aligned}
I_{\bar{y}} &= 2 \int_0^{b/2} \int_{2Lx/b}^L \left(y - \frac{2L}{3} \right)^2 dy dx \\
&= \frac{2}{3} \int_0^{b/2} \left[\left(y - \frac{2L}{3} \right)^3 \right]_{2Lx/b}^L dx \\
&= \frac{2}{3} \int_0^{b/2} \left[\frac{L^3}{27} - \left(\frac{2Lx}{b} - \frac{2L}{3} \right)^3 \right] dx \\
&= \frac{2}{3} \left[\frac{L^3x}{27} - \frac{b}{8L} \left(\frac{2Lx}{b} - \frac{2L}{3} \right)^4 \right]_0^{b/2} = \frac{L^3b}{36} \\
y_a &= \frac{2L}{3} - \frac{L^3b/36}{L^2b/6} = \frac{L}{2}
\end{aligned}$$

Section 13.5 Surface Area

1. $f(x, y) = 2x + 2y$

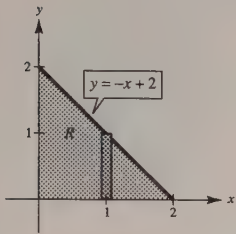
 R = triangle with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$

$f_x = 2, f_y = 2$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = 3$

$$S = \int_0^2 \int_0^{2-x} 3 \, dy \, dx = 3 \int_0^2 (2-x) \, dx$$

$$= \left[3\left(2x - \frac{x^2}{2}\right) \right]_0^2 = 6$$



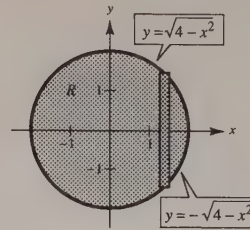
3. $f(x, y) = 8 + 2x + 2y$

$R = \{(x, y): x^2 + y^2 \leq 4\}$

$f_x = 2, f_y = 2$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = 3$

$$S = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 3 \, dy \, dx = \int_0^{2\pi} \int_0^2 3r \, dr \, d\theta = 12\pi$$



5. $f(x, y) = 9 - x^2$

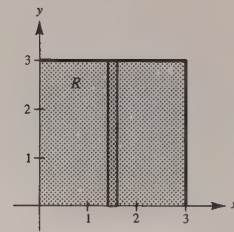
 R = square with vertices $(0, 0)$, $(3, 0)$, $(0, 3)$, $(3, 3)$

$f_x = -2x, f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2}$

$$S = \int_0^3 \int_0^3 \sqrt{1 + 4x^2} \, dy \, dx = \int_0^3 3\sqrt{1 + 4x^2} \, dx$$

$$= \left[\frac{3}{4}(2x\sqrt{1 + 4x^2} + \ln|2x + \sqrt{1 + 4x^2}|) \right]_0^3 = \frac{3}{4}(6\sqrt{37} + \ln|6 + \sqrt{37}|)$$



7. $f(x, y) = 2 + x^{3/2}$

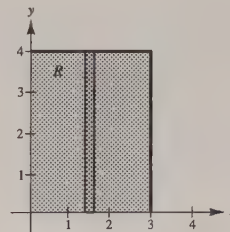
 R = rectangle with vertices $(0, 0)$, $(0, 4)$, $(3, 4)$, $(3, 0)$

$f_x = \frac{3}{2}x^{1/2}, f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \left(\frac{9}{4}\right)x} = \frac{\sqrt{4 + 9x}}{2}$

$$S = \int_0^3 \int_0^4 \frac{\sqrt{4 + 9x}}{2} \, dy \, dx = \int_0^3 4\left(\frac{\sqrt{4 + 9x}}{2}\right) \, dx$$

$$= \left[\frac{4}{27}(4 + 9x)^{3/2} \right]_0^3 = \frac{4}{27}(31\sqrt{31} - 8)$$



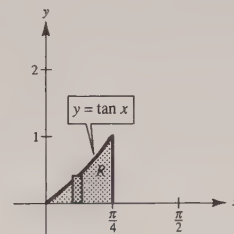
9. $f(x, y) = \ln|\sec x|$

$R = \left\{ (x, y): 0 \leq x \leq \frac{\pi}{4}, 0 \leq y \leq \tan x \right\}$

$f_x = \tan x, f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \tan^2 x} = \sec x$

$$S = \int_0^{\pi/4} \int_0^{\tan x} \sec x \, dy \, dx = \int_0^{\pi/4} \sec x \tan x \, dx = \left[\sec x \right]_0^{\pi/4} = \sqrt{2} - 1$$



11. $f(x, y) = \sqrt{x^2 + y^2}$

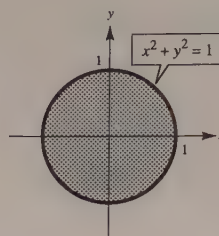
$$R = \{(x, y): 0 \leq f(x, y) \leq 1\}$$

$$0 \leq \sqrt{x^2 + y^2} \leq 1, x^2 + y^2 \leq 1$$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}}, f_y = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{2}$$

$$S = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{2} \, dy \, dx = \int_0^{2\pi} \int_0^1 \sqrt{2} \, r \, dr \, d\theta = \sqrt{2}\pi$$



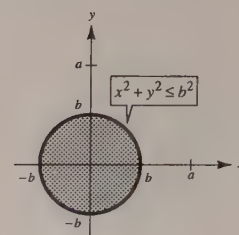
13. $f(x, y) = \sqrt{a^2 - x^2 - y^2}$

$$R = \{(x, y): x^2 + y^2 \leq b^2, b < a\}$$

$$f_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}, f_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$$

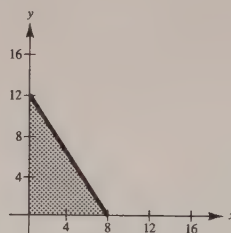
$$S = \int_{-b}^b \int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}} \, dy \, dx = \int_0^{2\pi} \int_0^b \frac{a}{\sqrt{a^2 - r^2}} \, r \, dr \, d\theta = 2\pi a(a - \sqrt{a^2 - b^2})$$



15. $z = 24 - 3x - 2y$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$$

$$S = \int_0^8 \int_0^{-(3/2)x+12} \sqrt{14} \, dy \, dx = 48\sqrt{14}$$

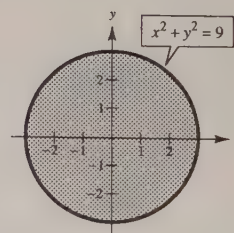


17. $z = \sqrt{25 - x^2 - y^2}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{25 - x^2 - y^2} + \frac{y^2}{25 - x^2 - y^2}} = \frac{5}{\sqrt{25 - x^2 - y^2}}$$

$$S = 2 \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{5}{\sqrt{25 - (x^2 + y^2)}} \, dy \, dx$$

$$= 2 \int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25 - r^2}} \, r \, dr \, d\theta = 20\pi$$

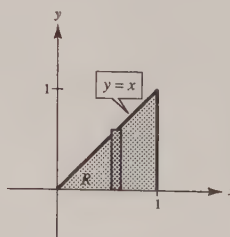


19. $f(x, y) = 2y + x^2$

$$R = \text{triangle with vertices } (0, 0), (1, 0), (1, 1)$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{5 + 4x^2}$$

$$S = \int_0^1 \int_0^x \sqrt{5 + 4x^2} \, dy \, dx = \frac{1}{12}(27 - 5\sqrt{5})$$



21. $f(x, y) = 4 - x^2 - y^2$

$$R = \{(x, y): 0 \leq f(x, y)\}$$

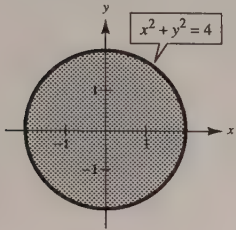
$$0 \leq 4 - x^2 - y^2, x^2 + y^2 \leq 4$$

$$f_x = -2x, f_y = -2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

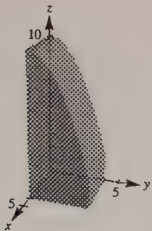
$$S = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + 4x^2 + 4y^2} dy dx$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} r dr d\theta = \frac{(17\sqrt{17} - 1)\pi}{6}$$



25. Surface area $> (4) \cdot (6) = 24$.

Matches (e)



29. $f(x, y) = x^3 - 3xy + y^3$

$$R = \text{square with vertices } (1, 1), (-1, 1), (-1, -1), (1, -1)$$

$$f_x = 3x^2 - 3y = 3(x^2 - y), f_y = -3x + 3y^2 = 3(y^2 - x)$$

$$S = \int_{-1}^1 \int_{-1}^1 \sqrt{1 + 9(x^2 - y)^2 + 9(y^2 - x)^2} dy dx$$

33. $f(x, y) = e^{xy}$

$$R = \{(x, y): 0 \leq x \leq 4, 0 \leq y \leq 10\}$$

$$f_x = ye^{xy}, f_y = xe^{xy}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2 e^{2xy} + x^2 e^{2xy}} = \sqrt{1 + e^{2xy}(x^2 + y^2)}$$

$$S = \int_0^4 \int_0^{10} \sqrt{1 + e^{2xy}(x^2 + y^2)} dy dx$$

23. $f(x, y) = 4 - x^2 - y^2$

$$R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

$$f_x = -2x, f_y = -2y$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$S = \int_0^1 \int_0^1 \sqrt{1 + 4x^2 + 4y^2} dy dx \approx 1.8616$$

27. $f(x, y) = e^x$

$$R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

$$f_x = e^x, f_y = 0$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + e^{2x}}$$

$$S = \int_0^1 \int_0^1 \sqrt{1 + e^{2x}} dy dx$$

$$= \int_0^1 \sqrt{1 + e^{2x}} \approx 2.0035$$

31. $f(x, y) = e^{-x} \sin y$

$$f_x = -e^{-x} \sin y, f_y = e^{-x} \cos y$$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + e^{-2x} \sin^2 y + e^{-2x} \cos^2 y}$$

$$= \sqrt{1 + e^{-2x}}$$

$$S = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + e^{-2x}} dy dx$$

35. See the definition on page 972.

$$37. f(x, y) = \sqrt{1 - x^2}, f_x = \frac{-x}{\sqrt{1 - x^2}}, f_y = 0$$

$$\begin{aligned} S &= \iint_R \sqrt{1 + f_x^2 + f_y^2} \, dA \\ &= 16 \int_0^1 \int_0^x \frac{1}{\sqrt{1 - x^2}} \, dy \, dx \\ &= 16 \int_0^1 \frac{x}{\sqrt{1 - x^2}} \, dx = \left[-16(1 - x^2)^{1/2} \right]_0^1 = 16 \end{aligned}$$

$$39. (a) V = \int_0^{50} \int_0^{\sqrt{50^2 - x^2}} \left(20 + \frac{xy}{100} - \frac{x+y}{5} \right) dy \, dx$$

$$\begin{aligned} &= \int_0^{50} \left[20\sqrt{50^2 - x^2} + \frac{x}{200}(50^2 - x^2) - \frac{x}{5}\sqrt{50^2 - x^2} - \frac{50^2 - x^2}{10} \right] dy \\ &= \left[10 \left(x\sqrt{50^2 - x^2} + 50^2 \arcsin \frac{x}{50} \right) + \frac{25}{4}x^2 - \frac{x^4}{800} + \frac{1}{15}(50^2 - x^2)^{3/2} - 250x + \frac{x^3}{30} \right]_0^{50} \\ &\approx 30,415.74 \text{ ft}^3 \end{aligned}$$

$$(b) z = 20 + \frac{xy}{100}$$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{y^2}{100^2} + \frac{x^2}{100^2}} = \frac{\sqrt{100^2 + x^2 + y^2}}{100}$$

$$\begin{aligned} S &= \frac{1}{100} \int_0^{50} \int_0^{\sqrt{50^2 - x^2}} \sqrt{100^2 + x^2 + y^2} \, dy \, dx \\ &= \frac{1}{100} \int_0^{\pi/2} \int_0^{50} \sqrt{100^2 + r^2} \, r \, dr \, d\theta \approx 2081.53 \text{ ft}^2 \end{aligned}$$

$$41. (a) V = \iint_R f(x, y) \, dA$$

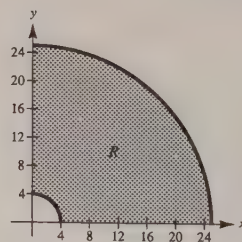
$$= 8 \iint_R \sqrt{625 - x^2 - y^2} \, dA \quad \text{where } R \text{ is the region in the first quadrant}$$

$$= 8 \int_0^{\pi/2} \int_4^{25} \sqrt{625 - r^2} \, r \, dr \, d\theta$$

$$= -4 \int_0^{\pi/2} \left[\frac{2}{3}(625 - r^2)^{3/2} \right]_4^{25} d\theta$$

$$= -\frac{8}{3} [0 - 609\sqrt{609}] \cdot \frac{\pi}{2}$$

$$= 812\pi\sqrt{609} \text{ cm}^3$$



$$(b) A = \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2} \, dA = 8 \iint_R \sqrt{1 + \frac{x^2}{625 - x^2 - y^2} + \frac{y^2}{625 - x^2 - y^2}} \, dA$$

$$= 8 \iint_R \frac{25}{\sqrt{625 - x^2 - y^2}} \, dA = 8 \int_0^{\pi/2} \int_4^{25} \frac{25}{\sqrt{625 - r^2}} \, r \, dr \, d\theta$$

$$= \lim_{b \rightarrow 25^-} \left[-200\sqrt{625 - r^2} \right]_4^b \cdot \frac{\pi}{2} = 100\pi\sqrt{609} \text{ cm}^2$$

Section 13.6 Triple Integrals and Applications

$$\begin{aligned}
 1. \int_0^3 \int_0^2 \int_0^1 (x + y + z) \, dx \, dy \, dz &= \int_0^3 \int_0^2 \left[\frac{1}{2}x^2 + xy + xz \right]_0^1 dy \, dz \\
 &= \int_0^3 \int_0^2 \left(\frac{1}{2} + y + z \right) dy \, dz = \int_0^3 \left[\frac{1}{2}y + \frac{1}{2}y^2 + yz \right]_0^2 dz = \left[3z + z^2 \right]_0^3 = 18
 \end{aligned}$$

$$\begin{aligned}
 3. \int_0^1 \int_0^x \int_0^{xy} x \, dz \, dy \, dx &= \int_0^1 \int_0^x \left[xz \right]_0^{xy} dy \, dx \\
 &= \int_0^1 \int_0^x x^2 y \, dy \, dx = \int_0^1 \left[\frac{x^2 y^2}{2} \right]_0^x dx = \int_0^1 \frac{x^4}{2} dx = \left[\frac{x^5}{10} \right]_0^1 = \frac{1}{10}
 \end{aligned}$$

$$\begin{aligned}
 5. \int_1^4 \int_0^1 \int_0^x 2ze^{-x^2} \, dy \, dx \, dz &= \int_1^4 \int_0^1 \left[(2ze^{-x^2})y \right]_0^x dx \, dz = \int_1^4 \int_0^1 2zxe^{-x^2} \, dx \, dz \\
 &= \int_1^4 \left[-ze^{-x^2} \right]_0^1 dz = \int_1^4 z(1 - e^{-1}) \, dz = \left[(1 - e^{-1}) \frac{z^2}{2} \right]_1^4 = \frac{15}{2} \left(1 - \frac{1}{e} \right)
 \end{aligned}$$

$$\begin{aligned}
 7. \int_0^4 \int_0^{\pi/2} \int_0^{1-x} x \cos y \, dz \, dy \, dx &= \int_0^4 \int_0^{\pi/2} \left[(x \cos y)z \right]_0^{1-x} dy \, dx = \int_0^4 \int_0^{\pi/2} x(1-x) \cos y \, dy \, dx \\
 &= \int_0^4 \left[x(1-x) \sin y \right]_0^{\pi/2} dx = \int_0^4 x(1-x) \, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^4 = 8 - \frac{64}{3} = \frac{-40}{3}
 \end{aligned}$$

$$9. \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{x^2} x \, dz \, dy \, dx = \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x^3 \, dy \, dx = \frac{128}{15}$$

$$\begin{aligned}
 11. \int_0^2 \int_0^{\sqrt{4-x^2}} \int_1^4 \frac{x^2 \sin y}{z} \, dz \, dy \, dx &= \int_0^2 \int_0^{\sqrt{4-x^2}} \left[x^2 \sin y \ln |z| \right]_1^4 dy \, dx \\
 &= \int_0^2 \left[x^2 \ln 4 (-\cos y) \right]_0^{\sqrt{4-x^2}} dx = \int_0^2 x^2 \ln 4 [1 - \cos \sqrt{4-x^2}] \, dx \approx 2.44167
 \end{aligned}$$

$$13. \int_0^4 \int_0^{4-x} \int_0^{4-x-y} dz \, dy \, dx$$

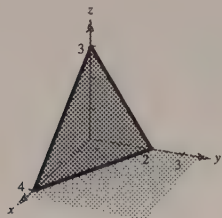
$$15. \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} dz \, dy \, dx$$

$$\begin{aligned}
 17. \int_{-2}^2 \int_0^{4-y^2} \int_0^x dz \, dx \, dy &= \int_{-2}^2 \int_0^{4-y^2} x \, dx \, dy \\
 &= \frac{1}{2} \int_{-2}^2 (4-y^2)^2 \, dy = \int_0^2 (16-8y^2+y^4) \, dy = \left[16y - \frac{8}{3}y^3 + \frac{1}{5}y^5 \right]_0^2 = \frac{256}{15}
 \end{aligned}$$

$$\begin{aligned}
 19. 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz \, dy \, dx &= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2} \, dy \, dx \\
 &= 4 \int_0^a \left[y \sqrt{a^2-x^2-y^2} + (a^2-x^2) \arcsin \left(\frac{y}{\sqrt{a^2-x^2}} \right) \right]_0^{\sqrt{a^2-x^2}} dx \\
 &= 4 \left(\frac{\pi}{2} \right) \int_0^a (a^2-x^2) \, dx = \left[2\pi \left(a^2x - \frac{1}{3}x^3 \right) \right]_0^a = \frac{4}{3} \pi a^3
 \end{aligned}$$

$$21. \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2} dz \, dy \, dx = \int_0^2 (4-x^2)^2 dx = \int_0^2 (16 - 8x^2 + x^4) dx = \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_0^2 = \frac{256}{15}$$

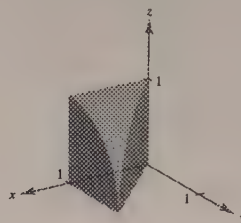
$$23. \text{Plane: } 3x + 6y + 4z = 12$$



$$\int_0^3 \int_0^{(12-4z)/3} \int_0^{(12-4z-3x)/6} dy \, dx \, dz$$

$$25. \text{Top cylinder: } y^2 + z^2 = 1$$

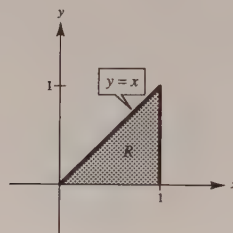
$$\text{Side plane: } x = y$$



$$\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz \, dy \, dx$$

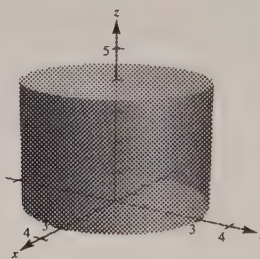
$$27. Q = \{(x, y, z): 0 \leq x \leq 1, 0 \leq y \leq x, 0 \leq z \leq 3\}$$

$$\begin{aligned} \iiint_Q xyz \, dV &= \int_0^3 \int_0^1 \int_0^x xyz \, dx \, dy \, dz = \int_0^3 \int_0^1 \int_0^x xyz \, dy \, dx \, dz \\ &= \int_0^3 \int_0^1 \int_0^1 xyz \, dx \, dz \, dy \\ &= \int_0^3 \int_0^1 \int_0^x xyz \, dy \, dz \, dx \\ &= \int_0^3 \int_0^1 \int_0^3 xyz \, dz \, dx \, dy \\ &= \int_0^3 \int_0^1 \int_0^3 xyz \, dz \, dy \, dx = \left(\frac{9}{16} \right) \end{aligned}$$



$$29. Q = \{(x, y, z): x^2 + y^2 \leq 9, 0 \leq z \leq 4\}$$

$$\begin{aligned} \iiint_Q xyz \, dV &= \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz \, dy \, dx \, dz \\ &= \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz \, dx \, dy \, dz \\ &= \int_{-3}^3 \int_0^4 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz \, dx \, dz \, dy \\ &= \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^4 xyz \, dz \, dx \, dy \\ &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^4 xyz \, dy \, dz \, dx \\ &= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^4 xyz \, dz \, dy \, dx (= 0) \end{aligned}$$



$$31. \quad m = k \int_0^6 \int_0^{4-(2x/3)} \int_0^{2-(y/2)-(x/3)} dz \, dy \, dx$$

$$= 8k$$

$$M_{yz} = k \int_0^6 \int_0^{4-(2x/3)} \int_0^{2-(y/2)-(x/3)} x \, dz \, dy \, dx$$

$$= 12k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{12k}{8k} = \frac{3}{2}$$

$$33. \quad m = k \int_0^4 \int_0^4 \int_0^{4-x} x \, dz \, dy \, dx = k \int_0^4 \int_0^4 x(4-x) \, dy \, dx$$

$$= 4k \int_0^4 (4x - x^2) \, dx = \frac{128k}{3}$$

$$M_{xy} = k \int_0^4 \int_0^4 \int_0^{4-x} xz \, dz \, dy \, dx = k \int_0^4 \int_0^4 x \frac{(4-x)^2}{2} \, dy \, dx$$

$$= 2k \int_0^4 (16x - 8x^2 + x^3) \, dx = \frac{128k}{3}$$

$$\bar{z} = \frac{M_{xy}}{m} = 1$$

$$35. \quad m = k \int_0^b \int_0^b \int_0^b xy \, dz \, dy \, dx = \frac{kb^5}{4}$$

$$M_{yz} = k \int_0^b \int_0^b \int_0^b x^2 y \, dz \, dy \, dx = \frac{kb^6}{6}$$

$$M_{xz} = k \int_0^b \int_0^b \int_0^b xy^2 \, dz \, dy \, dx = \frac{kb^6}{6}$$

$$M_{xy} = k \int_0^b \int_0^b \int_0^b xyz \, dz \, dy \, dx = \frac{kb^6}{8}$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{kb^6/6}{kb^5/4} = \frac{2b}{3}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{kb^6/6}{kb^5/4} = \frac{2b}{3}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{kb^6/8}{kb^5/4} = \frac{b}{2}$$

37. \bar{x} will be greater than 2, whereas \bar{y} and \bar{z} will be unchanged.

39. \bar{y} will be greater than 0, whereas \bar{x} and \bar{z} will be unchanged.

$$41. \quad m = \frac{1}{3} k \pi r^2 h$$

$$\bar{x} = \bar{y} = 0 \text{ by symmetry}$$

$$M_{xy} = 4k \int_0^r \int_0^{\sqrt{r^2-x^2}} \int_{h\sqrt{x^2+y^2}/r}^h z \, dz \, dy \, dx$$

$$= \frac{2kh^2}{r^2} \int_0^r \int_0^{\sqrt{r^2-x^2}} (r^2 - x^2 - y^2) \, dy \, dx$$

$$= \frac{4kh^2}{3r^2} \int_0^r (r^2 - x^2)^{3/2} \, dx$$

$$= \frac{k\pi r^2 h^2}{4}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r^2 h^2/4}{k\pi r^2 h/3} = \frac{3h}{4}$$

$$43. \quad m = \frac{128k\pi}{3}$$

$\bar{x} = \bar{y} = 0$ by symmetry

$$z = \sqrt{4^2 - x^2 - y^2}$$

$$\begin{aligned} M_{xy} &= 4k \int_0^4 \int_0^{\sqrt{4^2-x^2}} \int_0^{\sqrt{4^2-x^2-y^2}} z \, dz \, dy \, dx \\ &= 2k \int_0^4 \int_0^{\sqrt{4^2-x^2}} (4^2 - x^2 - y^2) \, dy \, dx = 2k \int_0^4 \left[16y - x^2y - \frac{1}{3}y^3 \right]_0^{\sqrt{4^2-x^2}} dx = \frac{4k}{3} \int_0^4 (4^2 - x^2)^{3/2} dx \\ &= \frac{1024k}{3} \int_0^{\pi/2} \cos^4 \theta \, d\theta \quad (\text{let } x = 4 \sin \theta) \\ &= 64\pi k \quad \text{by Wallis's Formula} \\ \bar{z} &= \frac{M_{xy}}{m} = \frac{64k\pi}{1} \cdot \frac{3}{128k\pi} = \frac{3}{2} \end{aligned}$$

$$45. \quad f(x, y) = \frac{5}{12}y$$

$$m = k \int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} dz \, dy \, dx = 200k$$

$$M_{yz} = k \int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} x \, dz \, dy \, dx = 1000k$$

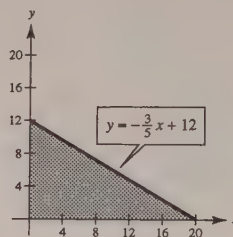
$$M_{xz} = k \int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} y \, dz \, dy \, dx = 1200k$$

$$M_{xy} = k \int_0^{20} \int_0^{-(3/5)x+12} \int_0^{(5/12)y} z \, dz \, dy \, dx = 250k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1000k}{200k} = 5$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1200k}{200k} = 6$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{250k}{200k} = \frac{5}{4}$$



$$\begin{aligned} 47. \quad (a) \quad I_x &= k \int_0^a \int_0^a \int_0^a (y^2 + z^2) \, dx \, dy \, dz = ka \int_0^a \int_0^a (y^2 + z^2) \, dy \, dz \\ &= ka \int_0^a \left[\frac{1}{3}y^3 + z^2y \right]_0^a dz = ka \int_0^a \left(\frac{1}{3}a^3 + az^2 \right) dz = \left[ka \left(\frac{1}{3}a^3z + \frac{1}{3}az^3 \right) \right]_0^a = \frac{2ka^5}{3} \end{aligned}$$

$$I_x = I_y = I_z = \frac{2ka^5}{3} \text{ by symmetry}$$

$$\begin{aligned} (b) \quad I_x &= k \int_0^a \int_0^a \int_0^a (y^2 + z^2)xyz \, dx \, dy \, dz = \frac{ka^2}{2} \int_0^a \int_0^a (y^3z + yz^3) \, dy \, dz \\ &= \frac{ka^2}{2} \int_0^a \left[\frac{y^4z}{4} + \frac{y^2z^3}{2} \right]_0^a dz = \frac{ka^4}{8} \int_0^a (a^2z + 2z^3) \, dz = \left[\frac{ka^4}{8} \left(\frac{a^2z^2}{2} + \frac{2z^4}{4} \right) \right]_0^a = \frac{ka^8}{8} \end{aligned}$$

$$I_x = I_y = I_z = \frac{ka^8}{8} \text{ by symmetry}$$

49. (a) $I_x = k \int_0^4 \int_0^4 \int_0^{4-x} (y^2 + z^2) dz dy dx = k \int_0^4 \int_0^4 \left[y^2(4-x) + \frac{1}{3}(4-x)^3 \right] dy dx$
 $= k \int_0^4 \left[\frac{y^3}{3}(4-x) + \frac{y}{3}(4-x)^3 \right]_0^4 dx = k \int_0^4 \left[\frac{64}{3}(4-x) + \frac{4}{3}(4-x)^3 \right] dx$
 $= k \left[-\frac{32}{3}(4-x)^2 - \frac{1}{3}(4-x)^4 \right]_0^4 = 256k$
 $I_y = k \int_0^4 \int_0^4 \int_0^{4-x} (x^2 + z^2) dz dy dx = k \int_0^4 \int_0^4 \left[x^2(4-x) + \frac{1}{3}(4-x)^3 \right] dy dx$
 $= 4k \int_0^4 \left[4x^2 - x^3 + \frac{1}{3}(4-x)^3 \right] dx = 4k \left[\frac{4}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(4-x)^4 \right]_0^4 = \frac{512k}{3}$
 $I_z = k \int_0^4 \int_0^4 \int_0^{4-x} (x^2 + y^2) dz dy dx = k \int_0^4 \int_0^4 (x^2 + y^2)(4-x) dy dx$
 $= k \int_0^4 \left[\left(x^2y + \frac{y^3}{3} \right) (4-x) \right]_0^4 dx = k \int_0^4 \left(4x^2 + \frac{64}{3} \right) (4-x) dx = 256k$
 (b) $I_x = k \int_0^4 \int_0^4 \int_0^{4-x} y(y^2 + z^2) dz dy dx = k \int_0^4 \int_0^4 \left[y^3(4-x) + \frac{1}{3}y(4-x)^3 \right] dy dx$
 $= k \int_0^4 \left[\frac{y^4}{4}(4-x) + \frac{y^2}{6}(4-x)^3 \right]_0^4 dx = k \int_0^4 \left[64(4-x) + \frac{8}{3}(4-x)^3 \right] dx$
 $= k \left[-32(4-x)^2 - \frac{2}{3}(4-x)^4 \right]_0^4 = \frac{2048k}{3}$
 $I_y = k \int_0^4 \int_0^4 \int_0^{4-x} y(x^2 + z^2) dz dy dx = k \int_0^4 \int_0^4 \left[x^2y(4-x) + \frac{1}{3}y(4-x)^3 \right] dy dx$
 $= 8k \int_0^4 \left[4x^2 - x^3 + \frac{1}{3}(4-x)^3 \right] dx = 8k \left[\frac{4}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(4-x)^4 \right]_0^4 = \frac{1024k}{3}$
 $I_z = k \int_0^4 \int_0^4 \int_0^{4-x} y(x^2 + y^2) dz dy dx = k \int_0^4 \int_0^4 (x^2y + y^3)(4-x) dx$
 $= k \int_0^4 \left[\left(\frac{x^2y^2}{2} + \frac{y^4}{4} \right) (4-x) \right]_0^4 dx = k \int_0^4 (8x^2 + 64)(4-x) dx$
 $= 8k \int_0^4 (32 - 8x + 4x^2 - x^3) dx = \left[8k \left(32x - 4x^2 + \frac{4}{3}x^3 - \frac{1}{4}x^4 \right) \right]_0^4 = \frac{2048k}{3}$

51. $I_{xy} = k \int_{-L/2}^{L/2} \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z^2 dz dx dy = k \int_{-L/2}^{L/2} \int_{-a}^a \frac{2}{3}(a^2 - x^2)\sqrt{a^2 - x^2} dx dy$
 $= \frac{2}{3} \int_{-L/2}^{L/2} k \left[\frac{a^2}{2} \left(x\sqrt{a^2 - x^2} + a^2 \arcsin \frac{x}{a} \right) - \frac{1}{8} \left(x(2x^2 - a^2)\sqrt{x^2 - a^2} + a^4 \arcsin \frac{x}{a} \right) \right]_{-a}^a dy$
 $= \frac{2k}{3} \int_{-L/2}^{L/2} 2 \left(\frac{a^4\pi}{4} - \frac{a^4\pi}{16} \right) dy = \frac{a^4\pi Lk}{4}$

Since $m = \pi a^2 Lk$, $I_{xy} = ma^2/4$.

—CONTINUED—

51. —CONTINUED—

$$\begin{aligned}
 I_{xz} &= k \int_{-L/2}^{L/2} \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2 dz dx dy = 2k \int_{-L/2}^{L/2} \int_{-a}^a y^2 \sqrt{a^2-x^2} dx dy \\
 &= 2k \int_{-L/2}^{L/2} \left[\frac{y^2}{2} \left(x\sqrt{a^2-x^2} + a^2 \arcsin \frac{x}{a} \right) \right]_{-a}^a dy = k\pi a^2 \int_{-L/2}^{L/2} y^2 dy = \frac{2k\pi a^2}{3} \left(\frac{L^3}{8} \right) = \frac{1}{12} mL^2
 \end{aligned}$$

$$\begin{aligned}
 I_{yz} &= k \int_{-L/2}^{L/2} \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2 dz dx dy = 2k \int_{-L/2}^{L/2} \int_{-a}^a x^2 \sqrt{a^2-x^2} dx dy \\
 &= 2k \int_{-L/2}^{L/2} \frac{1}{8} \left[x(2x^2 - a^2)\sqrt{a^2-x^2} + a^4 \arcsin \frac{x}{a} \right]_{-a}^a dy = \frac{ka^4\pi}{4} \int_{-L/2}^{L/2} dy = \frac{ka^4\pi L}{4} = \frac{ma^2}{4}
 \end{aligned}$$

$$I_x = I_{xy} + I_{xz} = \frac{ma^2}{4} + \frac{mL^2}{12} = \frac{m}{12}(3a^2 + L^2)$$

$$I_y = I_{xy} + I_{yz} = \frac{ma^2}{4} + \frac{ma^2}{4} = \frac{ma^2}{2}$$

$$I_z = I_{xz} + I_{yz} = \frac{mL^2}{12} + \frac{ma^2}{4} = \frac{m}{12}(3a^2 + L^2)$$

$$53. \int_{-1}^1 \int_{-1}^1 \int_0^{1-x} (x^2 + y^2) \sqrt{x^2 + y^2 + z^2} dz dy dx$$

55. See the definition, page 978.

See Theorem 13.4, page 979.

57. (a) The annular solid on the right has the greater density.

(b) The annular solid on the right has the greater moment of inertia.

(c) The solid on the left will reach the bottom first. The solid on the right has a greater resistance to rotational motion.

Section 13.7 Triple Integrals in Cylindrical and Spherical Coordinates

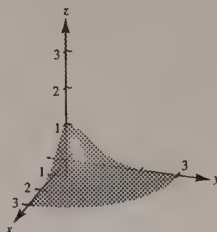
$$\begin{aligned}
 1. \int_0^4 \int_0^{\pi/2} \int_0^2 r \cos \theta dr d\theta dz &= \int_0^4 \int_0^{\pi/2} \left[\frac{r^2}{2} \cos \theta \right]_0^2 d\theta dz \\
 &= \int_0^4 \int_0^{\pi/2} 2 \cos \theta d\theta dz = \int_0^4 \left[2 \sin \theta \right]_0^{\pi/2} dz = \int_0^4 2 dz = 8
 \end{aligned}$$

$$\begin{aligned}
 3. \int_0^{\pi/2} \int_0^{2\cos^2\theta} \int_0^{4-r^2} r \sin \theta dz dr d\theta &= \int_0^{\pi/2} \int_0^{2\cos^2\theta} r(4-r^2) \sin \theta dr d\theta = \int_0^{\pi/2} \left[\left(2r^2 - \frac{r^4}{4} \right) \sin \theta \right]_0^{2\cos^2\theta} d\theta \\
 &= \int_0^{\pi/2} [8 \cos^4 \theta - 4 \cos^8 \theta] \sin \theta d\theta = \left[-\frac{8 \cos^5 \theta}{5} + \frac{4 \cos^9 \theta}{9} \right]_0^{\pi/2} = \frac{52}{45}
 \end{aligned}$$

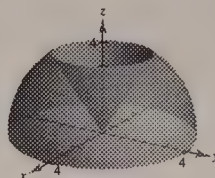
$$5. \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos \phi} \rho^2 \sin \phi d\rho d\phi d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/4} \cos^3 \phi \sin \phi d\phi d\theta = -\frac{1}{12} \int_0^{2\pi} [\cos^4 \phi]_0^{\pi/4} d\theta = \frac{\pi}{8}$$

$$7. \int_0^4 \int_0^z \int_0^{\pi/2} r e^r d\theta dr dz = \pi(e^4 + 3)$$

$$\begin{aligned}
 9. \int_0^{\pi/2} \int_0^3 \int_0^{e^{-r^2}} r \, dz \, dr \, d\theta &= \int_0^{\pi/2} \int_0^3 r e^{-r^2} \, dr \, d\theta \\
 &= \int_0^{\pi/2} \left[-\frac{1}{2} e^{-r^2} \right]_0^3 d\theta \\
 &= \int_0^{\pi/2} \frac{1}{2} (1 - e^{-9}) d\theta \\
 &= \frac{\pi}{4} (1 - e^{-9})
 \end{aligned}$$



$$\begin{aligned}
 11. \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta &= \frac{64}{3} \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \sin \phi \, d\phi \, d\theta \\
 &= \frac{64}{3} \int_0^{2\pi} \left[-\cos \phi \right]_{\pi/6}^{\pi/2} d\theta \\
 &= \frac{32\sqrt{3}}{3} \int_0^{2\pi} d\theta \\
 &= \frac{64\sqrt{3}\pi}{3}
 \end{aligned}$$



$$13. (a) \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2 \cos \theta \, dz \, dr \, d\theta = 0$$

$$(b) \int_0^{2\pi} \int_0^{\arctan(1/2)} \int_0^{4 \sec \phi} \rho^3 \sin^2 \phi \cos \theta \, d\rho \, d\phi \, d\theta + \int_0^{2\pi} \int_{\arctan(1/2)}^{\pi/2} \int_0^{\cot \phi \csc \phi} \rho^3 \sin^2 \phi \cos \theta \, d\rho \, d\phi \, d\theta = 0$$

$$15. (a) \int_0^{2\pi} \int_0^a \int_a^{a+\sqrt{a^2-r^2}} r^2 \cos \theta \, dz \, dr \, d\theta = 0$$

$$(b) \int_0^{\pi/4} \int_0^{2\pi} \int_{a \sec \phi}^{2a \cos \phi} \rho^3 \sin^2 \phi \cos \theta \, d\rho \, d\theta \, d\phi = 0$$

$$17. V = 4 \int_0^{\pi/2} \int_0^{a \cos \theta} \int_0^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^{a \cos \theta} r \sqrt{a^2-r^2} \, dr \, d\theta$$

$$= \frac{4}{3} a^3 \int_0^{\pi/2} (1 - \sin^3 \theta) \, d\theta = \frac{4}{3} a^3 \left[\theta + \frac{1}{3} \cos \theta (\sin^2 \theta + 2) \right]_0^{\pi/2} = \frac{4}{3} a^3 \left(\frac{\pi}{2} - \frac{2}{3} \right) = \frac{2a^3}{9} (3\pi - 4)$$

$$19. V = 2 \int_0^{\pi} \int_0^{a \cos \theta} \int_0^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta$$

$$= 2 \int_0^{\pi} \int_0^{a \cos \theta} r \sqrt{a^2-r^2} \, dr \, d\theta$$

$$= 2 \int_0^{\pi} \left[-\frac{1}{3} (a^2 - r^2)^{3/2} \right]_0^{a \cos \theta} d\theta$$

$$= \frac{2a^3}{3} \int_0^{\pi} (1 - \sin^3 \theta) \, d\theta$$

$$= \frac{2a^3}{3} \left[\theta + \cos \theta - \frac{\cos^3 \theta}{3} \right]_0^{\pi}$$

$$= \frac{2a^3}{9} (3\pi - 4)$$

$$21. m = \int_0^{2\pi} \int_0^2 \int_0^{9-r \cos \theta - 2r \sin \theta} (kr) \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 kr^2 (9 - r \cos \theta - 2r \sin \theta) \, dr \, d\theta$$

$$= \int_0^{2\pi} k \left[3r^3 - \frac{r^4}{4} \cos \theta - \frac{r^4}{2} \sin \theta \right]_0^2 d\theta$$

$$= \int_0^{2\pi} k [24 - 4 \cos \theta - 8 \sin \theta] d\theta$$

$$= k \left[24\theta - 4 \sin \theta + 8 \cos \theta \right]_0^{2\pi}$$

$$= k[48\pi + 8 - 8] = 48k\pi$$

$$23. \quad z = h - \frac{h}{r_0} \sqrt{x^2 + y^2} = \frac{h}{r_0} (r_0 - r)$$

$$\begin{aligned} V &= 4 \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r \, dz \, dr \, d\theta \\ &= \frac{4h}{r_0} \int_0^{\pi/2} \int_0^{r_0} (r_0 r - r^2) \, dr \, d\theta \\ &= \frac{4h}{r_0} \int_0^{\pi/2} \frac{r_0^3}{6} \, d\theta \\ &= \frac{4h}{r_0} \left(\frac{r_0^3}{6} \right) \left(\frac{\pi}{2} \right) = \frac{1}{3} \pi r_0^2 h \end{aligned}$$

$$\begin{aligned} 27. \quad I_z &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r^3 \, dz \, dr \, d\theta \\ &= \frac{4kh}{r_0} \int_0^{\pi/2} \int_0^{r_0} (r_0 r^3 - r^4) \, dr \, d\theta \\ &= \frac{4kh}{r_0} \left(\frac{r_0^5}{20} \right) \left(\frac{\pi}{2} \right) \\ &= \frac{1}{10} k \pi r_0^4 h \end{aligned}$$

Since the mass of the core is $m = kV = k(\frac{1}{3}\pi r_0^2 h)$ from Exercise 23, we have $k = 3m/\pi r_0^2 h$. Thus,

$$\begin{aligned} I_z &= \frac{1}{10} k \pi r_0^4 h \\ &= \frac{1}{10} \left(\frac{3m}{\pi r_0^2 h} \right) \pi r_0^4 h \\ &= \frac{3}{10} m r_0^2 \end{aligned}$$

$$31. \quad V = \int_0^{2\pi} \int_0^\pi \int_0^{4 \sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 16\pi^2$$

$$25. \quad \rho = k\sqrt{x^2 + y^2} = kr$$

$\bar{x} = \bar{y} = 0$ by symmetry

$$\begin{aligned} m &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r^2 \, dz \, dr \, d\theta \\ &= \frac{1}{6} k \pi r_0^3 h \\ M_{xy} &= 4k \int_0^{\pi/2} \int_0^{r_0} \int_0^{h(r_0-r)/r_0} r^2 z \, dz \, dr \, d\theta \\ &= \frac{1}{30} k \pi r_0^3 h^2 \\ \bar{z} &= \frac{M_{xy}}{m} = \frac{k \pi r_0^3 h^2 / 30}{k \pi r_0^3 h / 6} = \frac{h}{5} \end{aligned}$$

$$29. \quad m = k(\pi b^2 h - \pi a^2 h) = k\pi h(b^2 - a^2)$$

$$\begin{aligned} I_z &= 4k \int_0^{\pi/2} \int_a^b \int_0^h r^3 \, dz \, dr \, d\theta \\ &= 4kh \int_0^{\pi/2} \int_a^b r^3 \, dr \, d\theta \\ &= kh \int_0^{\pi/2} (b^4 - a^4) \, d\theta \\ &= \frac{k\pi(b^4 - a^4)h}{2} \\ &= \frac{k\pi(b^2 - a^2)(b^2 + a^2)h}{2} \\ &= \frac{1}{2} m(a^2 + b^2) \end{aligned}$$

$$\begin{aligned} 33. \quad m &= 8k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^3 \sin \phi \, d\rho \, d\theta \, d\phi \\ &= 2ka^4 \int_0^{\pi/2} \int_0^{\pi/2} \sin \phi \, d\theta \, d\phi \\ &= k\pi a^4 \int_0^{\pi/2} \sin \phi \, d\phi \\ &= \left[k\pi a^4 (-\cos \phi) \right]_0^{\pi/2} \\ &= k\pi a^4 \end{aligned}$$

$$35. \quad m = \frac{2}{3} k \pi r^3$$

$$\bar{x} = \bar{y} = 0 \text{ by symmetry}$$

$$M_{xy} = 4k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^r \rho^3 \cos \phi \sin \phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{1}{2} k r^4 \int_0^{\pi/2} \int_0^{\pi/2} \sin 2\phi \, d\theta \, d\phi$$

$$= \frac{k r^4 \pi}{4} \int_0^{\pi/2} \sin 2\phi \, d\phi$$

$$= \left[-\frac{1}{8} k \pi r^4 \cos 2\phi \right]_0^{\pi/2} = \frac{1}{4} k \pi r^4$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k \pi r^4 / 4}{2 k \pi r^3 / 3} = \frac{3r}{8}$$

$$39. \quad x = r \cos \theta \quad x^2 + y^2 = r^2$$

$$y = r \sin \theta \quad \tan \theta = \frac{y}{x}$$

$$z = z \quad z = z$$

$$43. \quad (a) \quad r = r_0: \text{ right circular cylinder about } z\text{-axis}$$

$$\theta = \theta_0: \text{ plane parallel to } z\text{-axis}$$

$$z = z_0: \text{ plane parallel to } xy\text{-plane}$$

$$37. \quad I_z = 4k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^{\cos \phi} \rho^4 \sin^3 \phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{4}{5} k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \cos^5 \phi \sin^3 \phi \, d\theta \, d\phi$$

$$= \frac{2}{5} k \pi \int_{\pi/4}^{\pi/2} \cos^5 \phi (1 - \cos^2 \phi) \sin \phi \, d\phi$$

$$= \left[\frac{2}{5} k \pi \left(-\frac{1}{6} \cos^6 \phi + \frac{1}{8} \cos^8 \phi \right) \right]_{\pi/4}^{\pi/2}$$

$$= \frac{k \pi}{192}$$

$$41. \quad \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{\theta_2(\theta)} \int_{h_1(r \cos \theta, r \sin \theta)}^{h_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r \, dz \, dr \, d\theta$$

$$(b) \quad \rho = \rho_0: \text{ sphere of radius } \rho_0$$

$$\theta = \theta_0: \text{ plane parallel to } z\text{-axis}$$

$$\phi = \phi_0: \text{ cone}$$

$$45. \quad 16 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2-z^2}} dw \, dz \, dy \, dx$$

$$= 16 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \sqrt{a^2-x^2-y^2-z^2} \, dz \, dy \, dx$$

$$= 16 \int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2-r^2}} \sqrt{(a^2-r^2)-z^2} \, dz (r \, dr \, d\theta)$$

$$= 16 \int_0^{\pi/2} \int_0^a \frac{1}{2} \left[z \sqrt{(a^2-r^2)-z^2} + (a^2-r^2) \arcsin \frac{z}{\sqrt{a^2-r^2}} \right]_0^{\sqrt{a^2-r^2}} r \, dr \, d\theta$$

$$= 8 \int_0^{\pi/2} \int_0^a \frac{\pi}{2} (a^2-r^2) r \, dr \, d\theta$$

$$= 4\pi \int_0^{\pi/2} \left[\frac{a^2 r^2}{2} - \frac{r^4}{4} \right]_0^a d\theta$$

$$= a^4 \pi \int_0^{\pi/2} d\theta = \frac{a^4 \pi^2}{2}$$

Section 13.8 Change of Variables: Jacobians

1. $x = -\frac{1}{2}(u - v)$

$$y = \frac{1}{2}(u + v)$$

$$\begin{aligned}\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} &= \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= -\frac{1}{2}\end{aligned}$$

5. $x = u \cos \theta - v \sin \theta$

$$y = u \sin \theta + v \cos \theta$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \cos^2 \theta + \sin^2 \theta = 1$$

7. $x = e^u \sin v$

$$y = e^u \cos v$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = (e^u \sin v)(-e^u \sin v) - (e^u \cos v)(e^u \cos v) = -e^{2u}$$

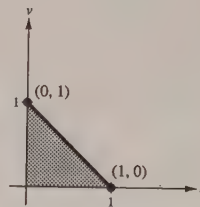
9. $x = 3u + 2v$

$$y = 3v$$

$$v = \frac{y}{3}$$

$$\begin{aligned}u &= \frac{x - 2v}{3} = \frac{x - 2(y/3)}{3} \\ &= \frac{x}{3} - \frac{2y}{9}\end{aligned}$$

(x, y)	(u, v)
$(0, 0)$	$(0, 0)$
$(3, 0)$	$(1, 0)$
$(2, 3)$	$(0, 1)$



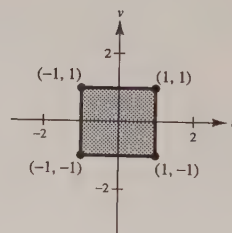
11. $x = \frac{1}{2}(u + v)$

$$y = \frac{1}{2}(u - v)$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$$

$$\iint_R 4(x^2 + y^2) dA = \int_{-1}^1 \int_{-1}^1 4\left[\frac{1}{4}(u + v)^2 + \frac{1}{4}(u - v)^2\right]\left(\frac{1}{2}\right) dv du$$

$$= \int_{-1}^1 \int_{-1}^1 (u^2 + v^2) dv du = \int_{-1}^1 2\left(u^2 + \frac{1}{3}\right) du = \left[2\left(\frac{u^3}{3} + \frac{u}{3}\right)\right]_{-1}^1 = \frac{8}{3}$$

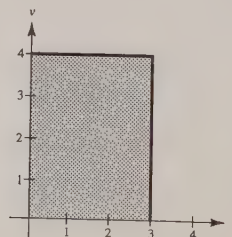


13. $x = u + v$

$$y = u$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = (1)(0) - (1)(1) = -1$$

$$\iint_R y(x - y) dA = \int_0^3 \int_0^4 uv(1) dv du = \int_0^3 8u du = 36$$



$$15. \iint_R e^{-xy/2} dA$$

$$R: y = \frac{x}{4}, y = 2x, y = \frac{1}{x}, y = \frac{4}{x}$$

$$x = \sqrt{v/u}, y = \sqrt{uv} \Rightarrow u = \frac{y}{x}, v = xy$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} \frac{v^{1/2}}{u^{3/2}} & \frac{1}{2} \frac{1}{u^{1/2}v^{1/2}} \\ \frac{1}{2} \frac{v^{1/2}}{u^{1/2}} & \frac{1}{2} \frac{u^{1/2}}{v^{1/2}} \end{vmatrix} = -\frac{1}{4} \left(\frac{1}{u} + \frac{1}{u} \right) = -\frac{1}{2u}$$

Transformed Region:

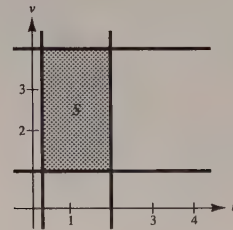
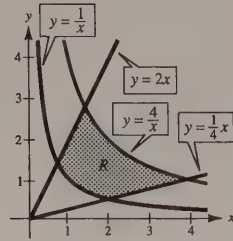
$$y = \frac{1}{x} \Rightarrow yx = 1 \Rightarrow v = 1$$

$$y = \frac{4}{x} \Rightarrow yx = 4 \Rightarrow v = 4$$

$$y = 2x \Rightarrow \frac{y}{x} = 2 \Rightarrow u = 2$$

$$y = \frac{x}{4} \Rightarrow \frac{y}{x} = \frac{1}{4} \Rightarrow u = \frac{1}{4}$$

$$\begin{aligned} \iint_R e^{-xy/2} dA &= \int_{1/4}^2 \int_1^4 e^{-v/2} \left(\frac{1}{2u} \right) dv du = - \int_{1/4}^2 \left[\frac{e^{-v/2}}{u} \right]_1^4 du = - \int_{1/4}^2 (e^{-2} - e^{-1/2}) \frac{1}{u} du \\ &= - \left[(e^{-2} - e^{-1/2}) \ln u \right]_{1/4}^2 = -(e^{-2} - e^{-1/2}) \left(\ln 2 - \ln \frac{1}{4} \right) = (e^{-1/2} - e^{-2}) \ln 8 \approx 0.9798 \end{aligned}$$



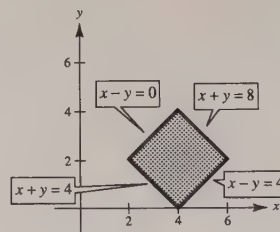
$$17. u = x + y = 4, \quad v = x - y = 0$$

$$u = x + y = 8, \quad v = x - y = 4$$

$$x = \frac{1}{2}(u + v), \quad y = \frac{1}{2}(u - v)$$

$$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}$$

$$\begin{aligned} \iint_R (x + y) e^{x-y} dA &= \int_4^8 \int_0^4 u e^v \left(\frac{1}{2} \right) dv du \\ &= \frac{1}{2} \int_4^8 u (e^4 - 1) du = \left[\frac{1}{4} u^2 (e^4 - 1) \right]_4^8 = 12(e^4 - 1) \end{aligned}$$



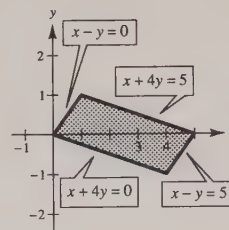
$$19. u = x + 4y = 0, \quad v = x - y = 0$$

$$u = x + 4y = 5, \quad v = x - y = 5$$

$$x = \frac{1}{5}(u + 4v), \quad y = \frac{1}{5}(u - v)$$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = \left(\frac{1}{5} \right) \left(-\frac{1}{5} \right) - \left(\frac{1}{5} \right) \left(\frac{4}{5} \right) = -\frac{1}{5}$$

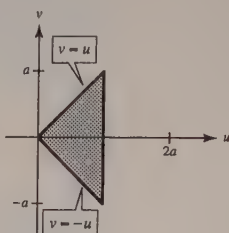
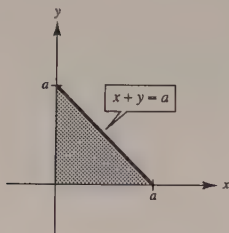
$$\begin{aligned} \iint_R \sqrt{(x-y)(x+4y)} dA &= \int_0^5 \int_0^5 \sqrt{uv} \left(\frac{1}{5} \right) du dv \\ &= \int_0^5 \left[\frac{1}{5} \left(\frac{2}{3} \right) u^{3/2} \sqrt{v} \right]_0^5 dv = \left[\frac{2\sqrt{5}}{3} \left(\frac{2}{3} \right) v^{3/2} \right]_0^5 = \frac{100}{9} \end{aligned}$$



21. $u = x + y, v = x - y, x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v)$

$$\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} = -\frac{1}{2}$$

$$\iint_R \sqrt{x+y} \, dA = \int_0^a \int_{-u}^u \sqrt{u} \left(\frac{1}{2}\right) dv \, du = \int_0^a u \sqrt{u} \, du = \left[\frac{2}{5} u^{5/2}\right]_0^a = \frac{2}{5} a^{5/2}$$



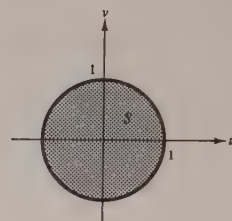
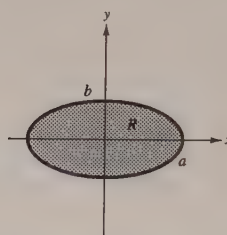
23. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, x = au, y = bv$

$$\frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} = 1$$

$$u^2 + v^2 = 1$$

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$u^2 + v^2 = 1$$



(b) $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$
 $= (a)(b) - (0)(0) = ab$

(c) $A = \iint_S ab \, dS$
 $= ab(\pi(1)^2) = \pi ab$

25. Jacobian $= \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$

27. $x = u(1-v), y = uv(1-w), z = uvw$

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} \\ &= (1-v)[u^2v(1-w) + u^2vw] + u[uv^2(1-w) + uv^2w] \\ &= (1-v)(u^2v) + u(uv^2) \\ &= u^2v \end{aligned}$$

29. $x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} &= \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\ \cos \phi & 0 & -\rho \sin \phi \end{vmatrix} \\ &= \cos \phi [-\rho^2 \sin \phi \cos \phi \sin^2 \theta - \rho^2 \sin \phi \cos \phi \cos^2 \theta] - \rho \sin \phi [\rho \sin^2 \phi \cos^2 \theta + \rho \sin^2 \phi \sin^2 \theta] \\ &= \cos \phi [-\rho^2 \sin \phi \cos \phi (\sin^2 \theta + \cos^2 \theta)] - \rho \sin \phi [\rho \sin^2 \phi (\cos^2 \theta + \sin^2 \theta)] \\ &= -\rho^2 \sin \phi \cos^2 \phi - \rho^2 \sin^3 \phi \\ &= -\rho^2 \sin \phi (\cos^2 \phi + \sin^2 \phi) \\ &= -\rho^2 \sin \phi \end{aligned}$$

Review Exercises for Chapter 13

$$1. \int_1^{x^2} x \ln y \, dy = \left[xy(-1 + \ln y) \right]_1^{x^2} = x^3(-1 + \ln x^2) + x = x - x^3 + x^3 \ln x^2$$

$$3. \int_0^1 \int_0^{1+x} (3x + 2y) \, dy \, dx = \int_0^1 \left[3xy + y^2 \right]_0^{1+x} dx = \int_0^1 (4x^2 + 5x + 1) \, dx = \left[\frac{4}{3}x^3 + \frac{5}{2}x^2 + x \right]_0^1 = \frac{29}{6}$$

$$5. \int_0^3 \int_0^{\sqrt{9-x^2}} 4x \, dy \, dx = \int_0^3 4x\sqrt{9-x^2} \, dx = \left[-\frac{4}{3}(9-x^2)^{3/2} \right]_0^3 = 36$$

$$7. \int_0^3 \int_0^{(3-x)/3} dy \, dx = \int_0^1 \int_0^{3-3y} dx \, dy$$

$$A = \int_0^1 \int_0^{3-3y} dx \, dy = \int_0^1 (3-3y) \, dy = \left[3y - \frac{3}{2}y^2 \right]_0^1 = \frac{3}{2}$$

$$9. \int_{-5}^3 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} dy \, dx = \int_{-5}^{-4} \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx \, dy + \int_{-4}^4 \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx \, dy + \int_4^5 \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx \, dy$$

$$A = 2 \int_{-5}^3 \int_0^{\sqrt{25-x^2}} dy \, dx = 2 \int_{-5}^3 \sqrt{25-x^2} \, dx = \left[x\sqrt{25-x^2} + 25 \arcsin \frac{x}{5} \right]_{-5}^3 = \frac{25\pi}{2} + 12 + 25 \arcsin \frac{3}{5} \approx 67.36$$

$$11. A = 4 \int_0^1 \int_0^{x\sqrt{1-x^2}} dy \, dx = 4 \int_0^1 x\sqrt{1-x^2} \, dx = \left[-\frac{4}{3}(1-x^2)^{3/2} \right]_0^1 = \frac{4}{3}$$

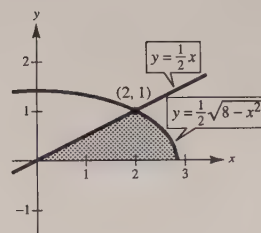
$$A = 4 \int_0^{1/2} \int_{\sqrt{(1-\sqrt{1-4y^2})/2}}^{\sqrt{(1+\sqrt{1-4y^2})/2}} dx \, dy$$

$$13. A = \int_2^5 \int_{x-3}^{\sqrt{x-1}} dy \, dx + 2 \int_1^2 \int_0^{\sqrt{x-1}} dy \, dx = \int_{-1}^2 \int_{y^2+1}^{y+3} dx \, dy = \frac{9}{2}$$

15. Both integrations are over the common region R shown in the figure. Analytically,

$$\int_0^1 \int_{2y}^{2\sqrt{2-y^2}} (x+y) \, dx \, dy = \frac{4}{3} + \frac{4}{3}\sqrt{2}$$

$$\int_0^2 \int_0^{x/2} (x+y) \, dy \, dx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}/2} (x+y) \, dy \, dx = \frac{5}{3} + \left(\frac{4}{3}\sqrt{2} - \frac{1}{3} \right) = \frac{4}{3} + \frac{4}{3}\sqrt{2}$$



$$17. V = \int_0^4 \int_0^{x^2+4} (x^2 - y + 4) \, dy \, dx$$

$$= \int_0^4 \left[x^2 y - \frac{1}{2} y^2 + 4y \right]_0^{x^2+4} dx$$

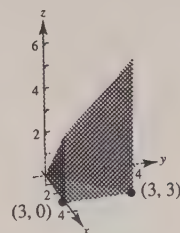
$$= \int_0^4 \left(\frac{1}{2} x^4 + 4x^2 + 8 \right) dx$$

$$= \left[\frac{1}{10} x^5 + \frac{4}{3} x^3 + 8x \right]_0^4 = \frac{3296}{15}$$

19. Volume \approx (base)(height)

$$\approx \frac{9}{2}(3) = \frac{27}{2}$$

Matches (c)



$$21. \int_0^\infty \int_0^\infty kxye^{-(x+y)} dy dx = \int_0^\infty \left[-kxe^{-(x+y)}(y+1) \right]_0^\infty dx = \int_0^\infty kxe^{-x} dx = \left[-k(x+1)e^{-x} \right]_0^\infty = k$$

Therefore, $k = 1$.

$$P = \int_0^1 \int_0^1 xye^{-(x+y)} dy dx \approx 0.070$$

23. True

25. True

$$27. \int_0^h \int_0^x \sqrt{x^2 + y^2} dy dx = \int_0^{\pi/4} \int_0^{h \sec \theta} r^2 dr d\theta$$

$$= \frac{h^3}{3} \int_0^{\pi/4} \sec^3 \theta d\theta = \frac{h^3}{6} \left[\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]_0^{\pi/4} = \frac{h^3}{6} [\sqrt{2} + \ln(\sqrt{2} + 1)]$$

$$29. V = 4 \int_0^h \int_0^{\pi/2} \int_1^{\sqrt{1+z^2}} r dr d\theta dz$$

$$= 2 \int_0^h \int_0^{\pi/2} (1 + z^2 - 1) d\theta dz$$

$$= \pi \int_0^h z^2 dz$$

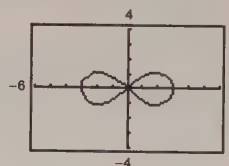
$$= \left[\pi \left(\frac{1}{3} z^3 \right) \right]_0^h = \frac{\pi h^3}{3}$$

$$31. (a) (x^2 + y^2)^2 = 9(x^2 - y^2)$$

$$(r^2)^2 = 9(r^2 \cos^2 \theta - r^2 \sin^2 \theta)$$

$$r^2 = 9(\cos^2 \theta - \sin^2 \theta) = 9 \cos 2\theta$$

$$r = 3\sqrt{\cos 2\theta}$$



$$(b) A = 4 \int_0^{\pi/4} \int_0^{3\sqrt{\cos 2\theta}} r dr d\theta = 9$$

$$(c) V = 4 \int_0^{\pi/4} \int_0^{3\sqrt{\cos 2\theta}} \sqrt{9 - r^2} r dr d\theta \approx 20.392$$

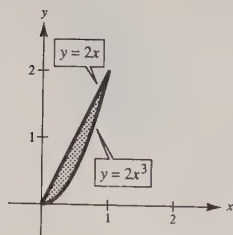
$$33. (a) m = k \int_0^1 \int_{2x^3}^{2x} xy dy dx = \frac{k}{4}$$

$$M_x = k \int_0^1 \int_{2x^3}^{2x} xy^2 dy dx = \frac{16k}{55}$$

$$M_y = k \int_0^1 \int_{2x^3}^{2x} x^2 y dy dx = \frac{8k}{45}$$

$$\bar{x} = \frac{M_y}{m} = \frac{32}{45}$$

$$\bar{y} = \frac{M_x}{m} = \frac{64}{55}$$



$$(b) m = k \int_0^1 \int_{2x^3}^{2x} (x^2 + y^2) dy dx = \frac{17k}{30}$$

$$M_x = k \int_0^1 \int_{2x^3}^{2x} y(x^2 + y^2) dy dx = \frac{392k}{585}$$

$$M_y = k \int_0^1 \int_{2x^3}^{2x} x(x^2 + y^2) dy dx = \frac{156k}{385}$$

$$\bar{x} = \frac{M_y}{m} = \frac{936}{1309}$$

$$\bar{y} = \frac{M_x}{m} = \frac{784}{663}$$

$$35. I_x = \iint_R y^2 \rho(x, y) dA = \int_0^a \int_0^b kxy^2 dy dx = \frac{1}{6} kb^3 a^2$$

$$I_y = \iint_R x^2 \rho(x, y) dA = \int_0^a \int_0^b kx^3 dy dx = \frac{1}{4} kba^4$$

$$I_0 = I_x + I_y = \frac{1}{6} kb^3 a^2 + \frac{1}{4} kba^4 = \frac{ka^2 b}{12} (2b^2 + 3a^2)$$

$$m = \iint_R \rho(x, y) dA = \int_0^a \int_0^b kx dy dx = \frac{1}{2} kba^2$$

$$\bar{x} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{(1/4)kba^4}{(1/2)kba^2}} = \sqrt{\frac{a^2}{2}} = \frac{a\sqrt{2}}{2}$$

$$\bar{y} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{(1/6)kb^3 a^2}{(1/2)kba^2}} = \sqrt{\frac{b^2}{3}} = \frac{b\sqrt{3}}{3}$$

$$37. S = \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2} dA$$

$$= 4 \int_0^4 \int_0^{\sqrt{16-x^2}} \sqrt{1 + 4x^2 + 4y^2} dy dx$$

$$= 4 \int_0^{\pi/2} \int_0^4 \sqrt{1 + 4r^2} r dr d\theta$$

$$= \left[\frac{1}{3} (65^{3/2} - 1) \theta \right]_0^{\pi/2} = \frac{\pi}{6} (65\sqrt{65} - 1)$$

$$39. f(x, y) = 9 - y^2$$

$$f_x = 0, f_y = -2y$$

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2} dA$$

$$= \int_0^3 \int_{-y}^y \sqrt{1 + 4y^2} dx dy$$

$$= \int_0^3 \left[\sqrt{1 + 4y^2} x \right]_{-y}^y dy$$

$$= \int_0^3 2y \sqrt{1 + 4y^2} dy = \frac{1}{4} \frac{2}{3} (1 + 4y^2)^{3/2} \Big|_0^3 = \frac{1}{6} [(37)^{3/2} - 1]$$

$$41. \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{x^2+y^2}^9 \sqrt{x^2 + y^2} dz dy dx = \int_0^{2\pi} \int_0^3 \int_{r^2}^9 r^2 dz dr d\theta$$

$$= \int_0^{2\pi} \int_0^3 (9r^2 - r^4) dr d\theta = \int_0^{2\pi} \left[3r^3 - \frac{r^5}{5} \right]_0^3 d\theta = \frac{162}{5} \int_0^{2\pi} d\theta = \frac{324\pi}{5}$$

$$43. \int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2) dx dy dz = \int_0^a \int_0^b \left(\frac{1}{3} c^3 + cy^2 + cz^2 \right) dy dz$$

$$= \int_0^a \left(\frac{1}{3} bc^3 + \frac{1}{3} b^3 c + bc z^2 \right) dz = \frac{1}{3} abc^3 + \frac{1}{3} ab^3 c + \frac{1}{3} a^3 bc = \frac{1}{3} abc(a^2 + b^2 + c^2)$$

$$45. \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (x^2 + y^2) dz dy dx = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^3 dz dr d\theta = \frac{8\pi}{15}$$

$$\begin{aligned}
 47. \quad V &= 4 \int_0^{\pi/2} \int_0^{2 \cos \theta} \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta \\
 &= 4 \int_0^{\pi/2} \int_0^{2 \cos \theta} r \sqrt{4-r^2} \, dr \, d\theta \\
 &= - \int_0^{\pi/2} \left[\frac{4}{3} (4-r^2)^{3/2} \right]_0^{2 \cos \theta} d\theta \\
 &= \frac{32}{3} \int_0^{\pi/2} (1 - \sin^3 \theta) \, d\theta \\
 &= \frac{32}{3} \left[\theta + \cos \theta - \frac{1}{3} \cos^3 \theta \right]_0^{\pi/2} = \frac{32}{3} \left(\frac{\pi}{2} - \frac{2}{3} \right)
 \end{aligned}$$

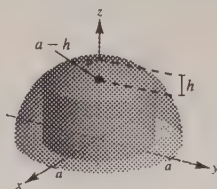
$$\begin{aligned}
 49. \quad m &= 4k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^{\cos \phi} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \\
 &= \frac{4}{3} k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \cos^3 \phi \sin \phi \, d\theta \, d\phi = \frac{2}{3} k \pi \int_{\pi/4}^{\pi/2} \cos^3 \phi \sin \phi \, d\phi = \left[-\frac{2}{3} k \pi \left(\frac{1}{4} \cos^4 \phi \right) \right]_{\pi/4}^{\pi/2} = \frac{k\pi}{24} \\
 M_{xy} &= 4k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^{\cos \phi} \rho^3 \cos \phi \sin \phi \, d\rho \, d\theta \, d\phi \\
 &= k \int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \cos^5 \phi \sin \phi \, d\theta \, d\phi = \frac{1}{2} k \pi \int_{\pi/4}^{\pi/2} \cos^5 \phi \sin \phi \, d\phi = \left[-\frac{1}{12} k \pi \cos^6 \phi \right]_{\pi/4}^{\pi/2} = \frac{k\pi}{96} \\
 \bar{z} &= \frac{M_{xy}}{m} = \frac{k\pi/96}{k\pi/24} = \frac{1}{4} \\
 \bar{x} = \bar{y} &= 0 \quad \text{by symmetry}
 \end{aligned}$$

$$\begin{aligned}
 51. \quad m &= k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \frac{k\pi a^3}{6} \\
 M_{xy} &= k \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \frac{k\pi a^4}{16} \\
 \bar{x} = \bar{y} = \bar{z} &= \frac{M_{xy}}{m} = \frac{k\pi a^4}{16} \left(\frac{6}{k\pi a^3} \right) = \frac{3a}{8}
 \end{aligned}$$

$$\begin{aligned}
 53. \quad I_z &= 4k \int_0^{\pi/2} \int_3^4 \int_0^{16-r^2} r^3 \, dz \, dr \, d\theta \\
 &= 4k \int_0^{\pi/2} \int_3^4 (16r^3 - r^5) \, dr \, d\theta = \frac{833\pi k}{3}
 \end{aligned}$$

$$\begin{aligned}
 55. \quad z &= f(x, y) = \sqrt{a^2 - x^2 - y^2} \\
 &= \sqrt{a^2 - r^2}
 \end{aligned}$$

$$0 \leq r \leq \sqrt{2ah - h^2}$$

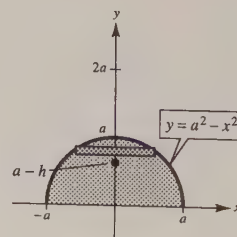


(a) Disc Method

$$\begin{aligned}
 V &= \pi \int_{a-h}^a (a^2 - y^2) dy \\
 &= \pi \left[a^2 y - \frac{y^3}{3} \right]_{a-h}^a = \pi \left[\left(a^3 - \frac{a^3}{3} \right) - \left(a^2(a-h) - \frac{(a-h)^3}{3} \right) \right] \\
 &= \pi \left[a^3 - \frac{a^3}{3} - a^3 + a^2 h + \frac{a^3}{3} - a^2 h + ah^2 - \frac{h^3}{3} \right] = \pi \left[ah^2 - \frac{h^3}{3} \right] = \frac{1}{3} \pi h^2 [3a - h]
 \end{aligned}$$

Equivalently, use spherical coordinates

$$V = \int_0^{2\pi} \int_0^{\cos^{-1}(a-h/a)} \int_{(a-h)\sec \phi}^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$



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55. —CONTINUED—

$$(b) M_{xy} = \int_0^{2\pi} \int_0^{\cos^{-1}(a-h/a)} \int_{(a-h)\sec\phi}^a (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{4} h^2 \pi (2a - h)^2$$

$$\bar{z} = \frac{M_{xy}}{V} = \frac{\frac{1}{4} h^2 \pi (2a - h)^2}{\frac{1}{3} h^2 \pi (3a - h)} = \frac{3(2a - h)^2}{4(3a - h)}$$

$$\text{centroid: } \left(0, 0, \frac{3(2a - h)^2}{4(3a - h)}\right)$$

$$(c) \text{ If } h = a, \bar{z} = \frac{3(a)^2}{4(2a)} = \frac{3}{8}a$$

$$\text{centroid of hemisphere: } \left(0, 0, \frac{3}{8}a\right)$$

$$(d) \lim_{h \rightarrow 0} \bar{z} = \lim_{h \rightarrow 0} \frac{3(2a - h)^2}{4(3a - h)} = \frac{3(4a^2)}{12a} = a$$

$$(e) x^2 + y^2 = \rho^2 \sin^2 \phi$$

$$I_z = \int_0^{2\pi} \int_0^{\cos^{-1}(a-h/a)} \int_{(a-h)\sec\phi}^a (\rho^2 \sin^2 \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{h^3}{30} (20a^2 - 15ah + 3h^2) \pi$$

$$(f) \text{ If } h = a, I_z = \frac{a^3 \pi}{30} (20a^2 - 15a^2 + 3a^2) = \frac{4}{15} a^5 \pi$$

$$57. \int_0^{2\pi} \int_0^{\pi} \int_0^{6 \sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Since $\rho = 6 \sin \phi$ represents (in the yz -plane) a circle of radius 3 centered at $(0, 3, 0)$, the integral represents the volume of the torus formed by revolving $(0 < \theta < 2\pi)$ this circle about the z -axis.

$$61. \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \frac{1}{2} \left(-\frac{1}{2}\right) - \frac{1}{2} \left(\frac{1}{2}\right) = -\frac{1}{2}$$

$$x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v) \Rightarrow u = x + y, v = x - y$$

Boundaries in xy -plane

$$x + y = 3$$

$$x + y = 5$$

$$x - y = -1$$

$$x - y = 1$$

Boundaries in uv -plane

$$u = 3$$

$$u = 5$$

$$v = -1$$

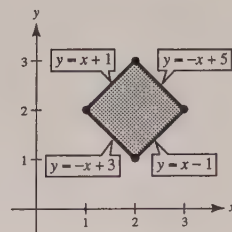
$$v = 1$$

$$\int_R \int \ln(x + y) dA = \int_3^5 \int_{-1}^1 \ln\left(\frac{1}{2}(u + v) + \frac{1}{2}(u - v)\right) \left(\frac{1}{2}\right) dv du = \int_3^5 \int_{-1}^1 \frac{1}{2} \ln u \, dv du = \int_3^5 \ln u \, du = \left[u \ln u - u\right]_3^5$$

$$= (5 \ln 5 - 5) - (3 \ln 3 - 3) = 5 \ln 5 - 3 \ln 3 - 2 \approx 2.751$$

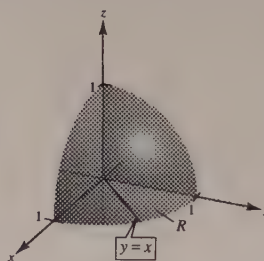
$$59. \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

$$= 1(-3) - 2(3) = -9$$



Problem Solving for Chapter 13

$$\begin{aligned}
 1. \quad (a) \quad V &= 16 \iint_R \sqrt{1-x^2} \, dA \\
 &= 16 \int_0^{\pi/4} \int_0^1 \sqrt{1-r^2 \cos^2 \theta} \, r \, dr \, d\theta \\
 &= -\frac{16}{3} \int_0^{\pi/4} \frac{1}{\cos^2 \theta} [(1 - \cos^2 \theta)^{3/2} - 1] \, d\theta \\
 &= -\frac{16}{3} \left[\sec \theta + \cos \theta - \tan \theta \right]_0^{\pi/4} \\
 &= 8(2 - \sqrt{2}) \approx 4.6863
 \end{aligned}$$



(b) Programs will vary.

$$3. \quad (a) \quad \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + c. \text{ Let } a^2 = 2 - u^2, u = v.$$

$$\text{Then } \int \frac{1}{(2 - u^2) + v^2} dv = \frac{1}{\sqrt{2 - u^2}} \arctan \frac{v}{\sqrt{2 - u^2}} + C.$$

$$\begin{aligned}
 (b) \quad I_1 &= \int_0^{\sqrt{2}/2} \left[\frac{2}{\sqrt{2 - u^2}} \arctan \frac{v}{\sqrt{2 - u^2}} \right]_{-u}^u du \\
 &= \int_0^{\sqrt{2}/2} \frac{2}{\sqrt{2 - u^2}} \left(\arctan \frac{u}{\sqrt{2 - u^2}} - \arctan \frac{-u}{\sqrt{2 - u^2}} \right) du \\
 &= \int_0^{\sqrt{2}/2} \frac{4}{\sqrt{2 - u^2}} \arctan \frac{u}{\sqrt{2 - u^2}} du
 \end{aligned}$$

$$\text{Let } u = \sqrt{2} \sin \theta, du = \sqrt{2} \cos \theta d\theta, 2 - u^2 = 2 - 2 \sin^2 \theta = 2 \cos^2 \theta.$$

$$\begin{aligned}
 I_1 &= 4 \int_0^{\pi/6} \frac{1}{\sqrt{2} \cos \theta} \arctan \left(\frac{\sqrt{2} \sin \theta}{\sqrt{2} \cos \theta} \right) \cdot \sqrt{2} \cos \theta d\theta \\
 &= 4 \int_0^{\pi/6} \arctan(\tan \theta) d\theta = \frac{4\theta^2}{2} \Big|_0^{\pi/6} = 2 \left(\frac{\pi}{6} \right)^2 = \frac{\pi^2}{18}
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad I_2 &= \int_{\sqrt{2}/2}^{\sqrt{2}} \left[\frac{2}{\sqrt{2 - u^2}} \arctan \frac{v}{\sqrt{2 - u^2}} \right]_{u - \sqrt{2}}^{-u + \sqrt{2}} du \\
 &= \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{2}{\sqrt{2 - u^2}} \left[\arctan \left(\frac{-u + \sqrt{2}}{\sqrt{2 - u^2}} \right) - \arctan \left(\frac{u - \sqrt{2}}{\sqrt{2 - u^2}} \right) \right] du \\
 &= \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{4}{\sqrt{2 - u^2}} \arctan \left(\frac{\sqrt{2} - u}{\sqrt{2 - u^2}} \right) du
 \end{aligned}$$

$$\text{Let } u = \sqrt{2} \sin \theta.$$

$$\begin{aligned}
 I_2 &= 4 \int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{2} \cos \theta} \arctan \left(\frac{\sqrt{2} - \sqrt{2} \sin \theta}{\sqrt{2} \cos \theta} \right) \cdot \sqrt{2} \cos \theta d\theta \\
 &= 4 \int_{\pi/6}^{\pi/2} \arctan \left(\frac{1 - \sin \theta}{\cos \theta} \right) d\theta
 \end{aligned}$$

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3. —CONTINUED—

$$\begin{aligned}
 \text{(d)} \quad \tan\left(\frac{1}{2}\left(\frac{\pi}{2} - \theta\right)\right) &= \sqrt{\frac{1 - \cos((\pi/2) - \theta)}{1 + \cos((\pi/2) - \theta)}} = \sqrt{\frac{1 - \sin \theta}{1 + \sin \theta}} \\
 &= \sqrt{\frac{(1 - \sin \theta)^2}{(1 + \sin \theta)(1 - \sin \theta)}} = \sqrt{\frac{(1 - \sin \theta)^2}{\cos^2 \theta}} = \frac{1 - \sin \theta}{\cos \theta}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad I_2 &= 4 \int_{\pi/6}^{\pi/2} \arctan\left(\frac{1 - \sin \theta}{\cos \theta}\right) d\theta = 4 \int_{\pi/6}^{\pi/2} \arctan\left(\tan\left(\frac{1}{2}\left(\frac{\pi}{2} - \theta\right)\right)\right) d\theta \\
 &= 4 \int_{\pi/6}^{\pi/2} \frac{1}{2}\left(\frac{\pi}{2} - \theta\right) d\theta = 2 \int_{\pi/6}^{\pi/2} \left(\frac{\pi}{2} - \theta\right) d\theta \\
 &= 2 \left[\frac{\pi}{2} \theta - \frac{\theta^2}{2} \right]_{\pi/6}^{\pi/2} = 2 \left[\left(\frac{\pi^2}{4} - \frac{\pi^2}{8} \right) - \left(\frac{\pi^2}{12} - \frac{\pi^2}{72} \right) \right] \\
 &= 2 \left[\frac{18 - 9 - 6 + 1}{72} \pi^2 \right] = \frac{4}{36} \pi^2 = \frac{\pi^2}{9}
 \end{aligned}$$

$$\text{(f)} \quad \frac{1}{1 - xy} = 1 + (xy) + (xy)^2 + \cdots \quad |xy| < 1$$

$$\begin{aligned}
 \int_0^1 \int_0^1 \frac{1}{1 - xy} dx dy &= \int_0^1 \int_0^1 [1 + (xy) + (xy)^2 + \cdots] dx dy \\
 &= \int_0^1 \int_0^1 \sum_{K=0}^{\infty} (xy)^K dx dy = \sum_{K=0}^{\infty} \int_0^1 \frac{x^{K+1} y^{K+1}}{K+1} \Big|_0^1 dy \\
 &= \sum_{K=0}^{\infty} \int_0^1 \frac{y^{K+1}}{K+1} dy = \sum_{K=0}^{\infty} \frac{y^{K+2}}{(K+1)^2} \Big|_0^1 \\
 &= \sum_{K=0}^{\infty} \frac{1}{(K+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}
 \end{aligned}$$

$$\text{(g)} \quad u = \frac{x+y}{\sqrt{2}}, v = \frac{y-x}{\sqrt{2}}$$

$$u - v = \frac{2x}{\sqrt{2}} \Rightarrow x = \frac{u-v}{\sqrt{2}}$$

$$u + v = \frac{2y}{\sqrt{2}} \Rightarrow y = \frac{u+v}{\sqrt{2}}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} = 1$$

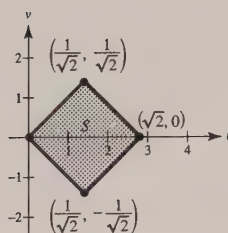
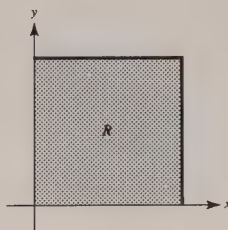
R S

$$(0, 0) \leftrightarrow (0, 0)$$

$$(1, 0) \leftrightarrow \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$$

$$(0, 1) \leftrightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$(1, 1) \leftrightarrow (\sqrt{2}, 0)$$



$$\int_0^1 \int_0^1 \frac{1}{1 - xy} dx dy = \int_0^{\sqrt{2}/2} \int_{-u}^u \frac{1}{1 - \frac{u^2}{2} + \frac{v^2}{2}} dv du + \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{u-\sqrt{2}}^{-u+\sqrt{2}} \frac{1}{1 - \frac{u^2}{2} + \frac{v^2}{2}} dv du$$

$$= I_1 + I_2 = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6}$$

5. Boundary in xy -plane

$$y = \sqrt{x}$$

$$y = \sqrt{2x}$$

$$y = \frac{1}{3}x^2$$

$$y = \frac{1}{4}x^2$$

Boundary in uv -plane

$$u = 1$$

$$u = 2$$

$$v = 3$$

$$v = 4$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{3}\left(\frac{v}{u}\right)^{2/3} & \frac{2}{3}\left(\frac{u}{v}\right)^{1/3} \\ \frac{2}{3}\left(\frac{v}{u}\right)^{1/3} & \frac{1}{3}\left(\frac{u}{v}\right)^{2/3} \end{vmatrix} = -\frac{1}{3}$$

$$A = \iint_R 1 \, dA = \iint_S 1 \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA = \frac{1}{3}$$

9. From Exercise 55, Section 13.3,

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$$

$$\text{Thus, } \int_0^{\infty} e^{-x^2/2} dx = \frac{\sqrt{2\pi}}{2} \text{ and } \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\int_0^{\infty} x^2 e^{-x^2} dx = \left[-\frac{1}{2} x e^{-x^2} \right]_0^{\infty} + \frac{1}{2} \int_0^{\infty} e^{-x^2} dx = \frac{1}{2} \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}$$

$$11. f(x, y) = \begin{cases} k e^{-(x+y)/a} & x \geq 0, y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dA &= \int_0^{\infty} \int_0^{\infty} k e^{-(x+y)/a} \, dx \, dy \\ &= k \int_0^{\infty} e^{-x/a} \, dx \cdot \int_0^{\infty} e^{-y/a} \, dy \end{aligned}$$

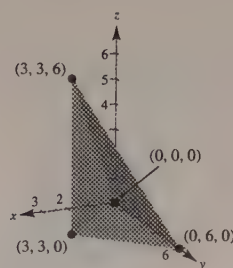
These two integrals are equal to

$$\int_0^{\infty} e^{-x/a} \, dx = \lim_{b \rightarrow \infty} \left[(-a) e^{-x/a} \right]_0^b = a.$$

Hence, assuming $a, k > 0$, you obtain

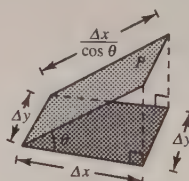
$$1 = ka^2 \quad \text{or} \quad a = \frac{1}{\sqrt{k}}$$

7.



$$V = \int_0^3 \int_0^{2x} \int_x^{6-x} dy \, dz \, dx = 18$$

$$13. A = l \cdot w = \left(\frac{\Delta x}{\cos \theta} \right) \Delta y = \sec \theta \Delta x \Delta y$$

Area in xy -plane: $\Delta x \Delta y$

CHAPTER 14

Vector Analysis

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CHAPTER 14

Vector Analysis

Section 14.1 Vector Fields

Solutions to Odd-Numbered Exercises

1. All vectors are parallel to y-axis.

Matches (c)

3. All vectors point outward.

Matches (b)

5. Vectors are parallel to x-axis for $y = n\pi$.

Matches (a)

7. $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$

$$\|\mathbf{F}\| = \sqrt{2}$$



9. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

$$\|\mathbf{F}\| = \sqrt{x^2 + y^2} = c$$

$$x^2 + y^2 = c^2$$



11. $\mathbf{F}(x, y, z) = 3y\mathbf{j}$

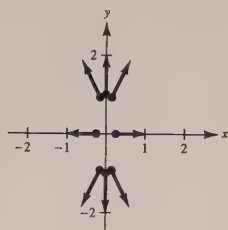
$$\|\mathbf{F}\| = 3|y| = c$$



13. $\mathbf{F}(x, y) = 4x\mathbf{i} + y\mathbf{j}$

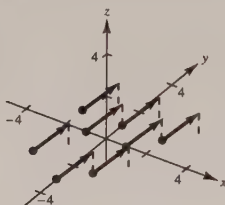
$$\|\mathbf{F}\| = \sqrt{16x^2 + y^2} = c$$

$$\frac{x^2}{c^2/16} + \frac{y^2}{c^2} = 1$$



15. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\|\mathbf{F}\| = \sqrt{3}$$



- 17.



- 19.



21. $f(x, y) = 5x^2 + 3xy + 10y^2$

$$f_x(x, y) = 10x + 3y$$

$$f_y(x, y) = 3x + 20y$$

$$\mathbf{F}(x, y) = (10x + 3y)\mathbf{i} + (3x + 20y)\mathbf{j}$$

23. $f(x, y, z) = z - ye^{x^2}$

$$f_x(x, y, z) = -2xye^{x^2}$$

$$f_y(x, y, z) = -e^{x^2}$$

$$f_z = 1$$

$$\mathbf{F}(x, y, z) = -2xye^{x^2}\mathbf{i} - e^{x^2}\mathbf{j} + \mathbf{k}$$

25. $g(x, y, z) = xy \ln(x + y)$

$$g_x(x, y, z) = y \ln(x + y) + \frac{xy}{x + y}$$

$$g_y(x, y, z) = x \ln(x + y) + \frac{xy}{x + y}$$

$$g_z(x, y, z) = 0$$

$$\mathbf{G}(x, y, z) = \left[\frac{xy}{x + y} + y \ln(x + y) \right] \mathbf{i} + \left[\frac{xy}{x + y} + x \ln(x + y) \right] \mathbf{j}$$

27. $\mathbf{F}(x, y) = 12xy\mathbf{i} + 6(x^2 + y)\mathbf{j}$

$M = 12xy$ and $N = 6(x^2 + y)$ have continuous first partial derivatives.

$$\frac{\partial N}{\partial x} = 12x = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative.}$$

31. $M = 15y^3, N = -5xy^2$

$$\frac{\partial N}{\partial x} = -5y^2 \neq \frac{\partial M}{\partial y} = 45y^2 \Rightarrow \text{Not conservative}$$

35. $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

$$\frac{\partial}{\partial y}[2xy] = 2x$$

$$\frac{\partial}{\partial x}[x^2] = 2x$$

Conservative

$$f_x(x, y) = 2xy$$

$$f_y(x, y) = x^2$$

$$f(x, y) = x^2y + K$$

37. $\mathbf{F}(x, y) = xe^{x^2y}(2y\mathbf{i} + x\mathbf{j})$

$$\frac{\partial}{\partial y}[2xye^{x^2y}] = 2xe^{x^2y} + 2x^3ye^{x^2y}$$

$$\frac{\partial}{\partial x}[x^2e^{x^2y}] = 2xe^{x^2y} + 2x^3ye^{x^2y}$$

Conservative

$$f_x(x, y) = 2xye^{x^2y}$$

$$f_y(x, y) = x^2e^{x^2y}$$

$$f(x, y) = e^{x^2y} + K$$

29. $\mathbf{F}(x, y) = \sin y\mathbf{i} + x \cos y\mathbf{j}$

$M = \sin y$ and $N = x \cos y$ have continuous first partial derivatives.

$$\frac{\partial N}{\partial x} = \cos y = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} \text{ is conservative.}$$

33. $M = \frac{2}{y}e^{2x/y}, N = \frac{-2x}{y^2}e^{2x/y}$

$$\frac{\partial N}{\partial x} = \frac{-2(y + 2x)}{y^3}e^{2x/y} = \frac{\partial M}{\partial y} \Rightarrow \text{Conservative}$$

39. $\mathbf{F}(x, y) = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{x}{x^2 + y^2}\right] = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial}{\partial x}\left[\frac{y}{x^2 + y^2}\right] = -\frac{2xy}{(x^2 + y^2)^2}$$

Conservative

$$f_x(x, y) = \frac{x}{x^2 + y^2}$$

$$f_y(x, y) = \frac{y}{x^2 + y^2}$$

$$f(x, y) = \frac{1}{2} \ln(x^2 + y^2) + K$$

41. $\mathbf{F}(x, y) = e^x(\cos y\mathbf{i} + \sin y\mathbf{j})$

$$\frac{\partial}{\partial y}[e^x \cos y] = -e^x \sin y$$

$$\frac{\partial}{\partial x}[e^x \sin y] = e^x \sin y$$

Not conservative

43. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}, (1, 2, 1)$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$$\text{curl } \mathbf{F}(1, 2, 1) = 2\mathbf{j} - \mathbf{k}$$

45. $\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j}, (0, 0, 3)$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & -e^x \cos y & 0 \end{vmatrix} = -2e^x \cos y\mathbf{k}$$

$$\text{curl } \mathbf{F}(0, 0, 3) = -2\mathbf{k}$$

$$47. \mathbf{F}(x, y, z) = \arctan\left(\frac{x}{y}\right)\mathbf{i} + \ln\sqrt{x^2 + y^2}\mathbf{j} + \mathbf{k}$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \arctan\left(\frac{x}{y}\right) & \frac{1}{2} \ln(x^2 + y^2) & 1 \end{vmatrix} = \left[\frac{x}{x^2 + y^2} - \frac{(-x/y^2)}{1 + (x/y)^2} \right] \mathbf{k} = \frac{2x}{x^2 + y^2} \mathbf{k}$$

$$49. \mathbf{F}(x, y, z) = \sin(x - y)\mathbf{i} + \sin(y - z)\mathbf{j} + \sin(z - x)\mathbf{k}$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(x - y) & \sin(y - z) & \sin(z - x) \end{vmatrix} = \cos(y - z)\mathbf{i} + \cos(z - x)\mathbf{j} + \cos(x - y)\mathbf{k}$$

$$51. \mathbf{F}(x, y, z) = \sin y \mathbf{i} - x \cos y \mathbf{j} + \mathbf{k}$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & -x \cos y & 1 \end{vmatrix} = -2 \cos y \mathbf{k} \neq \mathbf{0}$$

Not conservative

$$53. \mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = ye^z$$

$$f_y(x, y, z) = xe^z$$

$$f_z(x, y, z) = xye^z$$

$$f(x, y, z) = xye^z + K$$

$$55. \mathbf{F}(x, y, z) = \frac{1}{y}\mathbf{i} - \frac{x}{y^2}\mathbf{j} + (2z - 1)\mathbf{k}$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{y} & -\frac{x}{y^2} & 2z - 1 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = \frac{1}{y}$$

$$f_y(x, y, z) = -\frac{x}{y^2}$$

$$f_z(x, y, z) = 2z - 1$$

$$f(x, y, z) = \int \frac{1}{y} dx = \frac{x}{y} + g(y, z) + K_1$$

$$f(x, y, z) = \int -\frac{x}{y^2} dy = \frac{x}{y} + h(x, z) + K_2$$

$$\begin{aligned} f(x, y, z) &= \int (2z - 1) dz \\ &= z^2 - z + p(x, y) + K_3 \end{aligned}$$

$$f(x, y, z) = \frac{x}{y} + z^2 - z + K$$

$$57. \mathbf{F}(x, y, z) = 6x^2\mathbf{i} - xy^2\mathbf{j}$$

$$\begin{aligned} \text{div } \mathbf{F}(x, y, z) &= \frac{\partial}{\partial x}[6x^2] + \frac{\partial}{\partial y}[-xy^2] \\ &= 12x - 2xy \end{aligned}$$

59. $\mathbf{F}(x, y, z) = \sin x \mathbf{i} + \cos y \mathbf{j} + z^2 \mathbf{k}$

$$\operatorname{div} \mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[\sin x] + \frac{\partial}{\partial y}[\cos y] + \frac{\partial}{\partial z}[z^2] = \cos x - \sin y + 2z$$

61. $\mathbf{F}(x, y, z) = xyz \mathbf{i} + y \mathbf{j} + z \mathbf{k}$

$$\operatorname{div} \mathbf{F}(x, y, z) = yz + 1 + 1 = yz + 2$$

$$\operatorname{div} \mathbf{F}(1, 2, 1) = 4$$

63. $\mathbf{F}(x, y, z) = e^x \sin y \mathbf{i} - e^x \cos y \mathbf{j}$

$$\operatorname{div} \mathbf{F}(x, y, z) = e^x \sin y + e^x \sin y$$

$$\operatorname{div} \mathbf{F}(0, 0, 3) = 0$$

65. See the definition, page 1008. Examples include velocity fields, gravitational fields and magnetic fields.

67. See the definition on page 1014.

69. $\mathbf{F}(x, y, z) = \mathbf{i} + 2x \mathbf{j} + 3y \mathbf{k}$

$$\mathbf{G}(x, y, z) = x \mathbf{i} - y \mathbf{j} + z \mathbf{k}$$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 3y \\ x & -y & z \end{vmatrix} = (2xz + 3y^2) \mathbf{i} - (z - 3xy) \mathbf{j} + (-y - 2x^2) \mathbf{k}$$

$$\operatorname{curl}(\mathbf{F} \times \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz + 3y^2 & 3xy - z & -y - 2x^2 \end{vmatrix} = (-1 + 1) \mathbf{i} - (-4x - 2x) \mathbf{j} + (3y - 6y) \mathbf{k} = 6x \mathbf{j} - 3y \mathbf{k}$$

71. $\mathbf{F}(x, y, z) = xyz \mathbf{i} + y \mathbf{j} + z \mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy \mathbf{j} - xz \mathbf{k}$$

$$\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xy & -xz \end{vmatrix} = z \mathbf{j} + y \mathbf{k}$$

73. $\mathbf{F}(x, y, z) = \mathbf{i} + 2x \mathbf{j} + 3y \mathbf{k}$

$$\mathbf{G}(x, y, z) = x \mathbf{i} - y \mathbf{j} + z \mathbf{k}$$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 3y \\ x & -y & z \end{vmatrix} = (2xz + 3y^2) \mathbf{i} - (z - 3xy) \mathbf{j} + (-y - 2x^2) \mathbf{k}$$

$$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = 2z + 3x$$

75. $\mathbf{F}(x, y, z) = xyz \mathbf{i} + y \mathbf{j} + z \mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy \mathbf{j} - xz \mathbf{k}$$

$$\operatorname{div}(\operatorname{curl} \mathbf{F}) = x - x = 0$$

77. Let $\mathbf{F} = M \mathbf{i} + N \mathbf{j} + P \mathbf{k}$ and $\mathbf{G} = Q \mathbf{i} + R \mathbf{j} + S \mathbf{k}$ where M, N, P, Q, R , and S have continuous partial derivatives.

$$\mathbf{F} + \mathbf{G} = (M + Q) \mathbf{i} + (N + R) \mathbf{j} + (P + S) \mathbf{k}$$

$$\begin{aligned} \operatorname{curl}(\mathbf{F} + \mathbf{G}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M + Q & N + R & P + S \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y}(P + S) - \frac{\partial}{\partial z}(N + R) \right] \mathbf{i} - \left[\frac{\partial}{\partial x}(P + S) - \frac{\partial}{\partial z}(M + Q) \right] \mathbf{j} + \left[\frac{\partial}{\partial x}(N + R) - \frac{\partial}{\partial y}(M + Q) \right] \mathbf{k} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} + \left(\frac{\partial S}{\partial y} - \frac{\partial R}{\partial z} \right) \mathbf{i} - \left(\frac{\partial S}{\partial x} - \frac{\partial Q}{\partial z} \right) \mathbf{j} + \left(\frac{\partial R}{\partial x} - \frac{\partial Q}{\partial y} \right) \mathbf{k} \\ &= \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G} \end{aligned}$$

79. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = R\mathbf{i} + S\mathbf{j} + T\mathbf{k}$.

$$\begin{aligned}\operatorname{div}(\mathbf{F} + \mathbf{G}) &= \frac{\partial}{\partial x}(M + R) + \frac{\partial}{\partial y}(N + S) + \frac{\partial}{\partial z}(P + T) = \frac{\partial M}{\partial x} + \frac{\partial R}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial S}{\partial y} + \frac{\partial P}{\partial z} + \frac{\partial T}{\partial z} \\ &= \left[\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right] + \left[\frac{\partial R}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial T}{\partial z} \right] \\ &= \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}\end{aligned}$$

81. $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$

$$\begin{aligned}\nabla \times [\nabla f + (\nabla \times \mathbf{F})] &= \operatorname{curl}(\nabla f + (\nabla \times \mathbf{F})) \\ &= \operatorname{curl}(\nabla f) + \operatorname{curl}(\nabla \times \mathbf{F}) \quad (\text{Exercise 77}) \\ &= \operatorname{curl}(\nabla \times \mathbf{F}) \quad (\text{Exercise 78}) \\ &= \nabla \times (\nabla \times \mathbf{F})\end{aligned}$$

83. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, then $f\mathbf{F} = fM\mathbf{i} + fN\mathbf{j} + fP\mathbf{k}$.

$$\begin{aligned}\operatorname{div}(f\mathbf{F}) &= \frac{\partial}{\partial x}(fM) + \frac{\partial}{\partial y}(fN) + \frac{\partial}{\partial z}(fP) = f\frac{\partial M}{\partial x} + M\frac{\partial f}{\partial x} + f\frac{\partial N}{\partial y} + N\frac{\partial f}{\partial y} + f\frac{\partial P}{\partial z} + P\frac{\partial f}{\partial z} \\ &= f\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) + \left(\frac{\partial f}{\partial x}M + \frac{\partial f}{\partial y}N + \frac{\partial f}{\partial z}P\right) \\ &= f\operatorname{div} \mathbf{F} + \nabla f \cdot \mathbf{F}\end{aligned}$$

In Exercises 85 and 87, $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $f(x, y, z) = \|\mathbf{F}(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$.

85. $\ln f = \frac{1}{2} \ln(x^2 + y^2 + z^2)$

$$\nabla(\ln f) = \frac{x}{x^2 + y^2 + z^2}\mathbf{i} + \frac{y}{x^2 + y^2 + z^2}\mathbf{j} + \frac{z}{x^2 + y^2 + z^2}\mathbf{k} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2} = \frac{\mathbf{F}}{f^2}$$

87. $f^n = (\sqrt{x^2 + y^2 + z^2})^n$

$$\begin{aligned}\nabla f^n &= n(\sqrt{x^2 + y^2 + z^2})^{n-1} \frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + n(\sqrt{x^2 + y^2 + z^2})^{n-1} \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} \\ &\quad + n(\sqrt{x^2 + y^2 + z^2})^{n-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k} \\ &= n(\sqrt{x^2 + y^2 + z^2})^{n-2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = nf^{n-2}\mathbf{F}\end{aligned}$$

89. The winds are stronger over Phoenix. Although the winds over both cities are northeasterly, they are more towards the east over Atlanta.

Section 14.2 Line Integrals

1. $x^2 + y^2 = 9$

$$\frac{x^2}{9} + \frac{y^2}{9} = 1$$

$$\cos^2 t + \sin^2 t = 1$$

$$\cos^2 t = \frac{x^2}{9}$$

$$\sin^2 t = \frac{y^2}{9}$$

$$x = 3 \cos t$$

$$y = 3 \sin t$$

$$\mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j}$$

$$0 \leq t \leq 2\pi$$

3.
$$\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 3 \\ 3\mathbf{i} + (t-3)\mathbf{j}, & 3 \leq t \leq 6 \\ (9-t)\mathbf{i} + 3\mathbf{j}, & 6 \leq t \leq 9 \\ (12-t)\mathbf{j}, & 9 \leq t \leq 12 \end{cases}$$

5.
$$\mathbf{r}(t) = \begin{cases} t\mathbf{i} + \sqrt{t}\mathbf{j}, & 0 \leq t \leq 1 \\ (2-t)\mathbf{i} + (2-t)\mathbf{j}, & 1 \leq t \leq 2 \end{cases}$$

7. $\mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j}, 0 \leq t \leq 2; \mathbf{r}'(t) = 4\mathbf{i} + 3\mathbf{j}$

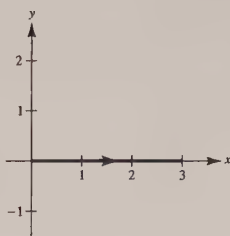
$$\int_C (x-y) ds = \int_0^2 (4t-3t) \sqrt{(4)^2 + (3)^2} dt = \int_0^2 5t dt = \left[\frac{5t^2}{2} \right]_0^2 = 10$$

9. $\mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + 8t \mathbf{k}, 0 \leq t \leq \frac{\pi}{2}; \mathbf{r}'(t) = \cos t \mathbf{i} - \sin t \mathbf{j} + 8\mathbf{k}$

$$\begin{aligned} \int_C (x^2 + y^2 + z^2) ds &= \int_0^{\pi/2} (\sin^2 t + \cos^2 t + 64t^2) \sqrt{(\cos t)^2 + (-\sin t)^2 + 64} dt \\ &= \int_0^{\pi/2} \sqrt{65}(1 + 64t^2) dt = \left[\sqrt{65} \left(t + \frac{64t^3}{3} \right) \right]_0^{\pi/2} = \sqrt{65} \left(\frac{\pi}{2} + \frac{8\pi^3}{3} \right) = \frac{\sqrt{65}\pi}{6} (3 + 16\pi^2) \end{aligned}$$

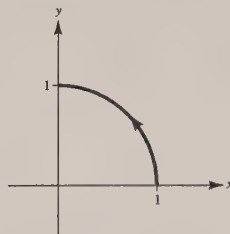
11. $\mathbf{r}(t) = t\mathbf{i}, 0 \leq t \leq 3$

$$\begin{aligned} \int_C (x^2 + y^2) ds &= \int_0^3 [t^2 + 0^2] \sqrt{1 + 0} dt \\ &= \int_0^3 t^2 dt \\ &= \left[\frac{1}{3} t^3 \right]_0^3 = 9 \end{aligned}$$



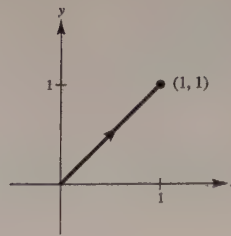
13. $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}, 0 \leq t \leq \frac{\pi}{2}$

$$\begin{aligned} \int_C (x^2 + y^2) ds &= \int_0^{\pi/2} [\cos^2 t + \sin^2 t] \sqrt{(-\sin t)^2 + (\cos t)^2} dt \\ &= \int_0^{\pi/2} dt = \frac{\pi}{2} \end{aligned}$$



15. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$, $0 \leq t \leq 1$

$$\begin{aligned}\int_C (x + 4\sqrt{y}) \, ds &= \int_0^1 (t + 4\sqrt{t})\sqrt{1+1} \, dt \\ &= \left[\sqrt{2} \left(\frac{t^2}{2} + \frac{8}{3} t^{3/2} \right) \right]_0^1 = \frac{19\sqrt{2}}{6}\end{aligned}$$



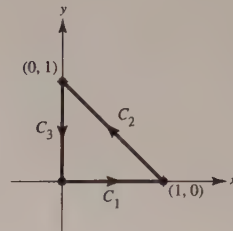
17. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 1 \\ (2-t)\mathbf{i} + (t-1)\mathbf{j}, & 1 \leq t \leq 2 \\ (3-t)\mathbf{j}, & 2 \leq t \leq 3 \end{cases}$

$$\int_{C_1} (x + 4\sqrt{y}) \, ds = \int_0^1 t \, dt = \frac{1}{2}$$

$$\begin{aligned}\int_{C_2} (x + 4\sqrt{y}) \, ds &= \int_1^2 [(2-t) + 4\sqrt{t-1}]\sqrt{1+1} \, dt \\ &= \sqrt{2} \left[2t - \frac{t^2}{2} + \frac{8}{3}(t-1)^{3/2} \right]_1^2 = \frac{19\sqrt{2}}{6}\end{aligned}$$

$$\int_{C_3} (x + 4\sqrt{y}) \, ds = \int_2^3 4\sqrt{3-t} \, dt = \left[-\frac{8}{3}(3-t)^{3/2} \right]_2^3 = \frac{8}{3}$$

$$\int_C (x + 4\sqrt{y}) \, ds = \frac{1}{2} + \frac{19\sqrt{2}}{6} + \frac{8}{3} = \frac{19 + 19\sqrt{2}}{6} = \frac{19(1 + \sqrt{2})}{6}$$



19. $\rho(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)$

$$\mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j} + 2t \mathbf{k}, \quad 0 \leq t \leq 4\pi$$

$$\mathbf{r}'(t) = -3 \sin t \mathbf{i} + 3 \cos t \mathbf{j} + 2 \mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + (2)^2} = \sqrt{13}$$

$$\begin{aligned}\text{Mass} &= \int_C \rho(x, y, z) \, ds = \int_0^{4\pi} \frac{1}{2} [(3 \cos t)^2 + (3 \sin t)^2 + (2t)^2] \sqrt{13} \, dt \\ &= \frac{\sqrt{13}}{2} \int_0^{4\pi} (9 + 4t^2) \, dt = \left[\frac{\sqrt{13}}{2} \left(9t + \frac{4t^3}{3} \right) \right]_0^{4\pi} \\ &= \frac{2\sqrt{13}\pi}{3} (27 + 64\pi^2) \approx 4973.8\end{aligned}$$

21. $\mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j}$

$$C: \mathbf{r}(t) = 4t\mathbf{i} + t\mathbf{j}, \quad 0 \leq t \leq 1$$

$$\mathbf{F}(t) = 4t^2\mathbf{i} + t\mathbf{j}$$

$$\mathbf{r}'(t) = 4\mathbf{i} + \mathbf{j}$$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (16t^2 + t) \, dt \\ &= \left[\frac{16}{3}t^3 + \frac{1}{2}t^2 \right]_0^1 = \frac{35}{6}\end{aligned}$$

23. $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

$$C: \mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{F}(t) = 6 \cos t \mathbf{i} + 8 \sin t \mathbf{j}$$

$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}$$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} (-12 \sin t \cos t + 16 \sin t \cos t) \, dt \\ &= \left[2 \sin^2 t \right]_0^{\pi/2} = 2\end{aligned}$$

25. $\mathbf{F}(x, y, z) = x^2y\mathbf{i} + (x - z)\mathbf{j} + xyz\mathbf{k}$

$C: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}, 0 \leq t \leq 1$

$\mathbf{F}(t) = t^4\mathbf{i} + (t - 2)\mathbf{j} + 2t^3\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 [t^4 + 2t(t - 2)] dt \\ &= \left[\frac{t^5}{5} + \frac{2t^3}{3} - 2t^2 \right]_0^1 = -\frac{17}{15}\end{aligned}$$

29. $\mathbf{F}(x, y) = -x\mathbf{i} - 2y\mathbf{j}$

$C: y = x^3$ from $(0, 0)$ to $(2, 8)$

$\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}, 0 \leq t \leq 2$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = -t\mathbf{i} - 2t^3\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = -t - 6t^5$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (-t - 6t^5) dt = \left[-\frac{1}{2}t^2 - t^6 \right]_0^2 = -66$$

31. $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$

C : counterclockwise around the triangle whose vertices are $(0, 0)$, $(1, 0)$, $(1, 1)$

$$\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 1 \\ \mathbf{i} + (t - 1)\mathbf{j}, & 1 \leq t \leq 2 \\ (3 - t)\mathbf{i} + (3 - t)\mathbf{j}, & 2 \leq t \leq 3 \end{cases}$$

On C_1 : $\mathbf{F}(t) = 2t\mathbf{i}, \mathbf{r}'(t) = \mathbf{i}$

$$\text{Work} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 2t dt = 1$$

On C_2 : $\mathbf{F}(t) = 2\mathbf{i} + (t - 1)\mathbf{j}, \mathbf{r}'(t) = \mathbf{j}$

$$\text{Work} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_1^2 (t - 1) dt = \frac{1}{2}$$

On C_3 : $\mathbf{F}(t) = 2(3 - t)\mathbf{i} + (3 - t)\mathbf{j}, \mathbf{r}'(t) = -\mathbf{i} - \mathbf{j}$

$$\text{Work} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_2^3 [-2(3 - t) - (3 - t)] dt = -\frac{3}{2}$$

$$\text{Total work} = \int_C \mathbf{F} \cdot d\mathbf{r} = 1 + \frac{1}{2} - \frac{3}{2} = 0$$

33. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 5z\mathbf{k}$

$C: \mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 2\pi$

$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \mathbf{k}$

$\mathbf{F}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} - 5t\mathbf{k}$

$\mathbf{F} \cdot \mathbf{r}' = -5t$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -5t dt = -10\pi^2$$

27. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + 6y\mathbf{j} + yz^2\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, 1 \leq t \leq 3$

$\mathbf{F}(t) = t^2 \ln t\mathbf{i} + 6t^2\mathbf{j} + t^2 \ln^2 t\mathbf{k}$

$d\mathbf{r} = \left(\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k} \right) dt$

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_1^3 [t^2 \ln t + 12t^3 + t(\ln t)^2] dt \\ &\approx 249.49\end{aligned}$$

35. $\mathbf{r}(t) = 3 \sin t\mathbf{i} + 3 \cos t\mathbf{j} + \frac{10}{2\pi}t\mathbf{k}, 0 \leq t \leq 2\pi$

$\mathbf{F} = 150\mathbf{k}$

$d\mathbf{r} = \left(3 \cos t\mathbf{i} - 3 \sin t\mathbf{j} + \frac{10}{2\pi}\mathbf{k} \right) dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \frac{1500}{2\pi} dt = \left[\frac{1500}{2\pi}t \right]_0^{2\pi} = 1500 \text{ ft} \cdot \text{lb}$$

37. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = 2t\mathbf{i} + (t-1)\mathbf{j}, 1 \leq t \leq 3$

$\mathbf{r}_1'(t) = 2\mathbf{i} + \mathbf{j}$

$\mathbf{F}(t) = 4t^2\mathbf{i} + 2t(t-1)\mathbf{j}$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_1^3 (8t^2 + 2t(t-1)) dt = \frac{236}{3}$$

Both paths join (2, 0) and (6, 2). The integrals are negatives of each other because the orientations are different.

(b) $\mathbf{r}_2(t) = 2(3-t)\mathbf{i} + (2-t)\mathbf{j}, 0 \leq t \leq 2$

$\mathbf{r}_2'(t) = -2\mathbf{i} - \mathbf{j}$

$\mathbf{F}(t) = 4(3-t)^2\mathbf{i} + 2(3-t)(2-t)\mathbf{j}$

$$\begin{aligned} \int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 [-8(3-t)^2 - 2(3-t)(2-t)] dt \\ &= -\frac{236}{3} \end{aligned}$$

39. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} - 2t\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} - 2\mathbf{j}$

$\mathbf{F}(t) = -2t\mathbf{i} - t\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = -2t + 2t = 0$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

41. $\mathbf{F}(x, y) = (x^3 - 2x^2)\mathbf{i} + \left(x - \frac{y}{2}\right)\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = (t^3 - 2t^2)\mathbf{i} + \left(t - \frac{t^2}{2}\right)\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = (t^3 - 2t^2) + 2t\left(t - \frac{t^2}{2}\right) = 0$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

43. $x = 2t, y = 10t, 0 \leq t \leq 1 \Rightarrow y = 5x \text{ or } x = \frac{y}{5}, 0 \leq y \leq 10$

$$\int_C (x + 3y^2) dy = \int_0^{10} \left(\frac{y}{5} + 3y^2\right) dy = \left[\frac{y^2}{10} + y^3\right]_0^{10} = 1010$$

45. $x = 2t, y = 10t, 0 \leq t \leq 1 \Rightarrow x = \frac{y}{5}, 0 \leq y \leq 10, dx = \frac{1}{5} dy$

$$\int_C xy dx + y dy = \int_0^{10} \left(\frac{y^2}{25} + y\right) dy = \left[\frac{y^3}{75} + \frac{y^2}{2}\right]_0^{10} = \frac{190}{3} \text{ OR}$$

$y = 5x, dy = 5 dx, 0 \leq x \leq 2$

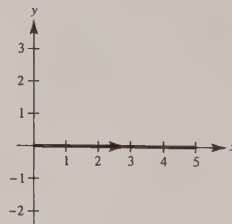
$$\int_C xy dx + y dy = \int_0^2 (5x^2 + 25x) dx = \left[\frac{5x^3}{3} + \frac{25x^2}{2}\right]_0^2 = \frac{190}{3}$$

47. $\mathbf{r}(t) = t\mathbf{i}, 0 \leq t \leq 5$

$x(t) = t, y(t) = 0$

$dx = dt, dy = 0$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^5 2t dt = 25$$



$$49. \mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 3 \\ 3\mathbf{i} + (t-3)\mathbf{j}, & 3 \leq t \leq 6 \end{cases}$$

$$C_1: x(t) = t, y(t) = 0,$$

$$dx = dt, dy = 0$$

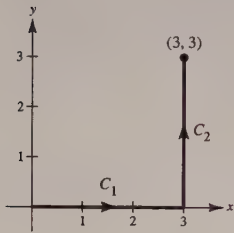
$$\int_{C_1} (2x - y) dx + (x + 3y) dy = \int_0^3 2t dt = 9$$

$$C_2: x(t) = 3, y(t) = t - 3$$

$$dx = 0, dy = dt$$

$$\int_{C_2} (2x - y) dx + (x + 3y) dy = \int_3^6 [3 + 3(t-3)] dt = \left[\frac{3t^2}{2} - 6t \right]_3^6 = \frac{45}{2}$$

$$\int_C (2x - y) dx + (x + 3y) dy = 9 + \frac{45}{2} = \frac{63}{2}$$



$$51. x(t) = t, y(t) = 1 - t^2, 0 \leq t \leq 1, dx = dt, dy = -2t dt$$

$$\begin{aligned} \int_C (2x - y) dx + (x + 3y) dy &= \int_0^1 [(2t - 1 + t^2) + (t + 3 - 3t^2)(-2t)] dt \\ &= \int_0^1 (6t^3 - t^2 - 4t - 1) dt = \left[\frac{3t^4}{2} - \frac{t^3}{3} - 2t^2 - t \right]_0^1 = -\frac{11}{6} \end{aligned}$$

$$53. x(t) = t, y(t) = 2t^2, 0 \leq t \leq 2$$

$$dx = dt, dy = 4t dt$$

$$\begin{aligned} \int_C (2x - y) dx + (x + 3y) dy &= \int_0^2 (2t - 2t^2) dt + \int_0^2 (t + 6t^2) 4t dt \\ &= \int_0^2 (24t^3 + 2t^2 + 2t) dt = \left[6t^4 + \frac{2}{3}t^3 + t^2 \right]_0^2 = \frac{316}{3} \end{aligned}$$

$$55. f(x, y) = h$$

C: line from (0, 0) to (3, 4)

$$\mathbf{r} = 3t\mathbf{i} + 4t\mathbf{j}, 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = 3\mathbf{i} + 4\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 5$$

Lateral surface area:

$$\int_C f(x, y) ds = \int_0^1 5h dt = 5h$$

$$57. f(x, y) = xy$$

C: $x^2 + y^2 = 1$ from (1, 0) to (0, 1)

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}, 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 1$$

Lateral surface area:

$$\begin{aligned} \int_C f(x, y) ds &= \int_0^{\pi/2} \cos t \sin t dt \\ &= \left[\frac{\sin^2 t}{2} \right]_0^{\pi/2} = \frac{1}{2} \end{aligned}$$

59. $f(x, y) = h$ C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + [1 - (1 - t)^2]\mathbf{j}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = -\mathbf{i} + 2(1 - t)\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1 - t)^2}$$

Lateral surface area:

$$\begin{aligned} \int_C f(x, y) \, ds &= \int_0^1 h \sqrt{1 + 4(1 - t)^2} \, dt \\ &= -\frac{h}{4} \left[2(1 - t) \sqrt{1 + 4(1 - t)^2} + \ln |2(1 - t) + \sqrt{1 + 4(1 - t)^2}| \right]_0^1 \\ &= \frac{h}{4} [2\sqrt{5} + \ln(2 + \sqrt{5})] \approx 1.4789h \end{aligned}$$

61. $f(x, y) = xy$ C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$ You could parameterize the curve C as in Exercises 59 and 60. Alternatively, let $x = \cos t$, then:

$$y = 1 - \cos^2 t = \sin^2 t$$

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin^2 t \mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + 2 \sin t \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + 4 \sin^2 t \cos^2 t} = \sin t \sqrt{1 + 4 \cos^2 t}$$

Lateral surface area:

$$\int_C f(x, y) \, ds = \int_0^{\pi/2} \cos t \sin^2 t (\sin t \sqrt{1 + 4 \cos^2 t}) \, dt = \int_0^{\pi/2} \sin^2 t [(1 + 4 \cos^2 t)^{1/2} \sin t \cos t] \, dt$$

Let $u = \sin^2 t$ and $dv = (1 + 4 \cos^2 t)^{1/2} \sin t \cos t$, then $du = 2 \sin t \cos t \, dt$ and $v = -\frac{1}{12}(1 + 4 \cos^2 t)^{3/2}$.

$$\begin{aligned} \int_C f(x, y) \, ds &= \left[-\frac{1}{12} \sin^2 t (1 + 4 \cos^2 t)^{3/2} \right]_0^{\pi/2} + \frac{1}{6} \int_0^{\pi/2} (1 + 4 \cos^2 t)^{3/2} \sin t \cos t \, dt \\ &= \left[-\frac{1}{12} \sin^2 t (1 + 4 \cos^2 t)^{3/2} - \frac{1}{120} (1 + 4 \cos^2 t)^{5/2} \right]_0^{\pi/2} \\ &= \left(-\frac{1}{12} - \frac{1}{120} \right) + \frac{1}{120} (5)^{5/2} = \frac{1}{120} (25\sqrt{5} - 11) \approx 0.3742 \end{aligned}$$

63. (a) $f(x, y) = 1 + y^2$

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi$$

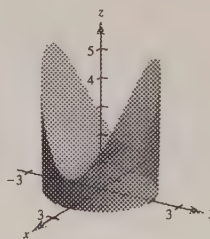
$$\mathbf{r}'(t) = -2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 2$$

$$\begin{aligned} S &= \int_C f(x, y) \, ds = \int_0^{2\pi} (1 + 4 \sin^2 t)(2) \, dt \\ &= \left[2t + 4(t - \sin t \cos t) \right]_0^{2\pi} = 12\pi \approx 37.70 \text{ cm}^2 \end{aligned}$$

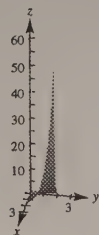
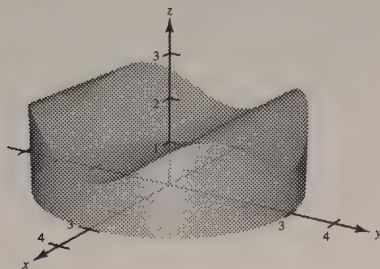
$$(b) \quad 0.2(12\pi) = \frac{12\pi}{5} \approx 7.54 \text{ cm}^3$$

(c)



65. $S \approx 25$

Matches b


 67. (a) Graph of: $\mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j} + (1 + \sin^2 2t) \mathbf{k}$, $0 \leq t \leq 2\pi$

 (b) Consider the portion of the surface in the first quadrant. The curve $z = 1 + \sin^2 2t$ is over the curve $\mathbf{r}_1(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j}$, $0 \leq t \leq \pi/2$. Hence, the total lateral surface area is

$$4 \int_C f(x, y) ds = 4 \int_0^{\pi/2} (1 + \sin^2 2t) 3 dt = 12 \left(\frac{3\pi}{4} \right) = 9\pi \text{ sq. cm}$$

 (c) The cross sections parallel to the xz -plane are rectangles of height $1 + 4(y/3)^2(1 - y^2/9)$ and base $2\sqrt{9 - y^2}$. Hence,

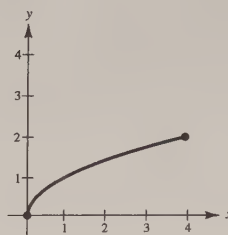
$$\text{Volume} = 2 \int_0^3 2\sqrt{9 - y^2} \left(1 + 4\frac{y^2}{9} \left(1 - \frac{y^2}{9} \right) \right) dy = \frac{27\pi}{2} \approx 42.412 \text{ cm}^3$$

69. See the definition of Line Integral, page 1020.

See Theorem 14.4.

71. The greater the height of the surface over the curve, the greater the lateral surface area. Hence,

$$z_3 < z_1 < z_2 < z_4.$$



73. False

$$\int_C xy ds = \sqrt{2} \int_0^1 t^2 dt$$

75. False, the orientations are different.

Section 14.3 Conservative Vector Fields and Independence of Path

1. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^2 + 2t^4) dt = \frac{11}{15}$$

(b) $\mathbf{r}_2(\theta) = \sin \theta \mathbf{i} + \sin^2 \theta \mathbf{j}, 0 \leq \theta \leq \frac{\pi}{2}$

$\mathbf{r}_2'(\theta) = \cos \theta \mathbf{i} + 2 \sin \theta \cos \theta \mathbf{j}$

$\mathbf{F}(t) = \sin^2 \theta \mathbf{i} + \sin^3 \theta \mathbf{j}$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} (\sin^2 \theta \cos \theta + 2 \sin^4 \theta \cos \theta) d\theta \\ &= \left[\frac{\sin^3 \theta}{3} + \frac{2 \sin^5 \theta}{5} \right]_0^{\pi/2} = \frac{11}{15} \end{aligned}$$

3. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(\theta) = \sec \theta \mathbf{i} + \tan \theta \mathbf{j}, 0 \leq \theta \leq \frac{\pi}{3}$

$\mathbf{r}_1'(\theta) = \sec \theta \tan \theta \mathbf{i} + \sec^2 \theta \mathbf{j}$

$\mathbf{F}(\theta) = \tan \theta \mathbf{i} - \sec \theta \mathbf{j}$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/3} (\sec \theta \tan^2 \theta - \sec^3 \theta) d\theta = \int_0^{\pi/3} [\sec \theta (\sec^2 \theta - 1) - \sec^3 \theta] d\theta \\ &= - \int_0^{\pi/3} \sec \theta d\theta = [-\ln|\sec \theta + \tan \theta|]_0^{\pi/3} = -\ln(2 + \sqrt{3}) \approx -1.317 \end{aligned}$$

(b) $\mathbf{r}_2(t) = \sqrt{t+1}\mathbf{i} + \sqrt{t}\mathbf{j}, 0 \leq t \leq 3$

$\mathbf{r}_2'(t) = \frac{1}{2\sqrt{t+1}}\mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$

$\mathbf{F}(t) = \sqrt{t}\mathbf{i} - \sqrt{t+1}\mathbf{j}$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^3 \left[\frac{\sqrt{t}}{2\sqrt{t+1}} - \frac{\sqrt{t+1}}{2\sqrt{t}} \right] dt = -\frac{1}{2} \int_0^3 \frac{1}{\sqrt{t}\sqrt{t+1}} dt = -\frac{1}{2} \int_0^3 \frac{1}{\sqrt{t^2 + t + (1/4) - (1/4)}} dt \\ &= -\frac{1}{2} \int_0^3 \frac{1}{\sqrt{[t + (1/2)]^2 - (1/4)}} dt = \left[-\frac{1}{2} \ln \left| \left(t + \frac{1}{2} \right) + \sqrt{t^2 + t} \right| \right]_0^3 \\ &= -\frac{1}{2} \left[\ln \left(\frac{7}{2} + 2\sqrt{3} \right) - \ln \left(\frac{1}{2} \right) \right] = -\frac{1}{2} \ln(7 + 4\sqrt{3}) \approx -1.317 \end{aligned}$$

5. $\mathbf{F}(x, y) = e^x \sin y \mathbf{i} + e^x \cos y \mathbf{j}$

$$\frac{\partial N}{\partial x} = e^x \cos y \quad \frac{\partial M}{\partial y} = e^x \cos y$$

Since $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, \mathbf{F} is conservative.

7. $\mathbf{F}(x, y) = \frac{1}{y}\mathbf{i} + \frac{x}{y^2}\mathbf{j}$

$$\frac{\partial N}{\partial x} = \frac{1}{y^2} \quad \frac{\partial M}{\partial y} = -\frac{1}{y^2}$$

Since $\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$, \mathbf{F} is not conservative.

9. $\mathbf{F}(x, y, z) = y^2z\mathbf{i} + 2xyz\mathbf{j} + xy^2\mathbf{k}$

$\text{curl } \mathbf{F} = \mathbf{0} \Rightarrow \mathbf{F} \text{ is conservative.}$

11. $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = 2t^3\mathbf{i} + t^2\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 4t^3 dt = 1$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^3\mathbf{j}, 0 \leq t \leq 1$

$\mathbf{r}_2'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = 2t^4\mathbf{i} + t^2\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 5t^4 dt = 1$

13. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1$

$\mathbf{r}_1'(t) = \mathbf{i} + \mathbf{j}$

$\mathbf{F}(t) = t\mathbf{i} - t\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 1$

$\mathbf{r}_2'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^2\mathbf{i} - t\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -t^2 dt = -\frac{1}{3}$

(c) $\mathbf{r}_3(t) = t\mathbf{i} + t^3\mathbf{j}, 0 \leq t \leq 1$

$\mathbf{r}_3'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = t^3\mathbf{i} - t\mathbf{j}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -2t^3 dt = -\frac{1}{2}$

15. $\int_C y^2 dx + 2xy dy$

Since $\partial M/\partial y = \partial N/\partial x = 2y$, $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ is conservative. The potential function is $f(x, y) = xy^2 + k$. Therefore, we can use the Fundamental Theorem of Line Integrals.

(a) $\int_C y^2 dx + 2xy dy = \left[xy^2 \right]_{(0,0)}^{(4,4)} = 64$

(b) $\int_C y^2 dx + 2xy dy = \left[xy^2 \right]_{(-1,0)}^{(1,0)} = 0$

(c) and (d) Since C is a closed curve, $\int_C y^2 dx + 2xy dy = 0$.

17. $\int_C 2xy dx + (x^2 + y^2) dy$

Since $\partial M/\partial y = \partial N/\partial x = 2x$,

$\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$ is conservative.

The potential function is $f(x, y) = x^2y + \frac{y^3}{3} + k$.

(a) $\int_C 2xy dx + (x^2 + y^2) dy = \left[x^2y + \frac{y^3}{3} \right]_{(5,0)}^{(0,4)} = \frac{64}{3}$

(b) $\int_C 2xy dx + (x^2 + y^2) dy = \left[x^2y + \frac{y^3}{3} \right]_{(2,0)}^{(0,4)} = \frac{64}{3}$

19. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

Since $\text{curl } \mathbf{F} = \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = xyz + k$.

(a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k}, 0 \leq t \leq 4$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[xyz \right]_{(0,2,0)}^{(4,2,4)} = 32$

(b) $\mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, 0 \leq t \leq 2$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[xyz \right]_{(0,0,0)}^{(4,2,4)} = 32$

21. $\mathbf{F}(x, y, z) = (2y + x)\mathbf{i} + (x^2 - z)\mathbf{j} + (2y - 4z)\mathbf{k}$

$\mathbf{F}(x, y, z)$ is not conservative.

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}, 0 \leq t \leq 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = (2t^2 + t)\mathbf{i} + (t^2 - 1)\mathbf{j} + (2t^2 - 4)\mathbf{k}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t^3 + 2t^2 - t) dt = \frac{2}{3}$

21. —CONTINUED—

$$(b) \mathbf{r}_2(t) = t\mathbf{i} + t\mathbf{j} + (2t - 1)^2\mathbf{k}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}_2'(t) = \mathbf{i} + \mathbf{j} + 4(2t - 1)\mathbf{k}$$

$$\mathbf{F}(t) = 3t\mathbf{i} + [t^2 - (2t - 1)^2]\mathbf{j} + [2t - 4(2t - 1)^2]\mathbf{k}$$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 [3t + t^2 - (2t - 1)^2 + 8t(2t - 1) - 16(2t - 1)^3] dt \\ &= \int_0^1 [17t^2 - 5t - (2t - 1)^2 - 16(2t - 1)^3] dt = \left[\frac{17t^3}{3} - \frac{5t^2}{2} - \frac{(2t - 1)^3}{6} - 2(2t - 1)^4 \right]_0^1 = \frac{17}{6} \end{aligned}$$

$$23. \mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$$

$\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = xye^z + k$.

$$(a) \mathbf{r}_1(t) = 4 \cos t\mathbf{i} + 4 \sin t\mathbf{j} + 3\mathbf{k}, \quad 0 \leq t \leq \pi$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[xye^z \right]_{(4, 0, 3)}^{(-4, 0, 3)} = 0$$

$$(b) \mathbf{r}_2(t) = (4 - 8t)\mathbf{i} + 3\mathbf{k}, \quad 0 \leq t \leq 1$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[xye^z \right]_{(4, 0, 3)}^{(-4, 0, 3)} = 0$$

$$25. \int_C (y\mathbf{i} + x\mathbf{j}) \cdot d\mathbf{r} = \left[xy \right]_{(0, 0)}^{(3, 8)} = 24$$

$$27. \int_C \cos x \sin y \, dx + \sin x \cos y \, dy = \left[\sin x \sin y \right]_{(0, -\pi)}^{(3\pi/2, \pi/2)} = -1$$

$$29. \int_C e^x \sin y \, dx + e^x \cos y \, dy = \left[e^x \sin y \right]_{(0, 0)}^{(2\pi, 0)} = 0$$

$$31. \int_C (y + 2z) \, dx + (x - 3z) \, dy + (2x - 3y) \, dz$$

$\mathbf{F}(x, y, z)$ is conservative and the potential function is $f(x, y, z) = xy - 3yz + 2xz$.

$$(a) \left[xy - 3yz + 2xz \right]_{(0, 0, 0)}^{(1, 1, 1)} = 0 - 0 = 0$$

$$(b) \left[xy - 3yz + 2xz \right]_{(0, 0, 0)}^{(0, 0, 1)} + \left[xy - 3yz + 2xz \right]_{(0, 0, 1)}^{(1, 1, 1)} = 0 + 0 = 0$$

$$(c) \left[xy - 3yz + 2xz \right]_{(0, 0, 0)}^{(1, 0, 0)} + \left[xy - 3yz + 2xz \right]_{(1, 0, 0)}^{(1, 1, 0)} + \left[xy - 3yz + 2xz \right]_{(1, 1, 0)}^{(1, 1, 1)} = 0 + 1 + (-1) = 0$$

$$33. \int_C -\sin x \, dx + z \, dy + y \, dz = \left[\cos x + yz \right]_{(0, 0, 0)}^{(\pi/2, 3, 4)} = 12 - 1 = 11$$

$$35. \mathbf{F}(x, y) = 9x^2y^2\mathbf{i} + (6x^3y - 1)\mathbf{j} \text{ is conservative.}$$

$$\text{Work} = \left[3x^3y^2 - y \right]_{(0, 0)}^{(5, 9)} = 30,366$$

$$37. \mathbf{r}(t) = 2 \cos 2\pi t \mathbf{i} + 2 \sin 2\pi t \mathbf{j}$$

$$\mathbf{r}'(t) = -4\pi \sin 2\pi t \mathbf{i} + 4\pi \cos 2\pi t \mathbf{j}$$

$$\mathbf{a}(t) = -8\pi^2 \cos 2\pi t \mathbf{i} - 8\pi^2 \sin 2\pi t \mathbf{j}$$

$$\mathbf{F}(t) = m\mathbf{a}(t) = \frac{1}{32}\mathbf{a}(t) = -\frac{\pi^2}{4}(\cos 2\pi t \mathbf{i} + \sin 2\pi t \mathbf{j})$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -\frac{\pi^2}{4}(\cos 2\pi t \mathbf{i} + \sin 2\pi t \mathbf{j}) \cdot 4\pi(-\sin 2\pi t \mathbf{i} + \cos 2\pi t \mathbf{j}) dt = -\pi^3 \int_C 0 dt = 0$$

39. Since the sum of the potential and kinetic energies remains constant from point to point, if the kinetic energy is decreasing at a rate of 10 units per minute, then the potential energy is increasing at a rate of 10 units per minute.

41. No. The force field is conservative.

43. See Theorem 14.5, page 1033.

45. (a) The direct path along the line segment joining $(-4, 0)$ to $(3, 4)$ requires less work than the path going from $(-4, 0)$ to $(-4, 4)$ and then to $(3, 4)$.

- (b) The closed curve given by the line segments joining $(-4, 0)$, $(-4, 4)$, $(3, 4)$, and $(-4, 0)$ satisfies $\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$.

47. False, it would be true if \mathbf{F} were conservative.

49. True

51. Let

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} = \frac{\partial f}{\partial y}\mathbf{i} - \frac{\partial f}{\partial x}\mathbf{j}.$$

$$\text{Then } \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{\partial f}{\partial x}\right) = -\frac{\partial^2 f}{\partial x^2}. \text{ Since}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \text{ we have } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Thus, \mathbf{F} is conservative. Therefore, by Theorem 14.7, we have

$$\int_C \left(\frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy \right) = \int_C (M dx + N dy) = \int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

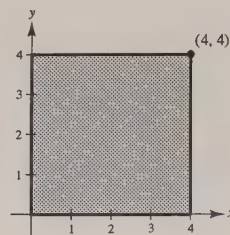
for every closed curve in the plane.

Section 14.4 Green's Theorem

$$1. \mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \leq t \leq 4 \\ 4\mathbf{i} + (t-4)\mathbf{j}, & 4 \leq t \leq 8 \\ (12-t)\mathbf{i} + 4\mathbf{j}, & 8 \leq t \leq 12 \\ (16-t)\mathbf{j}, & 12 \leq t \leq 16 \end{cases}$$

$$\begin{aligned} \int_C y^2 dx + x^2 dy &= \int_0^4 [0 dt + t^2(0)] + \int_4^8 [(t-4)^2(0) + 16 dt] \\ &\quad + \int_8^{12} [16(-dt) + (12-t)^2(0)] + \int_{12}^{16} [(16-t)^2(0) + 0(-dt)] \\ &= 0 + 64 - 64 + 0 = 0 \end{aligned}$$

$$\text{By Green's Theorem, } \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_0^4 \int_0^4 (2x - 2y) dy dx = \int_0^4 (8x - 16) dx = 0.$$

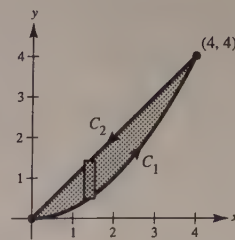


$$3. \mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2/4\mathbf{j}, & 0 \leq t \leq 4 \\ (8-t)\mathbf{i} + (8-t)\mathbf{j}, & 4 \leq t \leq 8 \end{cases}$$

$$\begin{aligned} \int_C y^2 dx + x^2 dy &= \int_0^4 \left[\frac{t^4}{16} (dt) + t^2 \left(\frac{t}{2} dt \right) \right] + \int_4^8 [(8-t)^2(-dt) + (8-t)^2(-dt)] \\ &= \int_0^4 \left[\frac{t^4}{16} + \frac{t^3}{2} \right] dt + \int_4^8 -2(8-t)^2 dt = \frac{224}{5} - \frac{128}{3} = \frac{32}{15} \end{aligned}$$

By Green's Theorem,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_0^4 \int_{x^2/4}^x (2x - 2y) dy dx = \int_0^4 \left(x^2 - \frac{x^3}{2} + \frac{x^4}{16} \right) dx = \frac{32}{15}.$$



$$5. C: x^2 + y^2 = 4$$

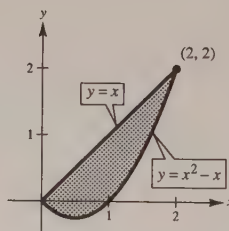
Let $x = 2 \cos t$ and $y = 2 \sin t$, $0 \leq t \leq 2\pi$.

$$\int_C xe^y dx + e^x dy = \int_0^{2\pi} [2 \cos t e^{2 \sin t} (-2 \sin t) + e^{2 \cos t} (2 \cos t)] dt \approx 19.99$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (e^x - xe^y) dy dx = \int_{-2}^2 \left[2\sqrt{4-x^2} e^x - xe^{\sqrt{4-x^2}} + xe^{-\sqrt{4-x^2}} \right] dx \approx 19.99$$

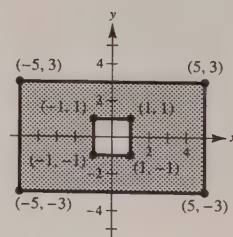
In Exercises 7 and 9, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$.

$$\begin{aligned} 7. \int_C (y-x) dx + (2x-y) dy &= \int_0^2 \int_{x^2-x}^x dy dx \\ &= \int_0^2 (2x - x^2) dx \\ &= \frac{4}{3} \end{aligned}$$



9. From the accompanying figure, we see that R is the shaded region. Thus, Green's Theorem yields

$$\begin{aligned} \int_C (y-x) dx + (2x-y) dy &= \iint_R 1 dA \\ &= \text{Area of } R \\ &= 6(10) - 2(2) \\ &= 56. \end{aligned}$$



11. Since the curves $y = 0$ and $y = 4 - x^2$ intersect at $(-2, 0)$ and $(2, 0)$, Green's Theorem yields

$$\begin{aligned} \int_C 2xy dx + (x+y) dy &= \iint_R (1-2x) dA = \int_{-2}^2 \int_0^{4-x^2} (1-2x) dy dx \\ &= \int_{-2}^2 \left[y - 2xy \right]_0^{4-x^2} dx \\ &= \int_{-2}^2 (4 - 8x - x^2 + 2x^3) dx \\ &= \left[4x - 4x^2 - \frac{x^3}{3} + \frac{x^4}{2} \right]_{-2}^2 \\ &= -\frac{8}{3} - \frac{8}{3} + 16 = \frac{32}{3}. \end{aligned}$$

13. Since R is the interior of the circle $x^2 + y^2 = a^2$, Green's Theorem yields

$$\begin{aligned}\int_C (x^2 - y^2) dx + 2xy dy &= \iint_R (2y + 2y) dA \\ &= \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 4y dy dx = 4 \int_{-a}^a 0 dx = 0.\end{aligned}$$

15. Since $\frac{\partial M}{\partial y} = \frac{2x}{x^2 + y^2} = \frac{\partial N}{\partial x}$,

we have path independence and

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = 0.$$

17. By Green's Theorem,

$$\begin{aligned}\int_C \sin x \cos y dx + (xy + \cos x \sin y) dy &= \iint_R [(y - \sin x \sin y) - (-\sin x \sin y)] dA \\ &= \int_0^1 \int_x^{\sqrt{x}} y dy dx = \frac{1}{2} \int_0^1 (x - x^2) dx = \frac{1}{2} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{12}.\end{aligned}$$

19. By Green's Theorem,

$$\begin{aligned}\int_C xy dx + (x + y) dy &= \iint_R (1 - x) dA \\ &= \int_0^{2\pi} \int_1^3 (1 - r \cos \theta) r dr d\theta = \int_0^{2\pi} \left(4 - \frac{26}{3} \cos \theta \right) d\theta = 8\pi.\end{aligned}$$

21. $\mathbf{F}(x, y) = xy\mathbf{i} + (x + y)\mathbf{j}$

$C: x^2 + y^2 = 4$

$$\text{Work} = \int_C xy dx + (x + y) dy = \iint_R (1 - x) dA = \int_0^{2\pi} \int_0^2 (1 - r \cos \theta) r dr d\theta = \int_0^{2\pi} \left(2 - \frac{8}{3} \cos \theta \right) d\theta = 4\pi$$

23. $\mathbf{F}(x, y) = (x^{3/2} - 3y)\mathbf{i} + (6x + 5\sqrt{y})\mathbf{j}$

C : boundary of the triangle with vertices $(0, 0)$, $(5, 0)$, $(0, 5)$

$$\text{Work} = \int_C (x^{3/2} - 3y) dx + (6x + 5\sqrt{y}) dy = \iint_R 9 dA = 9\left(\frac{1}{2}\right)(5)(5) = \frac{225}{2}$$

25. C : let $x = a \cos t$, $y = a \sin t$, $0 \leq t \leq 2\pi$. By Theorem 14.9, we have

$$A = \frac{1}{2} \int_C x dy - y dx = \frac{1}{2} \int_0^{2\pi} [a \cos t(a \cos t) - a \sin t(-a \sin t)] dt = \frac{1}{2} \int_0^{2\pi} a^2 dt = \left[\frac{a^2}{2} t \right]_0^{2\pi} = \pi a^2.$$

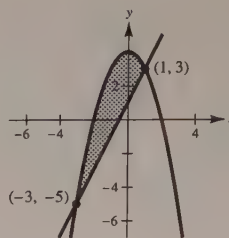
27. From the accompanying figure we see that

$$C_1: y = 2x + 1, \quad dy = 2 \, dx$$

$$C_2: y = 4 - x^2, \quad dy = -2x \, dx.$$

Thus, by Theorem 14.9, we have

$$\begin{aligned} A &= \frac{1}{2} \int_{-3}^1 [x(2) - (2x + 1)] \, dx + \frac{1}{2} \int_1^{-3} [x(-2x) - (4 - x^2)] \, dx \\ &= \frac{1}{2} \int_{-3}^1 (-1) \, dx + \frac{1}{2} \int_1^{-3} (-x^2 - 4) \, dx \\ &= \frac{1}{2} \int_{-3}^1 (-1) \, dx + \frac{1}{2} \int_{-3}^1 (x^2 + 4) \, dx = \frac{1}{2} \int_{-3}^1 (3 + x^2) \, dx = \frac{1}{2} \left[3x + \frac{x^3}{3} \right]_{-3}^1 = \frac{32}{3}. \end{aligned}$$



29. See Theorem 14.8, page 1042.

31. Answers will vary.

$$\mathbf{F}_1(x, y) = y\mathbf{i} + x\mathbf{j}$$

$$\mathbf{F}_2(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$$

$$\mathbf{F}_3(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$$

$$33. A = \int_{-2}^2 (4 - x^2) \, dx = \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3}$$

$$\bar{x} = \frac{1}{2A} \int_{C_1} x^2 \, dy + \frac{1}{2A} \int_{C_2} x^2 \, dy$$

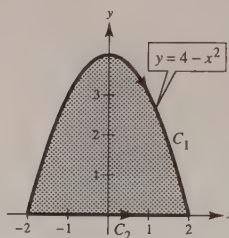
For C_1 , $dy = -2x \, dx$ and for C_2 , $dy = 0$. Thus,

$$\bar{x} = \frac{1}{2(32/3)} \int_2^{-2} x^2(-2x \, dx) = \left[\frac{3}{64} \left(-\frac{x^4}{2} \right) \right]_2^{-2} = 0.$$

To calculate \bar{y} , note that $y = 0$ along C_2 . Thus,

$$\bar{y} = \frac{-1}{2(32/3)} \int_2^{-2} (4 - x^2)^2 \, dx = \frac{3}{64} \int_{-2}^2 (16 - 8x^2 + x^4) \, dx = \frac{3}{64} \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_{-2}^2 = \frac{8}{5}.$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{5} \right)$$



35. Since $A = \int_0^1 (x - x^3) \, dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{4}$, we have $\frac{1}{2A} = 2$. On C_1 we have $y = x^3$, $dy = 3x^2 \, dx$ and on C_2 we have $y = x$, $dy = dx$. Thus,

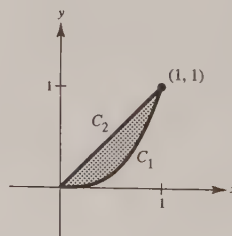
$$\bar{x} = 2 \int_C x^2 \, dy = 2 \int_{C_1} x^2(3x^2 \, dx) + 2 \int_{C_2} x^2 \, dx$$

$$= 6 \int_0^1 x^4 \, dx + 2 \int_1^0 x^2 \, dx = \frac{6}{5} - \frac{2}{3} = \frac{8}{15}$$

$$\bar{y} = -2 \int_C y^2 \, dx$$

$$= -2 \int_0^1 x^6 \, dx - 2 \int_1^0 x^2 \, dx = -\frac{2}{7} + \frac{2}{3} = \frac{8}{21}.$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{15}, \frac{8}{21} \right)$$



$$\begin{aligned}
 37. A &= \frac{1}{2} \int_0^{2\pi} a^2(1 - \cos \theta)^2 d\theta \\
 &= \frac{a^2}{2} \int_0^{2\pi} \left(1 - 2\cos \theta + \frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta = \frac{a^2}{2} \left[\frac{3\theta}{2} - 2\sin \theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{a^2}{2}(3\pi) = \frac{3\pi a^2}{2}
 \end{aligned}$$

39. In this case the inner loop has domain $\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}$. Thus,

$$\begin{aligned}
 A &= \frac{1}{2} \int_{2\pi/3}^{4\pi/3} (1 + 4\cos \theta + 4\cos^2 \theta) d\theta \\
 &= \frac{1}{2} \int_{2\pi/3}^{4\pi/3} (3 + 4\cos \theta + 2\cos 2\theta) d\theta = \frac{1}{2} \left[3\theta + 4\sin \theta + \sin 2\theta\right]_{2\pi/3}^{4\pi/3} = \pi - \frac{3\sqrt{3}}{2}.
 \end{aligned}$$

$$41. I = \int_C \frac{y dx - x dy}{x^2 + y^2}$$

(a) Let $\mathbf{F} = \frac{y}{x^2 + y^2} \mathbf{i} - \frac{x}{x^2 + y^2} \mathbf{j}$.

\mathbf{F} is conservative since $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$.

\mathbf{F} is defined and has continuous first partials everywhere except at the origin. If C is a circle (a closed path) that does not contain the origin, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = 0.$$

(b) Let $\mathbf{r} = a \cos t \mathbf{i} - a \sin t \mathbf{j}$, $0 \leq t \leq 2\pi$ be a circle C_1 oriented clockwise inside C (see figure). Introduce line segments C_2 and C_3 as illustrated in Example 6 of this section in the text. For the region inside C and outside C_1 , Green's Theorem applies. Note that since C_2 and C_3 have opposite orientations, the line integrals over them cancel. Thus, $C_4 = C_1 + C_2 + C + C_3$ and

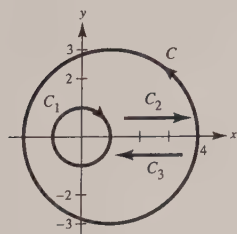
$$\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

But,

$$\begin{aligned}
 \int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \left[\frac{(-a \sin t)(-a \sin t)}{a^2 \cos^2 t + a^2 \sin^2 t} + \frac{(-a \cos t)(-a \cos t)}{a^2 \cos^2 t + a^2 \sin^2 t} \right] dt \\
 &= \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = \left[t \right]_0^{2\pi} = 2\pi.
 \end{aligned}$$

Finally, $\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -2\pi$.

Note: If C were orientated clockwise, then the answer would have been 2π .



43. Pentagon: $(0, 0), (2, 0), (3, 2), (1, 4), (-1, 1)$

$$A = \frac{1}{2}[(0 - 0) + (4 - 0) + (12 - 2) + (1 + 4) + (0 - 0)] = \frac{19}{2}$$

$$45. \int_C y^n dx + x^n dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

For the line integral, use the two paths

$$C_1: \mathbf{r}_1(x) = x\mathbf{i}, \quad -a \leq x \leq a$$

$$C_2: \mathbf{r}_2(x) = x\mathbf{i} + \sqrt{a^2 - x^2}\mathbf{j}, \quad x = a \text{ to } x = -a$$

$$\int_{C_1} y^n dx + x^n dy = 0$$

$$\int_{C_2} y^n dx + x^n dy = \int_a^{-a} \left[(a^2 - x^2)^{n/2} + x^n \frac{-x}{\sqrt{a^2 - x^2}} \right] dx$$

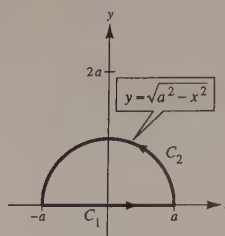
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_{-a}^a \int_0^{\sqrt{a^2 - x^2}} [nx^{n-1} - ny^{n-1}] dy dx$$

(a) For $n = 1, 3, 5, 7$, both integrals give 0.

(b) For n even, you obtain

$$n = 2: -\frac{4}{3}a^3 \quad n = 4: -\frac{16}{15}a^5 \quad n = 6: -\frac{32}{35}a^7 \quad n = 8: -\frac{256}{315}a^9$$

(c) If n is odd then the integral equals 0.



$$47. \int_C (f D_{\mathbf{N}} g - g D_{\mathbf{N}} f) ds = \int_C f D_{\mathbf{N}} g ds - \int_C g D_{\mathbf{N}} f ds$$

$$= \iint_R (f \nabla^2 g + \nabla f \cdot \nabla g) dA - \iint_R (g \nabla^2 f + \nabla g \cdot \nabla f) dA = \iint_R (f \nabla^2 g - g \nabla^2 f) dA$$

$$49. \mathbf{F} = M\mathbf{i} + N\mathbf{j}$$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_R (0) dA = 0$$

Section 14.5 Parametric Surfaces

$$1. \mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + uv\mathbf{k}$$

$$z = xy$$

Matches c.

$$3. \mathbf{r}(u, v) = 2 \cos v \cos u \mathbf{i} + 2 \cos v \sin u \mathbf{j} + 2 \sin v \mathbf{k}$$

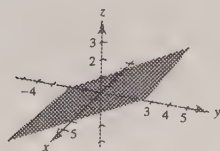
$$x^2 + y^2 + z^2 = 4$$

Matches b.

$$5. \mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \frac{v}{2}\mathbf{k}$$

$$y - 2z = 0$$

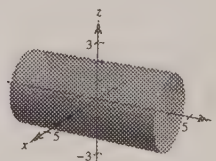
Plane



$$7. \mathbf{r}(u, v) = 2 \cos u \mathbf{i} + v\mathbf{j} + 2 \sin u \mathbf{k}$$

$$x^2 + z^2 = 4$$

Cylinder



For Exercises 9 and 11,

$$\mathbf{r}(u, v) = u^4 \cos v \mathbf{i} + u \sin v \mathbf{j} + u^2 \mathbf{k}, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi.$$

Eliminating the parameter yields

$$z = x^2 + y^2, \quad 0 \leq z \leq 4.$$



9. $\mathbf{s}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} - u^2 \mathbf{k}, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi$
 $z = -(x^2 + y^2)$

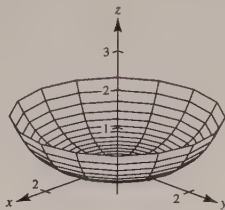
The paraboloid is reflected (inverted) through the xy -plane.

11. $\mathbf{s}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + u^2 \mathbf{k}, \quad 0 \leq u \leq 3, \quad 0 \leq v \leq 2\pi$
 The height of the paraboloid is increased from 4 to 9.

13. $\mathbf{r}(u, v) = 2u \cos v \mathbf{i} + 2u \sin v \mathbf{j} + u^4 \mathbf{k},$

$$0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi$$

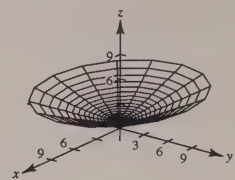
$$z = \frac{(x^2 + y^2)^2}{16}$$



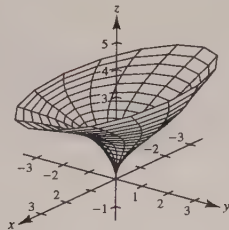
15. $\mathbf{r}(u, v) = 2 \sinh u \cos v \mathbf{i} + \sinh u \sin v \mathbf{j} + \cosh u \mathbf{k},$

$$0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi$$

$$\frac{z^2}{1} - \frac{x^2}{4} - \frac{y^2}{1} = 1$$



17. $\mathbf{r}(u, v) = (u - \sin u) \cos v \mathbf{i} + (1 - \cos u) \sin v \mathbf{j} + u \mathbf{k},$
 $0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$



19. $z = y$

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + v \mathbf{k}$$

23. $z = x^2$

$$\mathbf{r}(u, v) = u \mathbf{i} + v \mathbf{j} + u^2 \mathbf{k}$$

27. Function: $y = \frac{x}{2}, \quad 0 \leq x \leq 6$

Axis of revolution: x -axis

$$x = u, \quad y = \frac{u}{2} \cos v, \quad z = \frac{u}{2} \sin v$$

$$0 \leq u \leq 6, \quad 0 \leq v \leq 2\pi$$

21. $x^2 + y^2 = 16$

$$\mathbf{r}(u, v) = 4 \cos u \mathbf{i} + 4 \sin u \mathbf{j} + v \mathbf{k}$$

25. $z = 4$ inside $x^2 + y^2 = 9.$

$$\mathbf{r}(u, v) = v \cos u \mathbf{i} + v \sin u \mathbf{j} + 4 \mathbf{k}, \quad 0 \leq v \leq 3$$

29. Function: $x = \sin z, \quad 0 \leq z \leq \pi$

Axis of revolution: z -axis

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = u$$

$$0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$$

$$31. \mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + v\mathbf{k}, (1, -1, 1)$$

$$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j}, \mathbf{r}_v(u, v) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\text{At } (1, -1, 1), u = 0 \text{ and } v = 1.$$

$$\mathbf{r}_u(0, 1) = \mathbf{i} + \mathbf{j}, \mathbf{r}_v(0, 1) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\mathbf{N} = \mathbf{r}_u(0, 1) \times \mathbf{r}_v(0, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

$$\text{Tangent plane: } (x - 1) - (y + 1) - 2(z - 1) = 0$$

$$x - y - 2z = 0$$

(The original plane!)

$$33. \mathbf{r}(u, v) = 2u \cos v \mathbf{i} + 3u \sin v \mathbf{j} + u^2 \mathbf{k}, (0, 6, 4)$$

$$\mathbf{r}_u(u, v) = 2 \cos v \mathbf{i} + 3 \sin v \mathbf{j} + 2u \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -2u \sin v \mathbf{i} + 3u \cos v \mathbf{j}$$

$$\text{At } (0, 6, 4), u = 2 \text{ and } v = \pi/2.$$

$$\mathbf{r}_u\left(2, \frac{\pi}{2}\right) = 3\mathbf{j} + 4\mathbf{k}, \mathbf{r}_v\left(2, \frac{\pi}{2}\right) = -4\mathbf{i}$$

$$\mathbf{N} = \mathbf{r}_u\left(2, \frac{\pi}{2}\right) \times \mathbf{r}_v\left(2, \frac{\pi}{2}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 4 \\ -4 & 0 & 0 \end{vmatrix} = -16\mathbf{j} + 12\mathbf{k}$$

$$\text{Direction numbers: } 0, 4, -3$$

$$\text{Tangent plane: } 4(y - 6) - 3(z - 4) = 0$$

$$4y - 3z = 12$$

$$35. \mathbf{r}(u, v) = 2u\mathbf{i} - \frac{v}{2}\mathbf{j} + \frac{v}{2}\mathbf{k}, 0 \leq u \leq 2, 0 \leq v \leq 1$$

$$\mathbf{r}_u(u, v) = 2\mathbf{i}, \mathbf{r}_v(u, v) = -\frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\mathbf{j} - \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2}$$

$$A = \int_0^1 \int_0^2 \sqrt{2} \, du \, dv = 2\sqrt{2}$$

$$37. \mathbf{r}(u, v) = a \cos u \mathbf{i} + a \sin u \mathbf{j} + v \mathbf{k}, 0 \leq u \leq 2\pi, 0 \leq v \leq b$$

$$\mathbf{r}_u(u, v) = -a \sin u \mathbf{i} + a \cos u \mathbf{j}$$

$$\mathbf{r}_v(u, v) = \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin u & a \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = a \cos u \mathbf{i} + a \sin u \mathbf{j}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = a$$

$$A = \int_0^b \int_0^{2\pi} a \, du \, dv = 2\pi ab$$

$$39. \mathbf{r}(u, v) = au \cos v \mathbf{i} + au \sin v \mathbf{j} + u \mathbf{k}, 0 \leq u \leq b, 0 \leq v \leq 2\pi$$

$$\mathbf{r}_u(u, v) = a \cos v \mathbf{i} + a \sin v \mathbf{j} + \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -au \sin v \mathbf{i} + au \cos v \mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos v & a \sin v & 1 \\ -au \sin v & au \cos v & 0 \end{vmatrix} = -au \cos v \mathbf{i} - au \sin v \mathbf{j} + a^2 u \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = au\sqrt{1 + a^2}$$

$$A = \int_0^{2\pi} \int_0^b a\sqrt{1 + a^2} u \, du \, dv = \pi ab^2 \sqrt{1 + a^2}$$

$$41. \mathbf{r}(u, v) = \sqrt{u} \cos v \mathbf{i} + \sqrt{u} \sin v \mathbf{j} + u \mathbf{k}, \quad 0 \leq u \leq 4, \quad 0 \leq v \leq 2\pi$$

$$\mathbf{r}_u(u, v) = \frac{\cos v}{2\sqrt{u}} \mathbf{i} + \frac{\sin v}{2\sqrt{u}} \mathbf{j} + \mathbf{k}$$

$$\mathbf{r}_v(u, v) = -\sqrt{u} \sin v \mathbf{i} + \sqrt{u} \cos v \mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\cos v}{2\sqrt{u}} & \frac{\sin v}{2\sqrt{u}} & 1 \\ -\sqrt{u} \sin v & \sqrt{u} \cos v & 0 \end{vmatrix} = -\sqrt{u} \cos v \mathbf{i} - \sqrt{u} \sin v \mathbf{j} + \frac{1}{2} \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{u + \frac{1}{4}}$$

$$A = \int_0^{2\pi} \int_0^4 \sqrt{u + \frac{1}{4}} \, du \, dv = \frac{\pi}{6} (17\sqrt{17} - 1) \approx 36.177$$

43. See the definition, page 1051.

45. (a) From $(-10, 10, 0)$

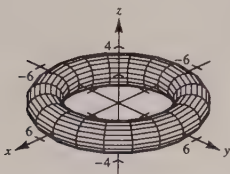
(b) From $(10, 10, 10)$

(c) From $(0, 10, 0)$

(d) From $(10, 0, 0)$

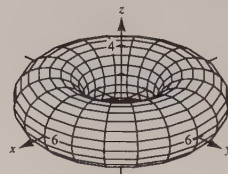
$$47. (a) \mathbf{r}(u, v) = (4 + \cos v) \cos u \mathbf{i} + (4 + \cos v) \sin u \mathbf{j} + \sin v \mathbf{k},$$

$$0 \leq u \leq 2\pi, \quad 0 \leq v \leq 2\pi$$



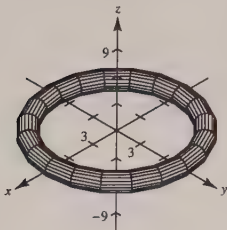
$$(b) \mathbf{r}(u, v) = (4 + 2 \cos v) \cos u \mathbf{i} + (4 + 2 \cos v) \sin u \mathbf{j} + 2 \sin v \mathbf{k},$$

$$0 \leq u \leq 2\pi, \quad 0 \leq v \leq 2\pi$$



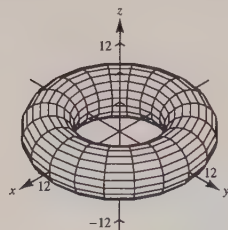
$$(c) \mathbf{r}(u, v) = (8 + \cos v) \cos u \mathbf{i} + (8 + \cos v) \sin u \mathbf{j} + \sin v \mathbf{k},$$

$$0 \leq u \leq 2\pi, \quad 0 \leq v \leq 2\pi$$



$$(d) \mathbf{r}(u, v) = (8 + 3 \cos v) \cos u \mathbf{i} + (8 + 3 \cos v) \sin u \mathbf{j} + 3 \sin v \mathbf{k},$$

$$0 \leq u \leq 2\pi, \quad 0 \leq v \leq 2\pi$$



The radius of the generating circle that is revolved about the z -axis is b , and its center is a units from the axis of revolution.

$$49. \mathbf{r}(u, v) = 20 \sin u \cos v \mathbf{i} + 20 \sin u \sin v \mathbf{j} + 20 \cos u \mathbf{k} \quad 0 \leq u \leq \pi/3, \quad 0 \leq v \leq 2\pi$$

$$\mathbf{r}_u = 20 \cos u \cos v \mathbf{i} + 20 \cos u \sin v \mathbf{j} - 20 \sin u \mathbf{k}$$

$$\mathbf{r}_v = -20 \sin u \sin v \mathbf{i} + 20 \sin u \cos v \mathbf{j}$$

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 \cos u \cos v & 20 \cos u \sin v & -20 \sin u \\ -20 \sin u \sin v & 20 \sin u \cos v & 0 \end{vmatrix} \\ &= 400 \sin^2 u \cos v \mathbf{i} + 400 \sin^2 u \sin v \mathbf{j} + 400(\cos u \sin u \cos^2 v + \cos u \sin u \sin^2 v) \mathbf{k} \\ &= 400[\sin^2 u \cos v \mathbf{i} + \sin^2 u \sin v \mathbf{j} + \cos u \sin u \mathbf{k}] \end{aligned}$$

$$\begin{aligned} \|\mathbf{r}_u \times \mathbf{r}_v\| &= 400 \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \cos^2 u \sin^2 u} \\ &= 400 \sqrt{\sin^4 u + \cos^2 u \sin^2 u} \\ &= 400 \sqrt{\sin^2 u} = 400 \sin u \end{aligned}$$

$$\begin{aligned} S &= \iint_S dS = \int_0^{2\pi} \int_0^{\pi/3} 400 \sin u \, du \, dv = \int_0^{2\pi} \left[-400 \cos u \right]_0^{\pi/3} dv \\ &= \int_0^{2\pi} 200 \, dv = 400\pi \, \text{m}^2 \end{aligned}$$

$$51. \mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + 2v \mathbf{k}, \quad 0 \leq u \leq 3, \quad 0 \leq v \leq 2\pi$$

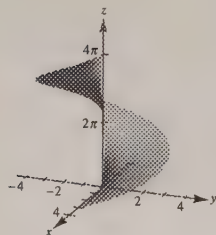
$$\mathbf{r}_u(u, v) = \cos v \mathbf{i} + \sin v \mathbf{j}$$

$$\mathbf{r}_v(u, v) = -u \sin v \mathbf{i} + u \cos v \mathbf{j} + 2 \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 2 \end{vmatrix} = 2 \sin v \mathbf{i} - 2 \cos v \mathbf{j} + u \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4 + u^2}$$

$$A = \int_0^{2\pi} \int_0^3 \sqrt{4 + u^2} \, du \, dv = \pi \left[3\sqrt{13} + 4 \ln \left(\frac{3 + \sqrt{13}}{2} \right) \right]$$



53. Essay

Section 14.6 Surface Integrals

$$1. S: z = 4 - x, \quad 0 \leq x \leq 4, \quad 0 \leq y \leq 4, \quad \frac{\partial z}{\partial x} = -1, \quad \frac{\partial z}{\partial y} = 0$$

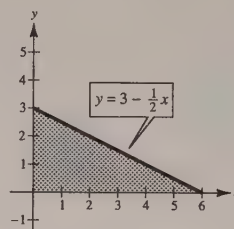
$$\begin{aligned} \iint_S (x - 2y + z) \, dS &= \int_0^4 \int_0^4 (x - 2y + 4 - x) \sqrt{1 + (-1)^2 + (0)^2} \, dy \, dx \\ &= \sqrt{2} \int_0^4 \int_0^4 (4 - 2y) \, dy \, dx = 0 \end{aligned}$$

3. $S: z = 10, x^2 + y^2 \leq 1, \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$

$$\begin{aligned} \iint_S (x - 2y + z) dS &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x - 2y + 10) \sqrt{1 + (0)^2 + (0)^2} dy dx \\ &= \int_0^{2\pi} \int_0^1 (r \cos \theta - 2r \sin \theta + 10) r dr d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{3} \cos \theta - \frac{2}{3} \sin \theta + 5 \right) d\theta \\ &= \left[\frac{1}{3} \sin \theta + \frac{2}{3} \cos \theta + 5\theta \right]_0^{2\pi} = 10\pi \end{aligned}$$

5. $S: z = 6 - x - 2y$, (first octant) $\frac{\partial z}{\partial x} = -1, \frac{\partial z}{\partial y} = -2$

$$\begin{aligned} \iint_S xy dS &= \int_0^6 \int_0^{3-(x/2)} xy \sqrt{1 + (-1)^2 + (-2)^2} dy dx \\ &= \sqrt{6} \int_0^6 \left[\frac{xy^2}{2} \right]_0^{3-(x/2)} dx \\ &= \frac{\sqrt{6}}{2} \int_0^6 x \left(9 - 3x + \frac{1}{4}x^2 \right) dx \\ &= \frac{\sqrt{6}}{2} \left[\frac{9x^2}{2} - x^3 + \frac{x^4}{16} \right]_0^6 = \frac{27\sqrt{6}}{2} \end{aligned}$$



7. $S: z = 9 - x^2, 0 \leq x \leq 2, 0 \leq y \leq x$,

$$\frac{\partial z}{\partial x} = -2x, \frac{\partial z}{\partial y} = 0$$

$$\iint_S xy dS = \int_0^2 \int_y^2 xy \sqrt{1 + 4x^2} dx dy = \frac{391\sqrt{17} + 1}{240}$$

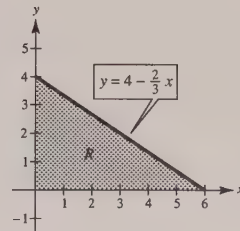
9. $S: z = 10 - x^2 - y^2, 0 \leq x \leq 2, 0 \leq y \leq 2$

$$\iint_S (x^2 - 2xy) dS = \int_0^2 \int_0^2 (x^2 - 2xy) \sqrt{1 + 4x^2 + 4y^2} dy dx \approx -11.47$$

11. $S: 2x + 3y + 6z = 12$ (first octant) $\Rightarrow z = 2 - \frac{1}{3}x - \frac{1}{2}y$

$$\rho(x, y, z) = x^2 + y^2$$

$$\begin{aligned} m &= \iint_R (x^2 + y^2) \sqrt{1 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{2}\right)^2} dA \\ &= \frac{7}{6} \int_0^6 \int_0^{4-(2x/3)} (x^2 + y^2) dy dx \\ &= \frac{7}{6} \int_0^6 \left[x^2 \left(4 - \frac{2}{3}x \right) + \frac{1}{3} \left(4 - \frac{2}{3}x \right)^3 \right] dx = \frac{7}{6} \left[\frac{4}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{8} \left(4 - \frac{2}{3}x \right)^4 \right]_0^6 = \frac{364}{3} \end{aligned}$$



13. $S: \mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \frac{v}{2}\mathbf{k}, 0 \leq u \leq 1, 0 \leq v \leq 2$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \left\| -\frac{1}{2}\mathbf{j} + \mathbf{k} \right\| = \frac{\sqrt{5}}{2}$$

$$\iint_S (y + 5) dS = \int_0^2 \int_0^1 (v + 5) \frac{\sqrt{5}}{2} du dv = 6\sqrt{5}$$

15. $S: \mathbf{r}(u, v) = 2 \cos u \mathbf{i} + 2 \sin u \mathbf{j} + v \mathbf{k}, 0 \leq u \leq \frac{\pi}{2}, 0 \leq v \leq 2$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|2 \cos u \mathbf{i} + 2 \sin u \mathbf{j}\| = 2$$

$$\iint_S xy dS = \int_0^2 \int_0^{\pi/2} 8 \cos u \sin u du dv = 8$$

17. $f(x, y, z) = x^2 + y^2 + z^2$

$$S: z = x + 2, x^2 + y^2 \leq 1$$

$$\begin{aligned} \iint_S f(x, y, z) dS &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [x^2 + y^2 + (x+2)^2] \sqrt{1+(1)^2+(0)^2} dy dx \\ &= \sqrt{2} \int_0^{2\pi} \int_0^1 [r^2 + (r \cos \theta + 2)^2] r dr d\theta \\ &= \sqrt{2} \int_0^{2\pi} \int_0^1 [r^2 + r^2 \cos^2 \theta + 4r \cos \theta + 4] r dr d\theta \\ &= \sqrt{2} \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{r^4}{4} \cos^2 \theta + \frac{4r^3}{3} \cos \theta + 2r^2 \right]_0^1 d\theta \\ &= \sqrt{2} \int_0^{2\pi} \left[\frac{9}{4} + \left(\frac{1}{4} \right) \frac{1 + \cos 2\theta}{2} + \frac{4}{3} \cos \theta \right] d\theta \\ &= \sqrt{2} \left[\frac{9}{4} \theta + \frac{1}{8} \left(\theta + \frac{1}{2} \sin 2\theta \right) + \frac{4}{3} \sin \theta \right]_0^{2\pi} = \sqrt{2} \left[\frac{18\pi}{4} + \frac{\pi}{4} \right] = \frac{19\sqrt{2}\pi}{4} \end{aligned}$$

19. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$$S: z = \sqrt{x^2 + y^2}, x^2 + y^2 \leq 4$$

$$\begin{aligned} \iint_S f(x, y, z) dS &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2 + (\sqrt{x^2 + y^2})^2} \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}} \right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}} \right)^2} dy dx \\ &= \sqrt{2} \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2} \sqrt{\frac{x^2 + y^2 + x^2 + y^2}{x^2 + y^2}} dy dx \\ &= 2 \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2} dy dx \\ &= 2 \int_0^{2\pi} \int_0^2 r^2 dr d\theta = 2 \int_0^{2\pi} \left[\frac{r^3}{3} \right]_0^2 d\theta = \left[\frac{16}{3} \theta \right]_0^{2\pi} = \frac{32\pi}{3} \end{aligned}$$

21. $f(x, y, z) = x^2 + y^2 + z^2$

$S: x^2 + y^2 = 9, 0 \leq x \leq 3, 0 \leq y \leq 3, 0 \leq z \leq 9$

Project the solid onto the yz -plane; $x = \sqrt{9 - y^2}, 0 \leq y \leq 3, 0 \leq z \leq 9$.

$$\begin{aligned} \iint_S f(x, y, z) dS &= \int_0^3 \int_0^9 [(9 - y^2) + y^2 + z^2] \sqrt{1 + \left(\frac{-y}{\sqrt{9 - y^2}}\right)^2 + (0)^2} dz dy \\ &= \int_0^3 \int_0^9 (9 + z^2) \frac{3}{\sqrt{9 - y^2}} dz dy = \int_0^3 \left[\frac{3}{\sqrt{9 - y^2}} \left(9z + \frac{z^3}{3} \right) \right]_0^9 dy \\ &= 324 \int_0^3 \frac{3}{\sqrt{9 - y^2}} dy = \left[972 \arcsin\left(\frac{y}{3}\right) \right]_0^3 = 972 \left(\frac{\pi}{2} - 0 \right) = 486\pi \end{aligned}$$

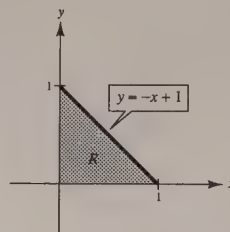
23. $\mathbf{F}(x, y, z) = 3z\mathbf{i} - 4\mathbf{j} + y\mathbf{k}$

$S: x + y + z = 1$ (first octant)

$G(x, y, z) = x + y + z - 1$

$\nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N} dS &= \iint_R \mathbf{F} \cdot \nabla G dA = \int_0^1 \int_0^{1-x} (3z - 4 + y) dy dx \\ &= \int_0^1 \int_0^{1-x} [3(1 - x - y) - 4 + y] dy dx \\ &= \int_0^1 \int_0^{1-x} (-1 - 3x - 2y) dy dx \\ &= \int_0^1 \left[-y - 3xy - y^2 \right]_0^{1-x} dx \\ &= - \int_0^1 [(1 - x) + 3x(1 - x) + (1 - x)^2] dx \\ &= - \int_0^1 (2 - 2x^2) dx = -\frac{4}{3} \end{aligned}$$



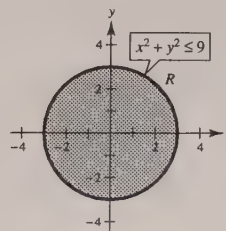
25. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$S: z = 9 - x^2 - y^2, 0 \leq z$

$G(x, y, z) = x^2 + y^2 + z - 9$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N} dS &= \iint_R \mathbf{F} \cdot \nabla G dA = \iint_R (2x^2 + 2y^2 + z) dA \\ &= \iint_R [2x^2 + 2y^2 + (9 - x^2 - y^2)] dA \\ &= \iint_R (x^2 + y^2 + 9) dA \\ &= \int_0^{2\pi} \int_0^3 (r^2 + 9) r dr d\theta \\ &= \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{9r^2}{2} \right]_0^3 d\theta = \frac{243\pi}{2} \end{aligned}$$

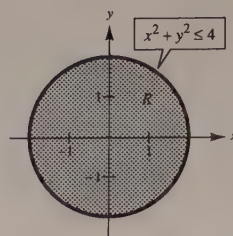


27. $\mathbf{F}(x, y, z) = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

$S: z = x^2 + y^2, x^2 + y^2 \leq 4$

$G(x, y, z) = -x^2 - y^2 + z$

$\nabla G(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$



$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N} \, dS &= \iint_R \mathbf{F} \cdot \nabla G \, dA = \iint_R (-8x + 6y + 5) \, dA \\ &= \int_0^{2\pi} \int_0^2 [-8r \cos \theta + 6r \sin \theta + 5] r \, dr \, d\theta \\ &= \int_0^{2\pi} \left[-\frac{8}{3} r^3 \cos \theta + 2r^3 \sin \theta + \frac{5}{2} r^2 \right]_0^2 d\theta \\ &= \int_0^{2\pi} \left[-\frac{64}{3} \cos \theta + 16 \sin \theta + 10 \right] d\theta \\ &= \left[-\frac{64}{3} \sin \theta - 16 \cos \theta + 10\theta \right]_0^{2\pi} = 20\pi \end{aligned}$$

29. $\mathbf{F}(x, y, z) = 4xy\mathbf{i} + z^2\mathbf{j} + yz\mathbf{k}$

S : unit cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

S_1 : The top of the cube

$\mathbf{N} = \mathbf{k}, z = 1$

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^1 \int_0^1 y(1) \, dy \, dx = \frac{1}{2}$$

S_2 : The bottom of the cube

$\mathbf{N} = -\mathbf{k}, z = 0$

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^1 \int_0^1 -y(0) \, dy \, dx = 0$$

S_4 : The back of the cube

$\mathbf{N} = -\mathbf{i}, x = 0$

$$\iint_{S_4} \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^1 \int_0^1 -4(0)y \, dy \, dx = 0$$

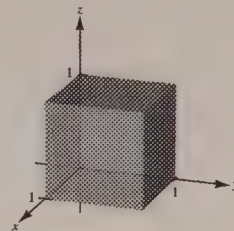
S_6 : The left side of the cube

$\mathbf{N} = -\mathbf{j}, y = 0$

$$\iint_{S_6} \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^1 \int_0^1 -z^2 \, dz \, dx = -\frac{1}{3}$$

Therefore,

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \frac{1}{2} + 0 + 2 + 0 + \frac{1}{3} - \frac{1}{3} = \frac{5}{2}.$$



S_3 : The front of the cube

$\mathbf{N} = \mathbf{i}, x = 1$

$$\iint_{S_3} \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^1 \int_0^1 4(1)y \, dy \, dz = 2$$

S_5 : The right side of the cube

$\mathbf{N} = \mathbf{j}, y = 1$

$$\iint_{S_5} \mathbf{F} \cdot \mathbf{N} \, dS = \int_0^1 \int_0^1 z^2 \, dz \, dx = \frac{1}{3}$$

31. The surface integral of f over a surface S , where S is given by $z = g(x, y)$, is defined as

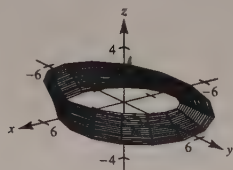
$$\iint_S f(x, y, z) \, dS = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i, y_i, z_i) \Delta S_i. \quad (\text{page 1061})$$

See Theorem 14.10, page 1061.

33. See the definition, page 1067.

See Theorem 14.11, page 1067.

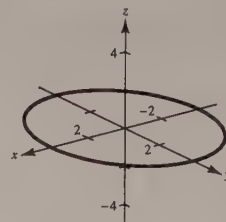
35. (a)



(b) If a normal vector at a point P on the surface is moved around the Möbius strip once, it will point in the opposite direction.

(c) $\mathbf{r}(u, 0) = 4 \cos(2u)\mathbf{i} + 4 \sin(2u)\mathbf{j}$

This is a circle.



(d) (construction)

(e) You obtain a strip with a double twist and twice as long as the original Möbius strip.

37. $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq a$

$$m = \iint_S k \, dS = k \iint_R \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2} \, dA = k \iint_R \sqrt{2} \, dA = \sqrt{2} k \pi a^2$$

$$I_z = \iint_S k(x^2 + y^2) \, dS = \iint_R k(x^2 + y^2) \sqrt{2} \, dA$$

$$= \sqrt{2} k \int_0^{2\pi} \int_0^a r^3 \, dr \, d\theta = \frac{\sqrt{2} k a^4}{4} (2\pi)$$

$$= \frac{\sqrt{2} k \pi a^4}{2} = \frac{a^2}{2} (\sqrt{2} k \pi a^2) = \frac{a^2 m}{2}$$

39. $x^2 + y^2 = a^2$, $0 \leq z \leq h$

$$\rho(x, y, z) = 1$$

$$y = \pm \sqrt{a^2 - x^2}$$

Project the solid onto the xz -plane.

$$I_z = 4 \iint_S (x^2 + y^2)(1) \, dS$$

$$= 4 \int_0^h \int_0^a [x^2 + (a^2 - x^2)] \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2 + (0)^2} \, dx \, dz$$

$$= 4a^3 \int_0^h \int_0^a \frac{1}{\sqrt{a^2 - x^2}} \, dx \, dz$$

$$= 4a^3 \int_0^h \left[\arcsin \frac{x}{a} \right]_0^a \, dz = 4a^3 \left(\frac{\pi}{2} \right) (h) = 2\pi a^3 h$$



41. $S: z = 16 - x^2 - y^2$, $z \geq 0$

$$\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$$

$$\iint_S \rho \mathbf{F} \cdot \mathbf{N} \, dS = \iint_R \rho \mathbf{F} \cdot (-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}) \, dA = \iint_R 0.5\rho z \mathbf{k} \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}) \, dA$$

$$= \iint_R 0.5\rho z \, dA = \iint_R 0.5\rho(16 - x^2 - y^2) \, dA$$

$$= 0.5\rho \int_0^{2\pi} \int_0^4 (16 - r^2)r \, dr \, d\theta = 0.5\rho \int_0^{2\pi} 64 \, d\theta = 64\pi\rho$$

Section 14.7 Divergence Theorem

1. **Surface Integral:** There are six surfaces to the cube, each with $dS = \sqrt{1} dA$.

$$z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z^2, \quad \int_{S_1} \int 0 dA = 0$$

$$z = a, \quad \mathbf{N} = \mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = z^2, \quad \int_{S_2} \int a^2 dA = \int_0^a \int_0^a a^2 dx dy = a^4$$

$$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = -2x, \quad \int_{S_3} \int 0 dA = 0$$

$$x = a, \quad \mathbf{N} = \mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = 2x, \quad \int_{S_4} \int 2a dy dz = \int_0^a \int_0^a 2a dy dz = 2a^3$$

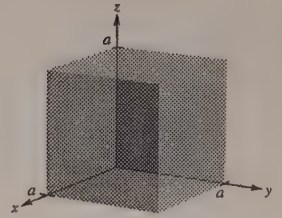
$$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = 2y, \quad \int_{S_5} \int 0 dA = 0$$

$$y = a, \quad \mathbf{N} = \mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = -2y, \quad \int_{S_6} \int -2a dA = \int_0^a \int_0^a -2a dz dx = -2a^3$$

Therefore, $\int_S \mathbf{F} \cdot \mathbf{N} dS = a^4 + 2a^3 - 2a^3 = a^4$.

Divergence Theorem: Since $\text{div } \mathbf{F} = 2z$, the Divergence Theorem yields

$$\iiint_Q \text{div } \mathbf{F} dV = \int_0^a \int_0^a \int_0^a 2z dz dy dx = \int_0^a \int_0^a a^2 dy dx = a^4.$$



3. **Surface Integral:** There are four surfaces to this solid.

$$z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z$$

$$\int_{S_1} \int 0 dS = 0$$

$$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = 2y - z, \quad dS = dA = dx dz$$

$$\int_{S_2} \int -z dS = \int_0^6 \int_0^{6-z} -z dx dz = \int_0^6 (z^2 - 6z) dz = -36$$

$$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = y - 2x, \quad dS = dA = dz dy$$

$$\int_{S_3} \int y dS = \int_0^3 \int_0^{6-2y} y dz dy = \int_0^3 (6y - 2y^2) dy = 9$$

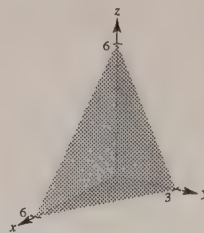
$$x + 2y + z = 6, \quad \mathbf{N} = \frac{\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{6}}, \quad \mathbf{F} \cdot \mathbf{N} = \frac{2x - 5y + 3z}{\sqrt{6}}, \quad dS = \sqrt{6} dA$$

$$\int_{S_4} \int (2x - 5y + 3z) dz dy = \int_0^3 \int_0^{6-2y} (18 - x - 11y) dx dy = \int_0^3 (90 - 90y + 20y^2) dy = 45$$

Therefore, $\int_S \mathbf{F} \cdot \mathbf{N} dS = 0 - 36 + 9 + 45 = 18$.

Divergence Theorem: Since $\text{div } \mathbf{F} = 1$, we have

$$\iiint_Q dV = (\text{Volume of solid}) = \frac{1}{3}(\text{Area of base}) \times (\text{Height}) = \frac{1}{3}(9)(6) = 18.$$



5. Since $\operatorname{div} \mathbf{F} = 2x + 2y + 2z$, we have

$$\begin{aligned} \iiint_Q \operatorname{div} \mathbf{F} \, dV &= \int_0^a \int_0^a \int_0^a (2x + 2y + 2z) \, dz \, dy \, dx \\ &= \int_0^a \int_0^a (2ax + 2ay + a^2) \, dy \, dx = \int_0^a (2a^2x + 2a^3) \, dx = \left[a^2x^2 + 2a^3x \right]_0^a = 3a^4. \end{aligned}$$

7. Since $\operatorname{div} \mathbf{F} = 2x - 2x + 2xyz = 2xyz$

$$\begin{aligned} \iiint_Q \operatorname{div} \mathbf{F} \, dV &= \iiint_Q 2xyz \, dV = \int_0^a \int_0^{2\pi} \int_0^{\pi/2} 2(\rho \sin \phi \cos \theta)(\rho \sin \phi \sin \theta)(\rho \cos \phi) \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho \\ &= \int_0^a \int_0^{2\pi} \int_0^{\pi/2} 2\rho^5 (\sin \theta \cos \theta)(\sin^3 \phi \cos \phi) \, d\phi \, d\theta \, d\rho \\ &= \int_0^a \int_0^{2\pi} \frac{1}{2} \rho^5 \sin \theta \cos \theta \, d\theta \, d\rho = \int_0^a \left[\left(\frac{\rho^5}{2} \right) \frac{\sin^2 \theta}{2} \right]_0^{2\pi} d\rho = 0. \end{aligned}$$

9. Since $\operatorname{div} \mathbf{F} = 3$, we have

$$\iiint_Q 3 \, dV = 3(\text{Volume of sphere}) = 3 \left[\frac{4}{3} \pi (2)^3 \right] = 32\pi.$$

11. Since $\operatorname{div} \mathbf{F} = 1 + 2y - 1 = 2y$, we have

$$\iiint_Q 2y \, dV = \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} 2y \, dx \, dy \, dz = \int_0^4 \int_{-3}^3 4y \sqrt{9-y^2} \, dy \, dz = \int_0^4 \left[-\frac{4}{3} (9-y^2)^{3/2} \right]_{-3}^3 dz = 0.$$

13. Since $\operatorname{div} \mathbf{F} = 3x^2 + x^2 + 0 = 4x^2$, we have

$$\iiint_Q 4x^2 \, dV = \int_0^6 \int_0^4 \int_0^{4-y} 4x^2 \, dz \, dy \, dx = \int_0^6 \int_0^4 4x^2(4-y) \, dy \, dx = \int_0^6 32x^2 \, dx = 2304.$$

15. $\mathbf{F}(x, y, z) = xy\mathbf{i} + 4y\mathbf{j} + xz\mathbf{k}$

$$\operatorname{div} \mathbf{F} = y + 4 + x$$

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{N} \, dS &= \iiint_Q \operatorname{div} \mathbf{F} \, dV = \iiint_Q (y + x + 4) \, dV \\ &= \int_0^3 \int_0^\pi \int_0^{2\pi} (\rho \sin \phi \sin \theta + \rho \sin \phi \cos \theta + 4) \rho^2 \sin \phi \, d\theta \, d\phi \, d\rho \\ &= \int_0^3 \int_0^\pi \int_0^{2\pi} [\rho^3 \sin^2 \phi \sin \theta + \rho^3 \sin^2 \phi \cos \theta + 4\rho^2 \sin \phi] \, d\theta \, d\phi \, d\rho \\ &= \int_0^3 \int_0^\pi \left[-\rho^3 \sin^2 \phi \cos \theta + \rho^3 \sin^2 \phi \sin \theta + 4\rho^2 \sin \phi \cdot \theta \right]_0^{2\pi} d\phi \, d\rho \\ &= \int_0^3 \int_0^\pi 8\pi \rho^2 \sin \phi \, d\phi \, d\rho \\ &= \int_0^3 \left[-8\pi \rho^2 \cos \phi \right]_0^\pi d\rho \\ &= \int_0^3 16\pi \rho^2 \, d\rho = \left[\frac{16\pi \rho^3}{3} \right]_0^3 = 144\pi. \end{aligned}$$

17. Using the Divergence Theorem, we have

$$\iint_S \mathbf{curl} \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q \operatorname{div}(\mathbf{curl} \mathbf{F}) dV$$

$$\mathbf{curl} \mathbf{F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy + z^2 & 2x^2 + 6yz & 2xz \end{vmatrix} = -6y\mathbf{i} - (2z - 2z)\mathbf{j} + (4x - 4x)\mathbf{k} = -6y\mathbf{i}$$

$$\operatorname{div}(\mathbf{curl} \mathbf{F}) = 0.$$

$$\text{Therefore, } \iiint_Q \operatorname{div}(\mathbf{curl} \mathbf{F}) dV = 0.$$

19. See Theorem 14.12, page 1073.

21. Using the triple integral to find volume, we need \mathbf{F} so that

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = 1.$$

Hence, we could have $\mathbf{F} = x\mathbf{i}$, $\mathbf{F} = y\mathbf{j}$, or $\mathbf{F} = z\mathbf{k}$.

$$\text{For } dA = dy dz \text{ consider } \mathbf{F} = x\mathbf{i}, x = f(y, z), \text{ then } \mathbf{N} = \frac{\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_y^2 + f_z^2}} \text{ and } dS = \sqrt{1 + f_y^2 + f_z^2} dy dz.$$

$$\text{For } dA = dz dx \text{ consider } \mathbf{F} = y\mathbf{j}, y = f(x, z), \text{ then } \mathbf{N} = \frac{f_x\mathbf{i} + \mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_x^2 + f_z^2}} \text{ and } dS = \sqrt{1 + f_x^2 + f_z^2} dz dx.$$

$$\text{For } dA = dx dy \text{ consider } \mathbf{F} = z\mathbf{k}, z = f(x, y), \text{ then } \mathbf{N} = \frac{f_x\mathbf{i} + f_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}} \text{ and } dS = \sqrt{1 + f_x^2 + f_y^2} dx dy.$$

$$\text{Correspondingly, we then have } V = \iint_S \mathbf{F} \cdot \mathbf{N} dS = \iint_S x dy dz = \iint_S y dz dx = \iint_S z dx dy.$$

23. Using the Divergence Theorem, we have $\iint_S \mathbf{curl} \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q \operatorname{div}(\mathbf{curl} \mathbf{F}) dV$. Let

$$\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

$$\mathbf{curl} \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}$$

$$\operatorname{div}(\mathbf{curl} \mathbf{F}) = \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 M}{\partial y \partial z} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0.$$

$$\text{Therefore, } \iint_S \mathbf{curl} \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q 0 dV = 0.$$

25. If $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\operatorname{div} \mathbf{F} = 3$.

$$\iint_S \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q \operatorname{div} \mathbf{F} dV = \iiint_Q 3 dV = 3V.$$

$$27. \iint_S f D_N g dS = \iint_S f \nabla g \cdot \mathbf{N} dS$$

$$= \iiint_Q \operatorname{div}(f \nabla g) dV = \iiint_Q (f \operatorname{div} \nabla g + \nabla f \cdot \nabla g) dV = \iiint_Q (f \nabla^2 g + \nabla f \cdot \nabla g) dV$$

Section 14.8 Stokes's Theorem

1. $\mathbf{F}(x, y, z) = (2y - z)\mathbf{i} + xyz\mathbf{j} + e^z\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y - z & xyz & e^z \end{vmatrix} = -xy\mathbf{i} - \mathbf{j} + (yz - 2)\mathbf{k}$$

3. $\mathbf{F}(x, y, z) = 2z\mathbf{i} - 4x^2\mathbf{j} + \arctan x\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & -4x^2 & \arctan x \end{vmatrix} = \left(2 - \frac{1}{1+x^2}\right)\mathbf{j} - 8x\mathbf{k}$$

5. $\mathbf{F}(x, y, z) = e^{x^2+y^2}\mathbf{i} + e^{y^2+z^2}\mathbf{j} + xyz\mathbf{k}$

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x^2+y^2} & e^{y^2+z^2} & xyz \end{vmatrix} \\ &= (xz - 2ze^{y^2+z^2})\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k} \\ &= z(x - 2e^{y^2+z^2})\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k} \end{aligned}$$

7. In this case, $M = -y + z$, $N = x - z$, $P = x - y$ and C is the circle $x^2 + y^2 = 1$, $z = 0$, $dz = 0$.

Line Integral: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (-y + z) dx + (x - z) dy + (x - y) dz = \int_C -y dx + x dy$

Letting $x = \cos t$, $y = \sin t$, we have $dx = -\sin t dt$, $dy = \cos t dt$ and

$$\int_C -y dx + x dy = \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 2\pi.$$

Double Integral: Consider $F(x, y, z) = x^2 + y^2 + z^2 - 1$.

Then

$$\mathbf{N} = \frac{\nabla F}{\|\nabla F\|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Since

$$z^2 = 1 - x^2 - y^2, \quad z_x = \frac{-2x}{2z} = \frac{-x}{z}, \quad \text{and} \quad z_y = \frac{-y}{z}, \quad dS = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} dA = \frac{1}{z} dA.$$

Now, since $\operatorname{curl} \mathbf{F} = 2\mathbf{k}$, we have

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} dS = \iint_R \int 2z \left(\frac{1}{z}\right) dA = \iint_R 2 dA = 2(\text{Area of circle of radius 1}) = 2\pi.$$

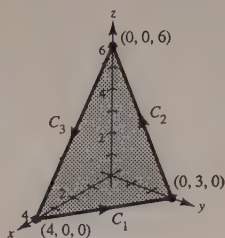
9. Line Integral: From the accompanying figure we see that for

$$C_1: z = 0, dz = 0$$

$$C_2: x = 0, dx = 0$$

$$C_3: y = 0, dy = 0.$$

$$\begin{aligned} \text{Hence, } \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C xyz \, dx + y \, dy + z \, dz \\ &= \int_{C_1} y \, dy + \int_{C_2} y \, dy + z \, dz + \int_{C_3} z \, dz \\ &= \int_0^3 y \, dy + \int_3^0 y \, dy + \int_0^6 z \, dz + \int_6^0 z \, dz = 0. \end{aligned}$$



Double Integral: $\text{curl } \mathbf{F} = xy\mathbf{j} - xz\mathbf{k}$

Considering $F(x, y, z) = 3x + 4y + 2z - 12$, then

$$\mathbf{N} = \frac{\nabla F}{\|\nabla F\|} = \frac{3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}{\sqrt{29}} \text{ and } dS = \sqrt{29} \, dA.$$

Thus,

$$\begin{aligned} \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS &= \iint_R (4xy - 2xz) \, dA \\ &= \int_0^4 \int_0^{(-3x+12)/4} \left[4xy - 2x \left(6 - 2y - \frac{3}{2}x \right) \right] dy \, dx \\ &= \int_0^4 \int_0^{(12-3x)/4} (8xy + 3x^2 - 12x) \, dy \, dx \\ &= \int_0^4 0 \, dx = 0. \end{aligned}$$

11. Let $A = (0, 0, 0)$, $B = (1, 1, 1)$ and $C = (0, 2, 0)$. Then $\mathbf{U} = \overrightarrow{AB} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{V} = \overrightarrow{AC} = 2\mathbf{j}$. Thus,

$$\mathbf{N} = \frac{\mathbf{U} \times \mathbf{V}}{\|\mathbf{U} \times \mathbf{V}\|} = \frac{-2\mathbf{i} + 2\mathbf{k}}{2\sqrt{2}} = \frac{-\mathbf{i} + \mathbf{k}}{\sqrt{2}}.$$

Surface S has direction numbers $-1, 0, 1$, with equation $z - x = 0$ and $dS = \sqrt{2} \, dA$. Since $\text{curl } \mathbf{F} = -3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, we have

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS = \iint_R \frac{1}{\sqrt{2}} (\sqrt{2}) \, dA = \iint_R dA = (\text{Area of triangle with } h = 1, b = 2) = 1.$$

13. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$, $S: z = 4 - x^2 - y^2$, $0 \leq z$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{vmatrix} = 2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}$$

$$G(x, y, z) = x^2 + y^2 + z - 4$$

$$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS &= \iint_R (4xy + 4yz + 2x) \, dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4xy + 4y(4 - x^2 - y^2) + 2x] \, dy \, dx \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4xy + 16y - 4x^2y - 4y^3 + 2x] \, dy \, dx \\ &= \int_{-2}^2 4x\sqrt{4-x^2} \, dx = 0 \end{aligned}$$

15. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + y\mathbf{j} + xz\mathbf{k}$, $S: z = \sqrt{4 - x^2 - y^2}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y & xz \end{vmatrix} = z\mathbf{j}$$

$$G(x, y, z) = z - \sqrt{4 - x^2 - y^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} \, dS = \iint_R \frac{yz}{\sqrt{4 - x^2 - y^2}} \, dA = \iint_R \frac{y\sqrt{4 - x^2 - y^2}}{\sqrt{4 - x^2 - y^2}} \, dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y \, dy \, dx = 0$$

17. $\mathbf{F}(x, y, z) = -\ln\sqrt{x^2 + y^2}\mathbf{i} + \arctan\frac{x}{y}\mathbf{j} + \mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -1/2 \ln(x^2 + y^2) & \arctan x/y & 1 \end{vmatrix} = \left[\frac{(1/y)}{1 + (x^2/y^2)} + \frac{y}{x^2 + y^2} \right] \mathbf{k} = \left[\frac{2y}{x^2 + y^2} \right] \mathbf{k}$$

$S: z = 9 - 2x - 3y$ over one petal of $r = 2 \sin 2\theta$ in the first octant.

$$G(x, y, z) = 2x + 3y + z - 9$$

$$\nabla G(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} \, dS &= \iint_R \frac{2y}{x^2 + y^2} \, dA \\ &= \int_0^{\pi/2} \int_0^{2 \sin 2\theta} \frac{2r \sin \theta}{r^2} r \, dr \, d\theta \\ &= \int_0^{\pi/2} \int_0^{4 \sin \theta \cos \theta} 2 \sin \theta \, dr \, d\theta \\ &= \int_0^{\pi/2} 8 \sin^2 \theta \cos \theta \, d\theta = \left[\frac{8 \sin^3 \theta}{3} \right]_0^{\pi/2} = \frac{8}{3} \end{aligned}$$

19. From Exercise 10, we have $\mathbf{N} = \frac{2x\mathbf{i} - \mathbf{k}}{\sqrt{1 + 4x^2}}$ and $dS = \sqrt{1 + 4x^2} \, dA$. Since $\operatorname{curl} \mathbf{F} = xy\mathbf{j} - xz\mathbf{k}$, we have

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} \, dS = \iint_R xz \, dA = \int_0^a \int_0^a x^3 \, dy \, dx = \int_0^a ax^3 \, dx = \left[\frac{ax^4}{4} \right]_0^a = \frac{a^5}{4}.$$

21. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 1 & -2 \end{vmatrix} = \mathbf{0}$$

Letting $\mathbf{N} = \mathbf{k}$, we have $\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} \, dS = 0$.

23. See Theorem 14.13, page 1081.

$$25. (a) \int_C f \nabla g \cdot d\mathbf{r} = \int_S \mathbf{curl}[f \nabla g] \cdot \mathbf{N} dS \text{ (Stokes's Theorem)}$$

$$f \nabla g = f \frac{\partial g}{\partial x} \mathbf{i} + f \frac{\partial g}{\partial y} \mathbf{j} + f \frac{\partial g}{\partial z} \mathbf{k}$$

$$\begin{aligned} \mathbf{curl}(f \nabla g) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(\partial g / \partial x) & f(\partial g / \partial y) & f(\partial g / \partial z) \end{vmatrix} \\ &= \left[\left[f \left(\frac{\partial^2 g}{\partial y \partial z} \right) + \left(\frac{\partial f}{\partial y} \right) \left(\frac{\partial g}{\partial z} \right) \right] - \left[f \left(\frac{\partial^2 g}{\partial z \partial y} \right) + \left(\frac{\partial f}{\partial z} \right) \left(\frac{\partial g}{\partial y} \right) \right] \right] \mathbf{i} \\ &\quad - \left[\left[f \left(\frac{\partial^2 g}{\partial x \partial z} \right) + \left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial g}{\partial z} \right) \right] - \left[f \left(\frac{\partial^2 g}{\partial z \partial x} \right) + \left(\frac{\partial f}{\partial z} \right) \left(\frac{\partial g}{\partial x} \right) \right] \right] \mathbf{j} \\ &\quad + \left[\left[f \left(\frac{\partial^2 g}{\partial x \partial y} \right) + \left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial g}{\partial y} \right) \right] - \left[f \left(\frac{\partial^2 g}{\partial y \partial x} \right) + \left(\frac{\partial f}{\partial y} \right) \left(\frac{\partial g}{\partial x} \right) \right] \right] \mathbf{k} \\ &= \left[\left(\frac{\partial f}{\partial y} \right) \left(\frac{\partial g}{\partial z} \right) - \left(\frac{\partial f}{\partial z} \right) \left(\frac{\partial g}{\partial y} \right) \right] \mathbf{i} - \left[\left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial g}{\partial z} \right) - \left(\frac{\partial f}{\partial z} \right) \left(\frac{\partial g}{\partial x} \right) \right] \mathbf{j} + \left[\left(\frac{\partial f}{\partial x} \right) \left(\frac{\partial g}{\partial y} \right) - \left(\frac{\partial f}{\partial y} \right) \left(\frac{\partial g}{\partial x} \right) \right] \mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix} = \nabla f \times \nabla g \end{aligned}$$

$$\text{Therefore, } \int_C f \nabla g \cdot d\mathbf{r} = \int_S \mathbf{curl}[f \nabla g] \cdot \mathbf{N} dS = \int_S [\nabla f \times \nabla g] \cdot \mathbf{N} dS.$$

$$(b) \int_C (f \nabla f) \cdot d\mathbf{r} = \int_S (\nabla f \times \nabla f) \cdot \mathbf{N} dS \text{ (using part a.)}$$

$$= 0 \text{ since } \nabla f \times \nabla f = \mathbf{0}.$$

$$\begin{aligned} (c) \int_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} &= \int_C (f \nabla g) \cdot d\mathbf{r} + \int_C (g \nabla f) \cdot d\mathbf{r} \\ &= \int_S (\nabla f \times \nabla g) \cdot \mathbf{N} dS + \int_S (\nabla g \times \nabla f) \cdot \mathbf{N} dS \text{ (using part a.)} \\ &= \int_S (\nabla f \times \nabla g) \cdot \mathbf{N} dS + \int_S -(\nabla f \times \nabla g) \cdot \mathbf{N} dS = 0 \end{aligned}$$

27. Let $\mathbf{C} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

$$\frac{1}{2} \int_C (\mathbf{C} \times \mathbf{r}) \cdot d\mathbf{r} = \frac{1}{2} \int_S \mathbf{curl}(\mathbf{C} \times \mathbf{r}) \cdot \mathbf{N} dS = \frac{1}{2} \int_S 2\mathbf{C} \cdot \mathbf{N} dS = \int_S \mathbf{C} \cdot \mathbf{N} dS$$

since

$$\mathbf{C} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\mathbf{i} - (az - cx)\mathbf{j} + (ay - bx)\mathbf{k}$$

and

$$\mathbf{curl}(\mathbf{C} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix} = 2(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 2\mathbf{C}.$$

Review Exercises for Chapter 14

1. $\mathbf{F}(x, y, z) = x\mathbf{i} + \mathbf{j} + 2z\mathbf{k}$



3. $f(x, y, z) = 8x^2 + xy + z^2$

$$\mathbf{F}(x, y, z) = (16x + y)\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$$

5. Since $\partial M/\partial y = -1/y^2 \neq \partial N/\partial x$, \mathbf{F} is not conservative.

7. Since $\partial M/\partial y = 12xy = \partial N/\partial x$, \mathbf{F} is conservative. From $M = \partial U/\partial x = 6xy^2 - 3x^2$ and $N = \partial U/\partial y = 6x^2y + 3y^2 - 7$, partial integration yields $U = 3x^2y^2 - x^3 + h(y)$ and $U = 3x^2y^2 + y^3 - 7y + g(x)$ which suggests $h(y) = y^3 - 7y$, $g(x) = -x^3$, and $U(x, y) = 3x^2y^2 - x^3 + y^3 - 7y + C$.

9. Since

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = 1 \neq \frac{\partial P}{\partial x}.$$

\mathbf{F} is not conservative.

11. Since

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2z} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{-1}{yz^2} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{x}{y^2z^2} = \frac{\partial P}{\partial y},$$

\mathbf{F} is conservative. From

$$M = \frac{\partial U}{\partial x} = \frac{1}{yz}, \quad N = \frac{\partial U}{\partial y} = \frac{-x}{y^2z}, \quad P = \frac{\partial U}{\partial z} = \frac{-x}{yz^2}$$

we obtain

$$U = \frac{x}{yz} + f(y, z), \quad U = \frac{x}{yz} + g(x, z), \quad U = \frac{x}{yz} + h(x, y) \Rightarrow f(x, y, z) = \frac{x}{yz} + K$$

13. Since $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$:

(a) $\operatorname{div} \mathbf{F} = 2x + 2y + 2z$

(b) $\operatorname{curl} \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

15. Since $\mathbf{F} = (\cos y + y \cos x)\mathbf{i} + (\sin x - x \sin y)\mathbf{j} + xyz\mathbf{k}$:

(a) $\operatorname{div} \mathbf{F} = -y \sin x - x \cos y + xy$

(b) $\operatorname{curl} \mathbf{F} = xz\mathbf{i} - yz\mathbf{j} + (\cos x - \sin y + \sin y - \cos x)\mathbf{k} = xz\mathbf{i} - yz\mathbf{j}$

17. Since $\mathbf{F} = \arcsin x \mathbf{i} + xy^2 \mathbf{j} + yz^2 \mathbf{k}$:

(a) $\operatorname{div} \mathbf{F} = \frac{1}{\sqrt{1-x^2}} + 2xy + 2yz$

(b) $\operatorname{curl} \mathbf{F} = z^2 \mathbf{i} + y^2 \mathbf{k}$

19. Since $\mathbf{F} = \ln(x^2 + y^2) \mathbf{i} + \ln(x^2 + y^2) \mathbf{j} + z \mathbf{k}$:

(a) $\operatorname{div} \mathbf{F} = \frac{2x}{x^2 + y^2} + \frac{2y}{x^2 + y^2} + 1$

$$= \frac{2x + 2y}{x^2 + y^2} + 1$$

(b) $\operatorname{curl} \mathbf{F} = \frac{2x - 2y}{x^2 + y^2} \mathbf{k}$

21. (a) Let $x = t, y = t, -1 \leq t \leq 2$, then $ds = \sqrt{2} dt$.

$$\int_C (x^2 + y^2) ds = \int_{-1}^2 2t^2 \sqrt{2} dt = \left[2\sqrt{2} \left(\frac{t^3}{3} \right) \right]_{-1}^2 = 6\sqrt{2}$$

(b) Let $x = 4 \cos t, y = 4 \sin t, 0 \leq t \leq 2\pi$, then $ds = 4 dt$.

$$\int_C (x^2 + y^2) ds = \int_0^{2\pi} 16(4 dt) = 128\pi$$

23. $x = \cos t + t \sin t, y = \sin t - t \cos t, 0 \leq t \leq 2\pi, \frac{dx}{dt} = t \cos t, \frac{dy}{dt} = t \sin t$

$$\begin{aligned} \int_C (x^2 + y^2) ds &= \int_0^{2\pi} [(\cos t + t \sin t)^2 + (\sin t - t \cos t)^2] \sqrt{t^2 \cos^2 t + t^2 \sin^2 t} dt = \int_0^{2\pi} [t^3 + t] dt \\ &= 2\pi^2(1 + 2\pi^2) \end{aligned}$$

25. (a) Let $x = 2t, y = -3t, 0 \leq t \leq 1$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^1 [7t(2) + (-7t)(-3)] dt = \int_0^1 35t dt = \frac{35}{2}$$

(b) $x = 3 \cos t, y = 3 \sin t, dx = -3 \sin t dt, dy = 3 \cos t dt, 0 \leq t \leq 2\pi$

$$\int_C (2x - y) dx + (x + 3y) dy = \int_0^{2\pi} (9 + 9 \sin t \cos t) dt = 18\pi$$

27. $\int_C (2x + y) ds, \mathbf{r}(t) = a \cos^3 t \mathbf{i} + a \sin^3 t \mathbf{j}, 0 \leq t \leq \frac{\pi}{2}$

$$x'(t) = -3a \cdot \cos^2 t \sin t$$

$$y'(t) = 3a \cdot \sin^2 t \cos t$$

$$\int_C (2x + y) ds = \int_0^{\pi/2} (2(a \cdot \cos^3 t) + a \cdot \sin^3 t) \sqrt{x'(t)^2 + y'(t)^2} dt = \frac{9a^2}{5}$$

29. $f(x, y) = 5 + \sin(x + y)$ $C: y = 3x$ from $(0, 0)$ to $(2, 6)$

$$\mathbf{r}(t) = t \mathbf{i} + 3t \mathbf{j}, 0 \leq t \leq 2$$

$$\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{10}$$

Lateral surface area:

$$\int_{C_1} f(x, y) ds = \int_0^2 [5 + \sin(t + 3t)] \sqrt{10} dt = \sqrt{10} \int_0^2 (5 + \sin 4t) dt = \frac{\sqrt{10}}{4} (41 - \cos 8) \approx 32.528$$

31. $d\mathbf{r} = (2t\mathbf{i} + 3t^2\mathbf{j}) dt$

$$\mathbf{F} = t^5\mathbf{i} + t^4\mathbf{j}, 0 \leq t \leq 1$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 5t^6 dt = \frac{5}{7}$$

35. Let $x = t, y = -t, z = 2t^2, -2 \leq t \leq 2, d\mathbf{r} = [\mathbf{i} - \mathbf{j} + 4t\mathbf{k}] dt$.

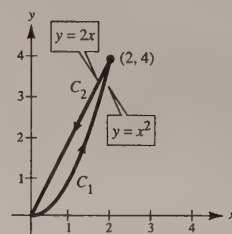
$$\mathbf{F} = (-t - 2t^2)\mathbf{i} + (2t^2 - t)\mathbf{j} + (2t)\mathbf{k}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-2}^2 4t^2 dt = \left[\frac{4t^3}{3} \right]_{-2}^2 = \frac{64}{3}$$

37. For $y = x^2, \mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 2$

For $y = 2x, \mathbf{r}_2(t) = (2 - t)\mathbf{i} + (4 - 2t)\mathbf{j}, 0 \leq t \leq 2$

$$\begin{aligned} \int_C xy dx + (x^2 + y^2) dy &= \int_{C_1} xy dx + (x^2 + y^2) dy + \int_{C_2} xy dx + (x^2 + y^2) dy \\ &= \frac{100}{3} + (-32) = \frac{4}{3} \end{aligned}$$



39. $\mathbf{F} = x\mathbf{i} - \sqrt{y}\mathbf{j}$ is conservative.

$$\text{Work} = \left[\frac{1}{2}x^2 - \frac{2}{3}y^{3/2} \right]_{(0,0)}^{(4,8)} = \frac{1}{2}(16) - \left(\frac{2}{3} \right) 8^{3/2} = \frac{8}{3}(3 - 4\sqrt{2})$$

41. $\int_C 2xyz dx + x^2z dy + x^2y dz = \left[x^2yz \right]_{(0,0,0)}^{(1,4,3)} = 12$

43. (a) $\int_C y^2 dx + 2xy dy = \int_0^1 [(1+t)^2(3) + 2(1+3t)(1+t)] dt$

$$= \int_0^1 3(t^2 + 2t + 1) + 2(3t^2 + 4t + 1) dt$$

$$= \int_0^1 (9t^2 + 14t + 5) dt$$

$$= \left[3t^3 + 7t^2 + 5t \right]_0^1 = 15$$

(b) $\int_C y^2 dx + 2xy dy = \int_1^4 \left[t(1) + 2(t)(\sqrt{t}) \frac{1}{2\sqrt{t}} \right] dt$

$$= \int_1^4 (t + t) dt$$

$$= \left[t^2 \right]_1^4 = 15$$

(c) $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j} = \nabla f$ where $f(x, y) = xy^2$.

Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4(2)^2 - 1(1)^2 = 15$$

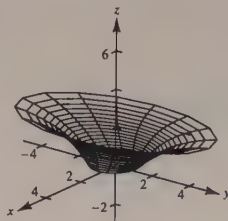
45. $\int_C y dx + 2x dy = \int_0^2 \int_0^2 (2 - 1) dy dx = \int_0^2 2 dx = 4$

47. $\int_C xy^2 dx + x^2y dy = \iint_R (2xy - 2xy) dA = 0$

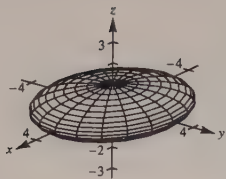
$$49. \int_C xy \, dx + x^2 \, dy = \int_0^1 \int_{x^2}^x x \, dy \, dx = \int_0^1 (x^2 - x^3) \, dx = \frac{1}{12}$$

$$51. \mathbf{r}(u, v) = \sec u \cos v \mathbf{i} + (1 + 2 \tan u) \sin v \mathbf{j} + 2u \mathbf{k}$$

$$0 \leq u \leq \frac{\pi}{3}, \quad 0 \leq v \leq 2\pi$$



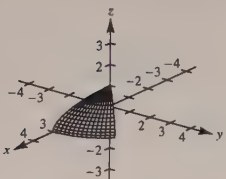
53. (a)



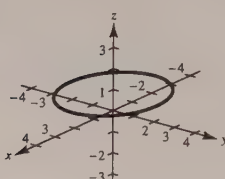
(b)



(c)



(d)



The space curve is a circle:

$$\mathbf{r}\left(u, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} \cos u \mathbf{i} + \frac{3\sqrt{2}}{2} \sin u \mathbf{j} + \frac{\sqrt{2}}{2} \mathbf{k}$$

$$(e) \mathbf{r}_u = -3 \cos v \sin u \mathbf{i} + 3 \cos v \cos u \mathbf{j}$$

$$\mathbf{r}_v = -3 \sin v \cos u \mathbf{i} - 3 \sin v \sin u \mathbf{j} + \cos v \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 \cos v \sin u & 3 \cos v \cos u & 0 \\ -3 \sin v \cos u & -3 \sin v \sin u & \cos v \end{vmatrix}$$

$$= (3 \cos^2 v \cos u) \mathbf{i} + (3 \cos^2 v \sin u) \mathbf{j} + (9 \cos v \sin v \sin^2 u + 9 \cos v \sin v \cos^2 u) \mathbf{k}$$

$$= (3 \cos^2 v \cos u) \mathbf{i} + (3 \cos^2 v \sin u) \mathbf{j} + (9 \cos v \sin v) \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{9 \cos^4 v \cos^2 u + 9 \cos^4 v \sin^2 u + 81 \cos^2 v \sin^2 v}$$

$$= \sqrt{9 \cos^4 v + 81 \cos^2 v \sin^2 v}$$

Using a Symbolic integration utility,

$$\int_{\pi/4}^{\pi/2} \int_0^{2\pi} \|\mathbf{r}_u \times \mathbf{r}_v\| \, du \, dv \approx 14.44$$

(f) Similarly,

$$\int_0^{\pi/4} \int_0^{\pi/2} \|\mathbf{r}_u \times \mathbf{r}_v\| \, dv \, du \approx 4.27$$

55. $S: \mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + (u - 1)(2 - u)\mathbf{k}, \quad 0 \leq u \leq 2, 0 \leq v \leq 2\pi$

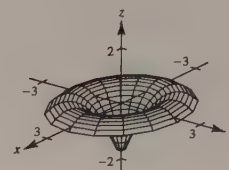
$$\mathbf{r}_u(u, v) = \cos v \mathbf{i} + \sin v \mathbf{j} + (3 - 2u)\mathbf{k}$$

$$\mathbf{r}_v(u, v) = -u \sin v \mathbf{i} + u \cos v \mathbf{j}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 3 - 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} = (2u - 3)u \cos v \mathbf{i} + (2u - 3)u \sin v \mathbf{j} + u \mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{(2u - 3)^2 + 1}$$

$$\begin{aligned} \iint_S (x + y) dS &= \int_0^2 \int_0^{2\pi} (u \cos v + u \sin v) u \sqrt{(2u - 3)^2 + 1} du dv \\ &= \int_0^2 \int_0^{2\pi} (\cos v + \sin v) u^2 \sqrt{(2u - 3)^2 + 1} dv du = 0 \end{aligned}$$



57. $\mathbf{F}(x, y, z) = x^2 \mathbf{i} + xy \mathbf{j} + z \mathbf{k}$

Q : solid region bounded by the coordinate planes and the plane $2x + 3y + 4z = 12$

Surface Integral: There are four surfaces for this solid.

$$z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z, \quad \iint_{S_1} 0 dS = 0$$

$$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = -xy, \quad \iint_{S_2} 0 dS = 0$$

$$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = -x^2, \quad \iint_{S_3} 0 dS = 0$$

$$2x + 3y + 4z = 12, \quad \mathbf{N} = \frac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}}, \quad dS = \sqrt{1 + \left(\frac{1}{4}\right) + \left(\frac{9}{16}\right)} dA = \frac{\sqrt{29}}{4} dA$$

$$\begin{aligned} \iint_{S_4} \mathbf{F} \cdot \mathbf{N} dS &= \frac{1}{4} \iint_R (2x^2 + 3xy + 4z) dA \\ &= \frac{1}{4} \int_0^6 \int_0^{4-(2x/3)} (2x^2 + 3xy + 12 - 2x - 3y) dy dx \\ &= \frac{1}{4} \int_0^6 \left[2x^2 \left(\frac{12 - 2x}{3} \right) + \frac{3x}{2} \left(\frac{12 - 2x}{3} \right)^2 + 12 \left(\frac{12 - 2x}{3} \right) - 2x \left(\frac{12 - 2x}{3} \right) - \frac{3}{2} \left(\frac{12 - 2x}{3} \right)^2 \right] dx \\ &= \frac{1}{6} \int_0^6 (-x^3 + x^2 + 24x + 36) dx = \frac{1}{6} \left[-\frac{x^4}{4} + \frac{x^3}{3} + 12x^2 + 36x \right]_0^6 = 66 \end{aligned}$$

Divergence Theorem: Since $\text{div } \mathbf{F} = 2x + x + 1 = 3x + 1$, Divergence Theorem yields

$$\begin{aligned} \iiint_Q \text{div } \mathbf{F} dV &= \int_0^6 \int_0^{(12-2x)/3} \int_0^{(12-2x-3y)/4} (3x + 1) dz dy dx \\ &= \int_0^6 \int_0^{(12-2x)/3} (3x + 1) \left(\frac{12 - 2x - 3y}{4} \right) dy dx \\ &= \frac{1}{4} \int_0^6 (3x + 1) \left[12y - 2xy - \frac{3}{2}y^2 \right]_0^{(12-2x)/3} dx \\ &= \frac{1}{4} \int_0^6 (3x + 1) \left[4(12 - 2x) - 2x \left(\frac{12 - 2x}{3} \right) - \frac{3}{2} \left(\frac{12 - 2x}{3} \right)^2 \right] dx \\ &= \frac{1}{4} \int_0^6 \frac{2}{3} (3x^3 - 35x^2 + 96x + 36) dx = \frac{1}{6} \left[\frac{3x^4}{4} - \frac{35x^3}{3} + 48x^2 + 36x \right]_0^6 = 66. \end{aligned}$$

59. $\mathbf{F}(x, y, z) = (\cos y + y \cos x)\mathbf{i} + (\sin x - x \sin y)\mathbf{j} + xyz\mathbf{k}$

S : portion of $z = y^2$ over the square in the xy -plane with vertices $(0, 0)$, $(a, 0)$, (a, a) , $(0, a)$

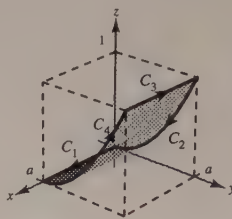
Line Integral: Using the line integral we have:

$$C_1: y = 0, \quad dy = 0$$

$$C_2: x = 0, \quad dx = 0, \quad z = y^2, \quad dz = 2y \, dy$$

$$C_3: y = a, \quad dy = 0, \quad z = a^2, \quad dz = 0$$

$$C_4: x = a, \quad dx = 0, \quad z = y^2, \quad dz = 2y \, dy$$



$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (\cos y + y \cos x) dx + (\sin x - x \sin y) dy + xyz dz \\ &= \int_{C_1} dx + \int_{C_2} 0 + \int_{C_3} (\cos a + a \cos x) dx + \int_{C_4} (\sin a - a \sin y) dy + ay^3(2y \, dy) \\ &= \int_0^a dx + \int_a^0 (\cos a + a \cos x) dx + \int_0^a (\sin a - a \sin y) dy + \int_0^a 2ay^4 dy \\ &= a + \left[x \cos a + a \sin x \right]_a^0 + \left[y \sin a + a \cos y \right]_0^a + \left[2a \frac{y^5}{5} \right]_0^a \\ &= a - a \cos a - a \sin a + a \sin a + a \cos a - a + \frac{2a^6}{5} = \frac{2a^6}{5} \end{aligned}$$

Double Integral: Considering $f(x, y, z) = z - y^2$, we have:

$$\mathbf{N} = \frac{\nabla f}{\|\nabla f\|} = \frac{-2y\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4y^2}}, \quad dS = \sqrt{1 + 4y^2} \, dA, \quad \text{and } \text{curl } \mathbf{F} = xz\mathbf{i} - yz\mathbf{j}.$$

Hence,

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS = \int_0^a \int_0^a 2y^2 z \, dy \, dx = \int_0^a \int_0^a 2y^4 \, dy \, dx = \int_0^a \frac{2a^5}{5} \, dx = \frac{2a^6}{5}.$$

Problem Solving for Chapter 14

1. (a) $\nabla T = \frac{-25}{(x^2 + y^2 + z^2)^{3/2}} [x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]$

$$\mathbf{N} = x\mathbf{i} + \sqrt{1 - x^2}\mathbf{k}$$

$$dS = \frac{1}{\sqrt{1 - x^2}} dA$$

$$\text{Flux} = \iint_S -k \nabla T \cdot \mathbf{N} \, dS$$

$$\begin{aligned} &= 25k \int_R \int \left[\frac{x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} + \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right] dA \\ &= 25k \int_{-1/2}^{1/2} \int_0^1 \left[\frac{x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} + \frac{1 - x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} \right] dy \, dx \\ &= 25k \int_{-1/2}^{1/2} \int_0^1 \frac{1}{(1 + y^2)^{3/2}(1 - x^2)^{1/2}} dy \, dx \\ &= 25k \int_0^1 \frac{1}{(1 + y^2)^{3/2}} dy \int_{-1/2}^{1/2} \frac{1}{(1 - x^2)^{1/2}} dx \\ &= 25k \left(\frac{\sqrt{2}}{2} \right) \left(\frac{\pi}{3} \right) = 25k \frac{\sqrt{2}\pi}{6} \end{aligned}$$

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1. —CONTINUED—

$$(b) \mathbf{r}(u, v) = \langle \cos u, v, \sin u \rangle$$

$$\mathbf{r}_u = \langle -\sin u, 0, \cos u \rangle, \mathbf{r}_v = \langle 0, 1, 0 \rangle$$

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -\cos u, 0, -\sin u \rangle$$

$$\begin{aligned} \nabla T &= \frac{-25}{(x^2 + y^2 + z^2)^{3/2}} [x\mathbf{i} + y\mathbf{j} + z\mathbf{k}] \\ &= \frac{-25}{(v^2 + 1)^{3/2}} [\cos u\mathbf{i} + v\mathbf{j} + \sin u\mathbf{k}] \end{aligned}$$

$$\nabla T \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \frac{-25}{(v^2 + 1)^{3/2}} (-\cos^2 u - \sin^2 u) = \frac{25}{(v^2 + 1)^{3/2}}$$

$$\text{Flux} = \int_0^1 \int_{\pi/3}^{2\pi/3} \frac{25k}{(v^2 + 1)^{3/2}} du dv = 25k \frac{\sqrt{2}\pi}{6}$$

$$3. \mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 2t \rangle$$

$$\mathbf{r}'(t) = \langle -3 \sin t, 3 \cos t, 2 \rangle, \|\mathbf{r}'(t)\| = \sqrt{13}$$

$$I_x = \int_C (y^2 + z^2) \rho ds = \int_0^{2\pi} (9 \sin^2 t + 4t^2) \sqrt{13} dt = \frac{1}{3} \sqrt{13} \pi (32\pi^2 + 27)$$

$$I_y = \int_C (x^2 + z^2) \rho ds = \int_0^{2\pi} (9 \cos^2 t + 4t^2) \sqrt{13} dt = \frac{1}{3} \sqrt{13} \pi (32\pi^2 + 27)$$

$$I_z = \int_C (x^2 + y^2) \rho ds = \int_0^{2\pi} (9 \cos^2 t + 9 \sin^2 t) \sqrt{13} dt = 18\pi \sqrt{13}$$

$$\begin{aligned} 5. \frac{1}{2} \int_C x dy - y dx &= \frac{1}{2} \int_0^{2\pi} [a(\theta - \sin \theta)(a \sin \theta) d\theta - a(1 - \cos \theta)(a(1 - \cos \theta)) d\theta] \\ &= \frac{1}{2} a^2 \int_0^{2\pi} [\theta \sin \theta - \sin^2 \theta - 1 + 2 \cos \theta - \cos^2 \theta] d\theta \\ &= \frac{1}{2} a^2 \int_0^{2\pi} (\theta \sin \theta + 2 \cos \theta - 2) d\theta \\ &= -3\pi a^2 \end{aligned}$$

Hence, the area is $3\pi a^2$.

$$7. (a) \mathbf{r}(t) = t\mathbf{j}, 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = \mathbf{j}$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (\mathbf{i} + \mathbf{j}) \cdot \mathbf{j} dt = \int_0^1 dt = 1$$

$$(b) \mathbf{r}(t) = (t - t^2)\mathbf{i} + t\mathbf{j}, 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = (1 - 2t)\mathbf{i} + \mathbf{j}$$

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 ((2t - t^2)\mathbf{i} + [(t - t^2)^2 + 1]\mathbf{j}) \cdot ((1 - 2t)\mathbf{i} + \mathbf{j}) dt \\ &= \int_0^1 [(1 - 2t)(2t - t^2) + (t^4 - 2t^3 + t^2 + 1)] dt \\ &= \int_0^1 (t^4 - 4t^2 + 2t + 1) dt = \frac{13}{15} \end{aligned}$$

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7. —CONTINUED—

$$(c) \quad \mathbf{r}(t) = c(t - t^2)\mathbf{i} + t\mathbf{j}, \quad 0 \leq t \leq 1$$

$$\mathbf{r}'(t) = c(1 - 2t)\mathbf{i} + \mathbf{j}$$

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{r} &= (c(t - t^2) + t)(c(1 - 2t)) + (c^2(t - t^2)^2 + 1)(1) \\ &= c^2t^4 - 2c^2t^2 + c^2t - 2ct^2 + ct + 1 \end{aligned}$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{30}c^2 - \frac{1}{6}c + 1$$

$$\frac{dW}{dc} = \frac{1}{15}c - \frac{1}{6} = 0 \Rightarrow c = \frac{5}{2}$$

$$\frac{d^2W}{dc^2} = \frac{1}{15} > 0 \quad c = \frac{5}{2} \text{ minimum.}$$

$$9. \quad \mathbf{v} \times \mathbf{r} = \langle a_1, a_2, a_3 \rangle \times \langle x, y, z \rangle$$

$$= \langle a_2z - a_3y, -a_1z + a_3x, a_1y - a_2x \rangle$$

$$\text{curl}(\mathbf{v} \times \mathbf{r}) = \langle 2a_1, 2a_2, 2a_3 \rangle = 2\mathbf{v}$$

By Stokes's Theorem,

$$\begin{aligned} \int_C (\mathbf{v} \times \mathbf{r}) \, d\mathbf{r} &= \iint_S \text{curl}(\mathbf{v} \times \mathbf{r}) \cdot \mathbf{N} \, dS \\ &= \iint_S 2\mathbf{v} \cdot \mathbf{N} \, dS. \end{aligned}$$

$$11. \quad \mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} = \frac{m}{(x^2 + y^2)^{5/2}}[3xy\mathbf{i} + (2y^2 - x^2)\mathbf{j}]$$

$$M = \frac{3mxy}{(x^2 + y^2)^{5/2}} = 3mxy(x^2 + y^2)^{-5/2}$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 3mxy \left[-\frac{5}{2}(x^2 + y^2)^{-7/2}(2y) \right] + (x^2 + y^2)^{-5/2}(3mx) \\ &= 3mx(x^2 + y^2)^{-7/2}[-5y^2 + (x^2 + y^2)] = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}} \end{aligned}$$

$$N = \frac{m(2y^2 - x^2)}{(x^2 + y^2)^{5/2}} = m(2y^2 - x^2)(x^2 + y^2)^{-5/2}$$

$$\begin{aligned} \frac{\partial N}{\partial x} &= m(2y^2 - x^2) \left[-\frac{5}{2}(x^2 + y^2)^{-7/2}(2x) \right] + (x^2 + y^2)^{-5/2}(-2mx) \\ &= mx(x^2 + y^2)^{-7/2}[(2y^2 - x^2)(-5) + (x^2 + y^2)(-2)] \\ &= mx(x^2 + y^2)^{-7/2}(3x^2 - 12y^2) = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}} \end{aligned}$$

Therefore, $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ and \mathbf{F} is conservative.

APPENDIX A

Appendix A.1 Additional Topics in Differential Equations

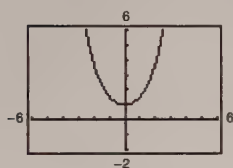
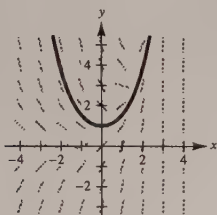
Solutions to Odd-Numbered Exercises

1.

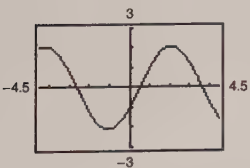
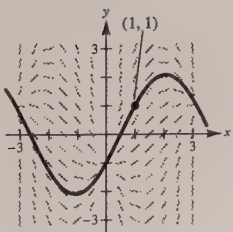
x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-2	Undef.	0	$\frac{1}{2}$	$\frac{2}{3}$	1

$$\frac{dy}{dx} = \frac{x}{y}. \text{ For } (x, y) = (-4, 2), \frac{dy}{dx} = \frac{-4}{2} = -2.$$

3. (a), (c)



5. (a), (c)



(b) $\frac{dy}{dx} = e^x - y$

$$\frac{dy}{dx} + y = e^x \quad \text{Integrating factor: } e^{\int dx} = e^x$$

$$e^x y' + e^x y = e^{2x}$$

$$(ye^x)' = \int e^{2x} dx$$

$$ye^x = \frac{1}{2}e^{2x} + C$$

$$y(0) = 1 \Rightarrow 1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$$

$$ye^x = \frac{1}{2}e^{2x} + \frac{1}{2}$$

$$y = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \frac{1}{2}(e^x + e^{-x})$$

(b) $\frac{dy}{dx} = \csc x + y \cot x$

$$\frac{dy}{dx} - (\cot x)y = \csc x$$

$$\text{Integrating factor: } e^{\int -\cot x dx} = e^{-\ln|\sin x|} = \csc x$$

$$\csc x \cdot y' - \csc x \cot x \cdot y = \csc^2 x$$

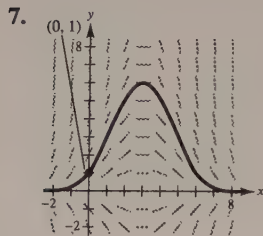
$$(y \csc x)' = \csc^2 x$$

$$y \csc x = \int \csc^2 x dx = -\cot x + C$$

$$y = -\cos x + C \sin x$$

$$y(1) = 1 \Rightarrow 1 = -\cos 1 + C \sin 1 \Rightarrow C = \frac{1 + \cos 1}{\sin 1}$$

$$\approx 1.83$$



9. $y' = x + y$, $y(0) = 2$, $n = 10$, $h = 0.1$

$$y_1 = y_0 + hF(x_0, y_0) = 2 + (0.1)(0 + 2) = 2.2$$

$$y_2 = y_1 + hF(x_1, y_1) = 2.2 + (0.1)(0.1 + 2.2) = 2.43, \text{ etc.}$$

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	2	2.2	2.43	2.693	2.992	3.332	3.715	4.146	4.631	5.174	5.781

11. $y' = 3x - 2y$, $y(0) = 3$, $n = 10$, $h = 0.05$

$$y_1 = y_0 + hF(x_0, y_0) = 3 + (0.05)(3(0) - 2(3)) = 2.7$$

$$y_2 = y_1 + hF(x_1, y_1) = 2.7 + (0.05)(3(0.05) - 2(2.7)) = 2.4375, \text{ etc.}$$

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
y_n	3	2.7	2.438	2.209	2.010	1.839	1.693	1.569	1.464	1.378	1.308

13. $y' = e^{xy}$, $y(0) = 1$, $n = 10$, $h = 0.1$

$$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.1)e^{0(1)} = 1.1$$

$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + (0.1)e^{(0.1)(1.1)} \approx 1.2116, \text{ etc.}$$

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	1	1.1	1.212	1.339	1.488	1.670	1.900	2.213	2.684	3.540	5.958

15. False

$$y' + xy = x^2 \text{ is first-order linear.}$$

17. $\frac{dy}{dx} + \left(\frac{1}{x}\right)y = 3x + 4$

$$\text{Integrating factor: } e^{\int (1/x) dx} = e^{\ln x} = x$$

$$xy = \int x(3x + 4) dx = x^3 + 2x^2 + C$$

$$y = x^2 + 2x + \frac{C}{x}$$

19. $\frac{dy}{dx} - 3x^2y = e^{x^3}$

$$\text{Integrating factor: } e^{-\int 3x^2 dx} = e^{-x^3}$$

$$ye^{-x^3} = \int dx$$

$$ye^{-x^3} = x + C$$

$$y = (x + C)e^{x^3}$$

21. $y' - y = \cos x$

Integrating factor: $e^{\int -1 dx} = e^{-x}$

$$ye^{-x} = \int e^{-x} \cos x dx$$

$$= \frac{1}{2} e^{-x} (-\cos x + \sin x) + C$$

$$y = \frac{1}{2} (\sin x - \cos x) + Ce^x$$

25. $(3y + \sin 2x) dx - dy = 0$

$$y' - 3y = \sin 2x$$

Integrating factor: $e^{\int -3 dx} = e^{-3x}$

$$ye^{-3x} = \int e^{-3x} \sin 2x dx$$

$$= \frac{1}{13} e^{-3x} (-3 \sin 2x - 2 \cos 2x) + C$$

$$y = -\frac{1}{13} (3 \sin 2x + 2 \cos 2x) + Ce^{3x}$$

29. $dy = (y \tan x + 2e^x) dx$

$$\frac{dy}{dx} - (\tan x)y = 2e^x$$

Integrating factor: $e^{-\int \tan x dx} = e^{\ln|\cos x|} = \cos x$

$$y \cos x = \int 2e^x \cos x dx = e^x (\cos x + \sin x) + C$$

$$y = e^x (1 + \tan x) + C \sec x$$

33. $y' \cos^2 x + y - 1 = 0$

$$y' + (\sec^2 x)y = \sec^2 x$$

Integrating factor: $e^{\int \sec^2 x dx} = e^{\tan x}$

$$ye^{\tan x} = \int \sec^2 x e^{\tan x} dx = e^{\tan x} + C$$

$$y = 1 + Ce^{-\tan x}$$

Initial condition: $y(0) = 5, C = 4$

Particular solution: $y = 1 + 4e^{-\tan x}$

23. $\frac{dy}{dx} = \frac{x+y}{x} = \frac{y}{x} + 1$

$$y' - \frac{1}{x}y = 1$$

Integrating factor: $e^{-\int 1/x dx} = e^{-\ln x} = 1/x$

$$\frac{1}{x}y' - \frac{1}{x^2}y = 1/x$$

$$\left(\frac{1}{x}y\right)' = 1/x$$

$$\frac{1}{x}y = \int \frac{1}{x} dx = \ln|x| + C$$

$$y = x \ln|x| + Cx$$

27. $(x-1)y' + y = x^2 - 1$

$$y' + \left(\frac{1}{x-1}\right)y = x+1$$

Integrating factor: $e^{\int 1/(x-1) dx} = e^{\ln|x-1|} = x-1$

$$y(x-1) = \int (x^2-1) dx = \frac{1}{3}x^3 - x + C_1$$

$$y = \frac{x^3 - 3x + C}{3(x-1)}$$

31. $y' - \left(\frac{a}{x}\right)y = bx^3$

Integrating factor: $e^{-\int (a/x) dx} = e^{-a \ln x} = x^{-a}$

$$yx^{-a} = \int bx^3(x^{-a}) dx = \frac{b}{4-a} x^{4-a} + C$$

$$y = \frac{bx^4}{4-a} + Cx^a$$

35. $y' + y \tan x = \sec x + \cos x$

Integrating factor: $e^{\int \tan x dx} = e^{\ln|\sec x|} = \sec x$

$$y \sec x = \int \sec x (\sec x + \cos x) dx = \tan x + x + C$$

$$y = \sin x + x \cos x + C \cos x$$

Initial condition: $y(0) = 1, 1 = C$

Particular solution: $y = \sin x + (x+1) \cos x$

37. $y' + \left(\frac{1}{x}\right)y = 0$

Integrating factor: $e^{\int (1/x) dx} = e^{\ln|x|} = x$

Separation of variables:

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\int \frac{1}{y} dy = \int -\frac{1}{x} dx$$

$$\ln y = -\ln x + \ln C$$

$$\ln xy = \ln C$$

$$xy = C$$

Initial condition: $y(2) = 2, C = 4$

Particular solution: $xy = 4$

39. $x dy = (x + y + 2) dx$

$$\frac{dy}{dx} - \left(\frac{1}{x}\right)y = \frac{x+2}{x}$$

Integrating factor: $e^{\int -(1/x) dx} = e^{-\ln|x|} = \frac{1}{x}$

$$y\left(\frac{1}{x}\right) = \int \frac{x+2}{x^2} dx = \ln|x| - \frac{2}{x} + C$$

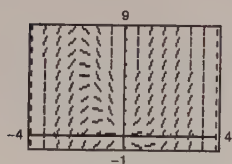
$$y = x \ln|x| - 2 + Cx$$

Initial condition: $y(1) = 10$

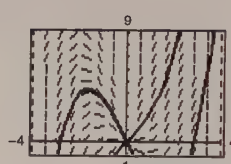
$$10 = -2 + C \Rightarrow C = 12$$

Particular solution: $y = x \ln|x| - 2 + 12x$

41. (a)



(c)



(b) $\frac{dy}{dx} - \frac{1}{x^2}y = x^2$

Integrating factor $e^{\int -(1/x^2) dx} = e^{-\ln x} = \frac{1}{x}$

$$\frac{1}{x}y' - \frac{1}{x^2}y = x$$

$$\left(\frac{1}{x}y\right)' = \int x dx = \frac{x^2}{2} + C$$

$$y = \frac{x^3}{2} + Cx$$

$$(-2, 4): 4 = \frac{-8}{2} - 2C \Rightarrow C = -4 \Rightarrow y = \frac{x^3}{2} - 4x = \frac{1}{2}x(x^2 - 8)$$

$$(2, 8): 8 = \frac{8}{2} + 2C \Rightarrow C = 2 \Rightarrow y = \frac{x^3}{2} + 2x = \frac{1}{2}x(x^2 + 4)$$

43. $L \frac{dI}{dt} + RI = E_0, I' + \frac{R}{L}I = \frac{E_0}{L}$

Integrating factor: $e^{\int (R/L) dt} = e^{Rt/L}$

$$I e^{Rt/L} = \int \frac{E_0}{L} e^{Rt/L} dt = \frac{E_0}{R} e^{Rt/L} + C$$

$$I = \frac{E_0}{R} + C e^{-Rt/L}$$

45. $L \frac{dI}{dt} + RI = E_0 \sin \omega t$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E_0}{L} \sin \omega t$$

Integrating factor: $e^{\int (R/L) dt} = e^{Rt/L}$

$$Ie^{Rt/L} = \int \frac{E_0}{L} e^{Rt/L} \sin \omega t dt$$

$$= \frac{E_0}{L} \left[\frac{L^2 e^{Rt/L}}{R^2 + L^2 \omega^2} \left(\frac{R}{L} \sin \omega t - \omega \cos \omega t \right) \right] + C = \frac{E_0 e^{Rt/L}}{R^2 + \omega^2 L^2} (R \sin \omega t - \omega L \cos \omega t) + C$$

$$I = \frac{E_0}{R^2 + \omega^2 L^2} (R \sin \omega t - \omega L \cos \omega t) + C e^{-Rt/L}$$

47. $\frac{dP}{dt} = kP + N, N \text{ constant}$

$$\frac{dP}{kP + N} = dt$$

$$\int \frac{1}{kP + N} dP = \int dt$$

$$\frac{1}{k} \ln(kP + N) = t + C_1$$

$$\ln(kP + N) = kt + C_2$$

$$kP + N = e^{kt+C_2}$$

$$P = \frac{C_3 e^{kt} - N}{k}$$

$$P = C e^{kt} - \frac{N}{k}$$

When $t = 0$: $P = P_0$

$$P_0 = C - \frac{N}{k} \Rightarrow C = P_0 + \frac{N}{k}$$

$$P = \left(P_0 + \frac{N}{k} \right) e^{kt} - \frac{N}{k}$$

49. (a) $A = \frac{P}{r}(e^{rt} - 1)$

$$A = \frac{100,000}{0.06}(e^{0.06(5)} - 1) \approx 583,098.01$$

(b) $A = \frac{250,000}{0.05}(e^{0.05(10)} - 1) \approx 3,243,606.35$

51. $\frac{dA}{dt} - rA = -P$

For this linear differential equation, we have $P(t) = -r$ and $Q(t) = -P$. Therefore, the integrating factor is

$u(x) = e^{\int -r dt} = e^{-rt}$ and the solution is

$$A = e^{rt} \int -P e^{-rt} dt = e^{rt} \left(\frac{P}{r} e^{-rt} + C \right) = \frac{P}{r} + C e^{rt}.$$

Since $A = A_0$ when $t = 0$, we have $C = A_0 - (P/r)$ which implies that

$$A = \frac{P}{r} + \left(A_0 - \frac{P}{r} \right) e^{rt}.$$

53. (a) $\frac{dQ}{dt} = q - kQ, q \text{ constant}$

(b) $Q' + kQ = q$

Let $P(t) = k, Q(t) = q$, then the integrating factor is $u(t) = e^{kt}$.

$$Q = e^{-kt} \int q e^{kt} dt = e^{-kt} \left(\frac{q}{k} e^{kt} + C \right) = \frac{q}{k} + C e^{-kt}$$

When $t = 0: Q = Q_0$

$$Q_0 = \frac{q}{k} + C \Rightarrow C = Q_0 - \frac{q}{k}$$

$$Q = \frac{q}{k} + \left(Q_0 - \frac{q}{k} \right) e^{-kt}$$

(c) $\lim_{t \rightarrow \infty} Q = \frac{q}{k}$

55. $y' - 2x = 0$

$$\int dy = \int 2x dx$$

$$y = x^2 + C$$

Matches c.

57. $y' - 2xy = 0$

$$\int \frac{dy}{y} = \int 2x dx$$

$$\ln y = x^2 + C_1$$

$$y = C e^{x^2}$$

Matches a.



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